Nonholonomic Lagrangians of second order: 
Equations of motion for the constrained Lagrangian

Mircea Crâșmăreanu


Abstract

The equations of motion for the associated Lagrangian to a non-holonomic Lagrangian of second order are computed. An example is given.

Subject Classification: 70H25, 70H35, 58F05.
Key words and phrases: Lagrangian of second order, Euler-Lagrange equations, nonholonomic constraints, constrained Lagrangian.

Introduction

In the last years there is an increasing interest in nonholonomic mechanics especially from a geometrical point of view. Following the methodology of [2], where are treated nonholonomic Lagrangians of first order, we obtain the equations of motion in terms of the associated constrained Lagrangian of a nonholonomic Lagrangian involving accelerations. This type of Lagrangian is illustrated by the spinning particle.

This paper is dedicated to the memory of Romanian Academician Gheorghe Vrâncianu (1900-1979) who introduces in 1926 the notion of nonholonomic spaces, in order to give a geometrical approach to nonholonomic
mechanics ([8], [9]). Note that the Romanian school of mathematics has an important contribution of this subject ([3], [4], [7], [8], [9], [10]).

1 Equations of motion

The starting point is a configuration-space given by a \( n \)-dimensional manifold \( Q \), for which we consider the tangent bundle of order two \( T^2Q ([5], [6]) \). For coordinates \( (q^i)_{1 \leq i \leq n} \) on \( Q \) we have the induced coordinates
\[
(q^i, \dot{q}^i = \frac{dq^i}{dt}, \ddot{q}^i = \frac{d^2q^i}{dt^2}) \text{ on } T^2Q.
\]
Let us suppose that the evolution of the considered system is described by the following objects:

1. a second-order Lagrangian, that is a smooth map \( L: T^2Q \to \mathbb{R} ([6]) \)
2. a set of \( p \) independent one-forms \( (\omega^a(q))_{1 \leq a \leq p} \) whose vanishing gives the constraints of the system.

This 1-forms defines an \( (n-p) \)-dimensional distribution \( D \) on \( Q \) i.e. \( (\omega^a(q)) \) is a local basis of the annihilator \( D^0 \) of \( D \). Also, this constraints means that the only allowable velocities are the tangent vectors belonging to \( D \) or in other words the motion is constrained to the submanifold \( D \).

The Lagrangian \( L \) gives the Euler-Lagrange equations of order two ([5]):
\[
\delta L = (EL)_i^{\text{free}} \delta q^i = 0 \tag{1.1a}
\]
with:
\[
(EL)_i^{\text{free}} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{q}^i} \right) \tag{1.1b}
\]
and supposing that the constraints are nonholonomic, we can choose a local coordinate chart and a local basis for the constraints such that ([2, p. 31]):
\[
\omega^a(q) = ds^a + \Lambda^a_\alpha(r, s) \, dr^\alpha, \ 1 \leq a \leq p \tag{1.2}
\]
where \( q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p \).

From (1.2) it results that:
\[
\delta s^a + \Lambda^a_\alpha \delta r^\alpha = 0 \tag{1.3}
\]
which, by substitution into (1.1) yields:
\[
\frac{\partial L}{\partial r^\alpha} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}^\alpha} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{r}^\alpha} \right) =
\]
\[ A^a_\alpha \left[ \frac{\partial L}{\partial s^a} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}^a} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{s}^a} \right) \right] = 0. \] (1.4)

Equations (1.4) combined with the constraint equations:

\[ s^a = - A^a_\alpha \dot{r}^\alpha \] (1.5a)
\[ \ddot{s}^a = - \frac{d}{dt} \left( A^a_\alpha \dot{r}^\alpha - A^a_\alpha \ddot{r}^\alpha \right) \] (1.5b)

gives a complete description of the equations of motion. Remark that another form for (1.5b) is:

\[ \ddot{s}^a = A^a_{\alpha \beta} \dot{r}^\alpha \dot{r}^\beta - A^a_\alpha \ddot{r}^\alpha \] (1.5b')

where:

\[ A^a_{\alpha \beta}(r, s) = \frac{\partial A^a_\alpha}{\partial s^b \partial r^\beta} A^b_\beta - \frac{\partial A^a_\alpha}{\partial r^\beta}. \] (1.6)

Following [2, p. 31] we define an associated constrained Lagrangian \( L_c \) by substituting the constraints (1.5) into the Lagrangian \( L \):

\[ L_c(r^\alpha, s^a, \dot{r}^\alpha, \ddot{r}^\alpha) \overset{def.}{=} L \left( r^\alpha, s^a, \dot{r}^\alpha, - A^a_\alpha \dot{r}^\alpha, \ddot{r}^\alpha, A^a_{\alpha \beta} \dot{r}^\alpha \dot{r}^\beta - A^a_\alpha \ddot{r}^\alpha \right). \] (1.7)

A direct coordinates calculation shows:

\[ \frac{\partial L_c}{\partial \dot{r}^\alpha} = \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L \partial A^b_\beta}{\partial s^b \partial \dot{r}^\alpha} \dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^a} \left( \frac{\partial A^b_\beta}{\partial r^\alpha} \dot{r}^\beta - \frac{\partial A^b_\beta}{\partial r^\alpha} \ddot{r}^\beta \right) \] (1.8a)
\[ \frac{\partial L_c}{\partial s^a} = \frac{\partial L}{\partial s^a} - \frac{\partial L \partial A^b_\beta}{\partial s^b \partial \dot{r}^\alpha} \dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^a} \left( \frac{\partial A^b_\beta}{\partial s^a} \dot{r}^\beta - \frac{\partial A^b_\beta}{\partial s^a} \ddot{r}^\beta \right) \] (1.8b)
\[ \frac{\partial L_c}{\partial \ddot{r}^\alpha} = \frac{\partial L}{\partial \ddot{r}^\alpha} - \frac{\partial L}{\partial \ddot{s}^a} A^b_\alpha + \frac{\partial L}{\partial \ddot{s}^a} \left( A^b_{\alpha \beta} + A^b_{\beta \alpha} \right) \dot{r}^\beta \] (1.8c)
\[ \frac{\partial L_c}{\partial \dot{r}^\alpha} = \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial \ddot{s}^a} A^b_\alpha. \] (1.8d)
A long, but straightforward computation gives the equations of motion for \( L_c \):

\[
(EL)^{\text{constraints}}_\alpha = \left( \frac{\partial L}{\partial s^b} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}^b} \right) \right) 1^b_{\alpha \beta} \dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^b} 2^b_{\alpha \beta \gamma} \dot{r}^\beta \dot{r}^\gamma
\] (1.9a)

where:

\[
(EL)^{\text{constraints}}_\alpha = \frac{\partial L_c}{\partial r^\alpha} - \frac{d}{dt} \left( \frac{\partial L_c}{\partial \dot{r}^\alpha} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L_c}{\partial \ddot{r}^\alpha} \right) - A^a_{\alpha} \frac{\partial L_c}{\partial s^a}
\] (1.9b)

\[
1^b_{\alpha \beta} = A^b_{\beta \alpha} - A^b_{\alpha \beta}
\] (1.10a)

\[
2^b_{\alpha \beta \gamma} = \frac{\partial A^b_{\beta \gamma}}{\partial r^\alpha} - \frac{\partial A^b_{\alpha \beta}}{\partial r^\gamma} + A^a_{\alpha} \frac{\partial A^b_{\beta \gamma}}{\partial s^a} - A^a_{\beta} \frac{\partial A^b_{\alpha \gamma}}{\partial s^a}. \] (1.10b)

Remark that the coefficients \( B \) does not depend of Lagrangian but only of constraints and \( 1^b_{\alpha \alpha} = 2^b_{\alpha \beta \alpha} = 0 \) for every \( \alpha \).

2 Example: the nonholonomic spinning particle

According to [5] the Lagrangian of classical spinning particle is:

\[
L(q, \dot{q}, \ddot{q}) = \frac{1}{2} \sum_{i=1}^{3} (\dot{q}^i)^2 - \frac{1}{2} \sum_{i=1}^{3} (\ddot{q}^i)^2.
\] (2.1)

The Euler-Lagrange equations for the free Lagrangian (2.1) are:

\[
(EL)^{\text{free}}_i = \frac{d^2 q^i}{dt^2} + \frac{d^4 q^i}{dt^4} = 0, \quad 1 \leq i \leq 3.
\] (2.2)

Consider the nonholonomic constraint of Rosenberg-Bates-Sniatycki type([1], [2, p. 84]):

\[
\dot{z} = y \dot{x}
\] (2.3)

which gives:

\[
\ddot{z} = \ddot{y} \dot{x} + y \ddot{x}
\] (2.4a)
The constrained Lagrangian is:

\[ L_c(y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + y^2 \dot{x}^2) - \frac{1}{2} [\ddot{x}^2 + \ddot{y}^2 + (\dot{x}\ddot{y} + y\dot{x})^2] \]  

(2.5)

We have: \( p = 1, s^1 = z, r^1 = x, r^2 = y \), \( A_{12} = 1, A_{21} = 0 \) and:

\[ \frac{\partial L_c}{\partial y} = y \dot{x}^2 - \ddot{x} \ddot{z} \]  

(2.6a)

\[ \frac{\partial L_c}{\partial \dot{x}} = \dot{x} + \dot{y}^2 - y \ddot{z}, \quad \frac{\partial L_c}{\partial \dot{y}} = \dot{y} - \dot{x} \ddot{z} \]  

(2.6b)

\[ \frac{\partial L_c}{\partial \ddot{x}} = -\dddot{x} - y \dddot{z}, \quad \frac{\partial L_c}{\partial \ddot{y}} = -\dddot{y} \]  

(2.6c)

where \( \dddot{z} \) is given by (2.4a).

Therefore:

\[ \frac{\partial L_c}{\partial x} - \frac{d}{dt} \left( \frac{\partial L_c}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L_c}{\partial \ddot{x}} \right) - A_1 \frac{\partial L_c}{\partial z} = 0 \]

\( (EL)_{\text{constraints}}^1 = -\frac{d^4 x}{dt^4} - \frac{d^2 x}{dt^2} - \dddot{x} \dddot{y} - 2\dddot{y} \dddot{z} - \dddot{y} \frac{d^3 z}{dt^3} - y \frac{d^4 z}{dt^4} \)  

(2.8a)

\( (EL)_{\text{constraints}}^2 = -\frac{d^4 y}{dt^4} - \frac{d^2 y}{dt^2} + \dddot{x} \dddot{y} + \dddot{x} \frac{d^3 z}{dt^3} \)  

(2.8b)

The right hand side of (1.9a) is:

\[ (EL)_{\text{constraints}} = -\dddot{y} \left( \dddot{z} + \frac{d^3 z}{dt^3} \right) \]  

(2.9a)
\[(EL)_{\text{constraints}}^2 = \dot{x} \left( \dot{z} + \frac{d^3 z}{dt^3} \right) \]  

(2.9b)

and then the equations (1.9a - b) gives:

\[(EL)_{\text{constraints}}^1 : \frac{d^4 x}{dt^4} + \frac{d^2 x}{dt^2} + y^2 \frac{d^2 x}{dt^2} + \frac{dy}{dt} \frac{dz}{dt} + y \frac{d^4 z}{dt^4} = 0 \]  

(2.10a)

\[(EL)_{\text{constraints}}^2 : \frac{d^4 y}{dt^4} + \frac{d^2 y}{dt^2} = 0. \]  

(2.10b)

**References**


Faculty of Mathematics  
University ”Al. I. Cuza”  
Iași, 6600, Romania  
E-mail: mcrasm@uaic.ro