NONHOLONOMIC LAGRANGIANS OF SECOND ORDER:
Equations of motion for the constrained Lagrangian

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Abstract

The equations of motion for the associated Lagrangian to a non-holonomic Lagrangian of second order are computed. An example is given.

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Introduction

In the last years there is an increasing interest in nonholonomic mechanics especially from a geometrical point of view. Following the methodology of [2], where are treated nonholonomic Lagrangians of first order, we obtain the equations of motion in terms of the associated constrained Lagrangian of a nonholonomic Lagrangian involving accelerations. This type of Lagrangian is illustrated by the spinning particle.

This paper is dedicated to the memory of Romanian Academician Gheorghe Vrânceanu(1900-1979) who introduces in 1926 the notion of *nonholonomic spaces*, in order to give a geometrical approach to nonholonomic mechanics([8],[9]). Note that the Romanian school of mathematics has an important contribution of this subject([3], [4], [7], [8], [9], [10]).

1 Equations of motion

The starting point is a configuration-space given by a n-dimensional manifold Q, for which we consider the tangent bundle of order two T^2Q([5], [6]). For coordinates $(q^i)_{1\leq i\leq n}$ on Q we have the induced coordinates $\left(q^i, \dot{q}^i = \frac{dq^i}{dt}, \ddot{q}^i = \frac{d^2q^i}{dt^2}\right)$ on T^2Q.

Let us suppose that the evolution of the considered system is described by the following objects:

1. a second-order Lagrangian, that is a smooth map $L : T^2Q \longrightarrow \mathbb{R}$([6])

2. a set of p independent one-forms $(\omega^a(q))_{1\leq a\leq p}$ whose vanishing gives the constraints of the system.

This 1-forms defines an $(n-p)$-dimensional distribution D on Q i.e. $(\omega^a(q))$ is a local basis of the annihilator D^0 of D. Also, this constraints means that the only allowable velocities are the tangent vectors belonging to D or in other words the motion is constrained to the submanifold D.

The Lagrangian L gives the Euler-Lagrange equations of order two([5]):

$$\delta L = (EL)_i^{free}\,\delta q^i = 0 \tag{1.1a}$$

with:

$$(EL)_i^{free} = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) \tag{1.1b}$$

and supposing that the constraints are nonholonomic, we can choose a local coordinate chart and a local basis for the constraints such that([2, p. 31]):

$$\omega^a(q) = ds^a + \overset{1}{A}{}^a_\alpha(r,s)\,dr^\alpha,\ 1\leq a\leq p \tag{1.2}$$

where $q = (r,s) \in \mathbb{R}^{n-p}\times\mathbb{R}^p$.

From (1.2) it results that:

$$\delta s^a + \overset{1}{A}{}^a_\alpha\,\delta r^\alpha = 0 \tag{1.3}$$

which, by substitution into (1.1) yields:

$$\frac{\partial L}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^\alpha}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{r}^\alpha}\right) =$$

$$= \overset{1}{A}{}^{a}_{\alpha}\left[\frac{\partial L}{\partial s^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}^a}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{s}^a}\right)\right]. \tag{1.4}$$

Equations (1.4) combined with the constraint equations:

$$\dot{s}^a = - \overset{1}{A}{}^{a}_{\alpha}\,\dot{r}^\alpha \tag{1.5a}$$

$$\ddot{s}^a = -\frac{d}{dt}(\overset{1}{A}{}^{a}_{\alpha})\dot{r}^\alpha - \overset{1}{A}{}^{a}_{\alpha}\,\ddot{r}^\alpha \tag{1.5b}$$

gives a complete description of the equations of motion. Remark that another form for (1.5b) is:

$$\ddot{s}^a = \overset{2}{A}{}^{a}_{\alpha\beta}\,\dot{r}^\alpha\dot{r}^\beta - \overset{1}{A}{}^{a}_{\alpha}\,\ddot{r}^\alpha \tag{1.5b'}$$

where:

$$\overset{2}{A}{}^{a}_{\alpha\beta}\,(r,s) = \frac{\partial\,\overset{1}{A}{}^{a}_{\alpha}}{\partial s^b}\,\overset{1}{A}{}^{b}_{\beta} - \frac{\partial\,\overset{1}{A}{}^{a}_{\alpha}}{\partial r^\beta}. \tag{1.6}$$

Following [2, p. 31] we define an associated *constrained* Lagrangian L_c by substituting the constraints (1.5) into the Lagrangian L:

$$L_c\left(r^\alpha, s^a, \dot{r}^\alpha, \ddot{r}^\alpha\right) \overset{def.}{=} L\left(r^\alpha, s^a, \dot{r}^\alpha, -\overset{1}{A}{}^{a}_{\alpha}\,\dot{r}^\alpha, \ddot{r}^\alpha, \overset{2}{A}{}^{a}_{\alpha\beta}\,\dot{r}^\alpha\dot{r}^\beta - \overset{1}{A}{}^{a}_{\alpha}\,\ddot{r}^\alpha\right). \tag{1.7}$$

A direct coordinates calculation shows:

$$\frac{\partial L_c}{\partial r^\alpha} = \frac{\partial L}{\partial r^\alpha} - \frac{\partial L}{\partial \dot{s}^b}\frac{\partial\,\overset{1}{A}{}^{b}_{\beta}}{\partial r^\alpha}\dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^b}\left(\frac{\partial\,\overset{2}{A}{}^{b}_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\beta\dot{r}^\gamma - \frac{\partial\,\overset{1}{A}{}^{b}_{\beta}}{\partial r^\alpha}\ddot{r}^\beta\right) \tag{1.8a}$$

$$\frac{\partial L_c}{\partial s^a} = \frac{\partial L}{\partial s^a} - \frac{\partial L}{\partial \dot{s}^b}\frac{\partial\,\overset{1}{A}{}^{b}_{\beta}}{\partial s^a}\dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^b}\left(\frac{\partial\,\overset{2}{A}{}^{b}_{\beta\gamma}}{\partial s^a}\dot{r}^\beta\dot{r}^\gamma - \frac{\partial\,\overset{1}{A}{}^{b}_{\beta}}{\partial s^a}\ddot{r}^\beta\right) \tag{1.8b}$$

$$\frac{\partial L_c}{\partial \dot{r}^\alpha} = \frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial \dot{s}^b}\,\overset{1}{A}{}^{b}_{\alpha} + \frac{\partial L}{\partial \ddot{s}^b}(\overset{2}{A}{}^{b}_{\alpha\beta} + \overset{2}{A}{}^{b}_{\beta\alpha})\dot{r}^\beta \tag{1.8c}$$

$$\frac{\partial L_c}{\partial \ddot{r}^\alpha} = \frac{\partial L}{\partial \ddot{r}^\alpha} - \frac{\partial L}{\partial \ddot{s}^b}\,\overset{1}{A}{}^{b}_{\alpha}\,. \tag{1.8d}$$

A long, but straightforward computation gives the equations of motion for L_c:

$$(EL)_\alpha^{constraints} = \left(\frac{\partial L}{\partial \dot{s}^b} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{s}^b}\right)\right) \overset{1}{B}{}^b_{\alpha\beta}\, \dot{r}^\beta + \frac{\partial L}{\partial \ddot{s}^b}\, \overset{2}{B}{}^b_{\alpha\beta\gamma}\, \dot{r}^\beta \dot{r}^\gamma \qquad (1.9a)$$

where:

$$(EL)_\alpha^{constraints} = \frac{\partial L_c}{\partial r^\alpha} - \frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{r}^\alpha}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial \ddot{r}^\alpha}\right) - \overset{1}{A}{}^a_\alpha\, \frac{\partial L_c}{\partial s^a} \qquad (1.9b)$$

$$\overset{1}{B}{}^b_{\alpha\beta} = \overset{2}{A}{}^b_{\beta\alpha} - \overset{2}{A}{}^b_{\alpha\beta} \qquad (1.10a)$$

$$\overset{2}{B}{}^b_{\alpha\beta\gamma} = \frac{\partial\, \overset{2}{A}{}^b_{\beta\gamma}}{\partial r^\alpha} - \frac{\partial\, \overset{2}{A}{}^b_{\beta\alpha}}{\partial r^\gamma} + \overset{1}{A}{}^a_\gamma\, \frac{\partial\, \overset{2}{A}{}^b_{\beta\alpha}}{\partial s^a} - \overset{1}{A}{}^a_\alpha\, \frac{\partial\, \overset{2}{A}{}^b_{\beta\gamma}}{\partial s^a}. \qquad (1.10b)$$

Remark that the coefficients B does not depend of Lagrangian but only of constraints and $\overset{1}{B}{}^b_{\alpha\alpha} = \overset{2}{B}{}^b_{\alpha\beta\alpha} = 0$ for every α.

2 Example: the nonholonomic spinning particle

According to [5] the Lagrangian of classical spinning particle is:

$$L(q, \dot{q}, \ddot{q}) = \frac{1}{2}\sum_{i=1}^{3}\left(\dot{q}^i\right)^2 - \frac{1}{2}\sum_{i=1}^{3}\left(\ddot{q}^i\right)^2. \qquad (2.1)$$

The Euler-Lagrange equations for the free Lagrangian (2.1) are:

$$(EL)_i^{free} = \frac{d^2 q^i}{dt^2} + \frac{d^4 q^i}{dt^4} = 0,\ 1 \leq i \leq 3. \qquad (2.2)$$

Consider the nonholonomic constraint of Rosenberg-Bates-Sniatycki type([1], [2, p. 84]):

$$\dot{z} = y\dot{x} \qquad (2.3)$$

which gives:

$$\ddot{z} = \dot{y}\dot{x} + y\ddot{x} \qquad (2.4a)$$

$$\overset{1}{A}{}^{1}_{1}= -y,\ \ \overset{1}{A}{}^{1}_{2}= 0. \tag{2.4b}$$

The constrained Lagrangian is:

$$L_c\left(y,\dot{x},\dot{y},\ddot{x},\ddot{y}\right)=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+y^2\dot{x}^2\right)-\frac{1}{2}\left[\ddot{x}^2+\ddot{y}^2+\left(\dot{x}\dot{y}+y\ddot{x}\right)^2\right] \tag{2.5}$$

We have: $p=1, s^1=z, r^1=x, r^2=y,\ \overset{2}{A}{}^{1}_{12}=1, \overset{2}{A}{}^{1}_{21}=0$ and:

$$\frac{\partial L_c}{\partial y}=y\dot{x}^2-\ddot{x}\ddot{z} \tag{2.6a}$$

$$\frac{\partial L_c}{\partial \dot{x}}=\dot{x}+\dot{x}y^2-\dot{y}\ddot{z},\ \ \frac{\partial L_c}{\partial \dot{y}}=\dot{y}-\dot{x}\ddot{z} \tag{2.6b}$$

$$\frac{\partial L_c}{\partial \ddot{x}}=-\ddot{x}-y\ddot{z},\ \ \frac{\partial L_c}{\partial \ddot{y}}=-\ddot{y} \tag{2.6c}$$

where $\ddot{z}$ is given by $(2.4a)$.

Therefore:

$$\frac{\partial L_c}{\partial x}-\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{x}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial \ddot{x}}\right)-\overset{1}{A}{}^{1}_{1}\frac{\partial L_c}{\partial z}=$$

$$=-\frac{d}{dt}\left(\dot{x}+\dot{x}y^2-\dot{y}\ddot{z}\right)-\frac{d^2}{dt^2}\left(\ddot{x}+y\ddot{z}\right) \tag{2.7a}$$

$$\frac{\partial L_c}{\partial y}-\frac{d}{dt}\left(\frac{\partial L_c}{\partial \dot{y}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L_c}{\partial \ddot{y}}\right)-\overset{1}{A}{}^{1}_{2}\frac{\partial L_c}{\partial z}=$$

$$=y\dot{x}^2-\ddot{x}\ddot{z}-\frac{d}{dt}\left(\dot{y}-\dot{x}\ddot{z}\right)-\frac{d^2}{dt^2}\ddot{y} \tag{2.7b}$$

that is:

$$(EL)_1^{constraints}=-\frac{d^4x}{dt^4}-\frac{d^2x}{dt^2}-\ddot{x}y^2-2\dot{y}\dot{z}-\dot{y}\frac{d^3z}{dt^3}-y\frac{d^4z}{dt^4} \tag{2.8a}$$

$$(EL)_2^{constraints}=-\frac{d^4y}{dt^4}-\frac{d^2y}{dt^2}+\dot{x}^2y+\dot{x}\frac{d^3z}{dt^3}. \tag{2.8b}$$

The right hand side of $(1.9a)$ is:

$$(EL)_1^{constraints}=-\dot{y}\left(\dot{z}+\frac{d^3z}{dt^3}\right) \tag{2.9a}$$

$$(EL)_2^{constraints} = \dot{x}\left(\dot{z} + \frac{d^3z}{dt^3}\right) \qquad (2.9b)$$

and then the equations $(1.9a - b)$ gives:

$$(EL)_1^{constraints} : \frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + y^2\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{dz}{dt} + y\frac{d^4z}{dt^4} = 0 \qquad (2.10a)$$

$$(EL)_2^{constraints} : \frac{d^4y}{dt^4} + \frac{d^2y}{dt^2} = 0. \qquad (2.10b)$$

References

[1] Bates, L., Śniatycki, J., *Nonholonomic reduction*, Reports on Mathematical Physics, 32(1993), 99-115.

[2] Bloch, A. M., Krishnaprasad, P. S, Marsden, J. E., Murray, R. M., *Nonholonomic Mechanical Systems with Symmetries*, Arch. Rational Mech. Anal., 136(1996), 21-99.

[3] Haimovici, M., *La geometrie des systemes mechaniques non holonomes*, Acad. Română, Fil. Iaşi, Stud. Cerc. Sti., Ser. I 5, No. 3-4(1954), 49-84.

[4] Haimovici, M., *Sur la representation geometrique des systemes mechaniques non holonomes*, Begriff des Raumes in der Geometrie, Ber. Riemann-Tagung. Schriftenreihe des Forsch. Inst. Math. Deutschen Akad. der Wiss. zu Berlin, 1(1957), 280-285.

[5] de León, M., De Diego, D. M., *Symmetries and constants of motion for higher-order Lagrangian systems*, J. Math. Phys., 36(1995), 4138-4161.

[6] Miron, R., *The Geometry of Higher-Order Lagrange Spaces(Applications to Mechanics and Physics)*, Fundamental Theories of Physics, no. 82, Kluwer Academic Publishers, 1997.

[7] Vâlcovici, V., *Une extension des liaison non holonomes et des principles variationnels*, Ber. Verh. Sachs. Acad. Wiss. Leipzig, Math. Nat. Kl. 102(1958), 1-39.

[8] Vrânceanu, Gh., *Sur les espaces nonholonomes*, C. R. Acad. Sci. Paris, 183(1926), 852-854.

[9] Vrânceanu, Gh., *Sur le calcul différential absolu pour les variétés non-holonomes*, C. R. Acad. Sci. Paris, 183(1926), 1083-1085.

[10] Vrânceanu, Gh., *Les espaces non holonomes*, Mémorial Sci. Math., 76, 1936.

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