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The geometry of tangent conjugate connections

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Abstract

The notion of conjugate connection is introduced in the almost tangent geometry and its properties are studied from a global point of view. Two variants for this type of connections are also considered in order to find the linear connections making parallel a given almost tangent structure.

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Introduction

Let F be a tensor field of $(1,1)$-type on a given smooth manifold M. An interesting object in the geometry of pair (M,F) is provided by the class of F-linear connections i.e. linear connections ∇ making F parallel: $\nabla F = 0$. In order to determine this class, in [9] is introduced the notion of F-conjugate connection associated to a fixed (non-necessary F-connection) ∇. By denoting $\nabla^{(F)}$ this F-conjugate connection we have studied the geometry of $(M, F, \nabla, \nabla^{(F)})$ until now for two cases: almost complex structures in [1] and almost product structures in [2].

The present work is devoted to another remarkable type of tensor fields of $(1,1)$-type, namely *almost tangent structures*. These structures were introduced by Clark and Bruckheimer [5] and Eliopoulos [10] around 1960 and have been investigated by several authors, see [3], [6]-[8], [16], [18]. As it is well-known, the tangent bundle of a manifold carries a canonical integrable almost tangent structure, hence the name. This tangent structure plays an important rôle in the Lagrangian description of analytical mechanics, [7]-[8], [12].

Recall that we are interested in the class of J-linear connections since, according to [15, p. 120], the existence of a symmetric (torsion-free) one in this class implies the

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integrability of J in the sense of G-structures as is discussed below; for example, J-linear connections of Levi-Civita type are studied in [11]. An important difference between the former structures (almost complex, almost product) and the later (almost tangent) is given by the fact that an almost tangent structure J is a degenerate tensor field due to its nilpotence $J^2 = 0$, see the following Section. An example where this difference is obvious is the duality property $(\nabla^{(F)})^{(F)} = \nabla$ which holds for a non-degenerate F while for almost tangent structures we have ii) of our Proposition 2.1.

The content of paper is as follows. After a short survey in almost tangent geometry we introduce the tangent conjugate connection $\nabla^{(J)}$ in Section 2 following the pattern of [1]-[2]. Its properties are studied following the same way as in the cited papers; for example the difference $\nabla^{(J)} - \nabla$ is expressed again in terms of two tensor fields of $(1,2)$-types called *structural* and *virtual* tensor fields. We study also the behavior of the tangent conjugate connections for a family of anti-commuting almost tangent structures. In the last two Sections we generalize $\nabla^{(J)}$, firstly through an exponential process and secondly with a general tensor field of $(1,2)$-type.

1. Almost tangent geometry revisited

Let M be a smooth, m-dimensional real manifold for which we denote: $C^\infty(M)$-the real algebra of smooth real functions on M, $\Gamma(TM)$-the Lie algebra of vector fields on M, $T^r_s(M)$-the $C^\infty(M)$-module of tensor fields of (r,s)-type on M. An element of $T^1_1(M)$ is usually called *vector 1-form* or *affinor*.

Recall the concept of almost tangent geometry:

1.1. Definition. $J \in T^1_1(M)$ is called *almost tangent structure* on M if it has constant rank and:

$$ImJ = \ker J. \tag{1.1}$$

The pair (M, J) is called *almost tangent manifold.*

The name is motivated by the fact that (1.1) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. Denoting $rankJ = n$ it results $m = 2n$. If in addition, we suppose that J is integrable i.e.:

$$N_J(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y] = 0 \tag{1.2}$$

then J is called *tangent structure* and (M, J) is called *tangent manifold.*

From [17, p. 3246] we get some features of tangent manifolds:
(i) the distribution $ImJ\ (= \ker J)$ defines a foliation denoted $V(M)$ and called *the vertical distribution.*

1.2. Example. $M = \mathbb{R}^2$, $J_e(x,y) = (0,x)$ is a tangent structure with $\ker J_e$ the Y-axis, hence the name. The subscript e comes from "Euclidean".
(ii) there exists an atlas on M with local coordinates $(x,y) = \left(x^i, y^i\right)_{1\leq i\leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes dx^i$ i.e.:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1.3}$$

We call *canonical coordinates* the above (x,y) and the change of canonical coordinates $(x,y) \to (\widetilde{x}, \widetilde{y})$ is given by:

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x) \\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^a} y^a + B^i(x). \end{cases} \tag{1.4}$$

It results an alternative description in terms of G-structures. Namely, a tangent structure is a G-structure with:

$$G = \{C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n, \mathbb{R}); \quad A \in GL(n, \mathbb{R}), B \in gl(n, \mathbb{R})\} \tag{1.5}$$

and G is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$ i.e. $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure J of $M = TN$ is an example of tangent structure having exactly the expression (1.3) if (x^i) are the coordinates on N and (y^i) are the coordinates in the fibers of $TN \to N$. Also, J_e of Example 1.2 has the above expression (1.3) with $n = 1$, whence it is integrable. A third class of examples is obtained by duality: if J is an (integrable) endomorphism with $J^2 = 0$ then its dual $J^* : \Gamma(T^*M) \to \Gamma(T^*M)$, given by $J^*\alpha := \alpha \circ J$ for $\alpha \in \Gamma(T^*M)$, is (integrable) endomorphism with $(J^*)^2 = 0$.

2. Basic properties of tangent conjugate connections

Let ∇ be a linear connection on the almost tangent manifold (M, J) and define the *tangent conjugate connection* of ∇ by:

$$\nabla^{(J)} := \nabla - J \circ \nabla J. \tag{2.1}$$

Remark that $\nabla^{(J)}$ coincides with ∇ if and only if $\nabla J \subseteq \ker J = ImJ$ which means the inclusion $\nabla(\Gamma(TM) \times \ker J) \subseteq \ker J = ImJ$, in particular if ∇ is a J-linear connection; for another case see i) of Proposition 2.3. For any $X, Y \in \Gamma(TM)$ we get:

$$\nabla^{(J)}_X Y = \nabla_X Y - J(\nabla_X JY). \tag{2.2}$$

A first set of properties for this linear connection are given by:

2.1. Proposition. *The tangent conjugate connection $\nabla^{(J)}$ satisfies:*
i) $\nabla^{(J)} J = \nabla J$, which means that ∇ and $\nabla^{(J)}$ are simultaneous J-linear connections or not;
ii) $\nabla^{2(J)} =: (\nabla^{(J)})^{(J)} = 2\nabla^{(J)} - \nabla$; more generally $\nabla^{n(J)} = n\nabla^{(J)} - (n-1)\nabla$ for $n \in \mathbb{N}^*$;
iii) its torsion is $T_{\nabla^{(J)}} = T_\nabla - J \circ d^\nabla J$ where d^∇ is the exterior covariant derivative induced by ∇, namely $(d^\nabla J)(X, Y) := (\nabla_X J)Y - (\nabla_Y J)X$;
iv) its curvature is

$$\begin{aligned} R_{\nabla^{(J)}}(X, Y, Z) = R_\nabla(X, Y, Z) - \nabla_X J(\nabla_Y JZ) + \nabla_Y J(\nabla_X JZ) - \\ -J[\nabla_X J(\nabla_Y Z) - \nabla_Y J(\nabla_X Z) - \nabla_{[X,Y]} JZ]. \end{aligned} \tag{2.3}$$

In particular:

$$R_{\nabla^{(J)}}(X, Y, JZ) = R_\nabla(X, Y, JZ) - J[\nabla_X J(\nabla_Y JZ) - \nabla_Y J(\nabla_X JZ)]. \tag{2.4}$$

Proof The general part of ii) follows by induction while for iii) a direct calculus yields $T_{\nabla^{(J)}}(X, Y) = T_\nabla(X, Y) - J(\nabla_X JY - \nabla_Y JX)$. □

Let $f : M \to M$ be a *tangentomorphism*, that is an automorphism of the G-structure defined by J:

$$f_* \circ J = J \circ f_*. \tag{2.5}$$

Recall that f is an *affine transformation for* ∇ if for any $X, Y \in \Gamma(TM)$:

$$f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y. \tag{2.6}$$

These notions are connected by:

2.2. Proposition. *If the tangentomorphism f is an affine transformation for ∇ then f is also affine transformation for $\nabla^{(J)}$.*

Proof We have:

$$f_*(\nabla^{(J)}_X Y) = f_*(\nabla_X Y) - (f_* \circ J)(\nabla_X JY) = \nabla_{f_*X} f_*Y - J(f_*(\nabla_X JY)) =$$

$$= \nabla_{f_*X} f_*Y - J((\nabla_{f_*X} f_*(JY))) = \nabla_{f_*X} f_*Y - J((\nabla_{f_*X} J(f_*Y))) = \nabla^{(J)}_{f_*X} f_*Y$$

which yields the conclusion. □

A second class of properties for the tangent conjugate connection is provided by:

2.3. Proposition. *i) If J is ∇-recurrent i.e. $\nabla J = \eta \otimes J$ for η a 1-form, then $\nabla^{(J)} = \nabla$.*
ii) If ∇ is symmetric and $\nabla J = \eta \otimes I$ then $\nabla^{(J)} = \nabla - \eta \otimes J$ and $\nabla^{(J)}$ is a quarter-symmetric connection.

Proof i) In this case we have $J \circ \nabla J = 0$.
ii) Recall after [1, p. 122] that the quarter-symmetry means the existence of a 1-form π and a tensor field F of $(1,1)$-type such that $T_{\nabla^{(J)}} = F \wedge \pi := F \otimes \pi - \pi \otimes F$. From Proposition 2.1 we have $T_{\nabla^{(J)}}(X,Y) = T_\nabla(X,Y) - \eta(X)JY + \eta(Y)JX$, and the hypothesis $T_\nabla = 0$ yields the previous equation with $F = J$ and $\pi = \eta$. □

2.4. Example. Let N be a smooth n-dimensional manifold and $M = TN$ its tangent bundle; hence $m = 2n$. Let $\{x^i; 1 \leq i \leq n\}$ be a local system of coordinates on N and consider its lift to M given by $\{x^i, y^i; 1 \leq i \leq n\}$ with y^i the coordinates on the fibres of TN. The canonical almost tangent structure J of M has the local expression (1.3) and it is integrable. Fix a general linear connection ∇ on M with local Christoffel symbols Γ as follows:

$$\begin{cases} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^{(1)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(2)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \Gamma^{(3)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(4)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^{(5)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(6)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \Gamma^{(7)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(8)k}_{ij} \frac{\partial}{\partial y^k}. \end{cases} \tag{2.7}$$

Then its tangent conjugate connection has the expression:

$$\begin{cases} \nabla^{(J)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^{(1)k}_{ij} \frac{\partial}{\partial x^k} + \left(\Gamma^{(2)k}_{ij} - \Gamma^{(3)k}_{ij}\right) \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \Gamma^{(3)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(4)k}_{ij} \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^j} = \Gamma^{(5)k}_{ij} \frac{\partial}{\partial x^k} + \left(\Gamma^{(6)k}_{ij} - \Gamma^{(7)k}_{ij}\right) \frac{\partial}{\partial y^k} \\ \nabla^{(J)}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \Gamma^{(7)k}_{ij} \frac{\partial}{\partial x^k} + \Gamma^{(8)k}_{ij} \frac{\partial}{\partial y^k}. \end{cases} \tag{2.8}$$

A special case is important in applications: the initial connection ∇ is called *distinguished* or *d-connection* if it preserves the linear structure of the fibres of M which means that:

$$\Gamma^{(2)} = \Gamma^{(3)} = \Gamma^{(6)} = \Gamma^{(7)} = 0. \tag{2.9}$$

It results that ∇ is a J-connection and then its tangent conjugate connection is $\nabla^{(J)} = \nabla$.

3. The structural and the virtual tensor fields

Remark that the tangent conjugate connection $\nabla^{(J)}$ of ∇ can be written in another form as:

$$\nabla^{(J)} = \nabla + C^J_\nabla - B^J_\nabla \tag{3.1}$$

where:

$$\begin{cases} C^J_\nabla(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y + (\nabla_X J)JY] \\ B^J_\nabla(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y - (\nabla_X J)JY]. \end{cases} \tag{3.2}$$

which we call respectively, the *structural* and the *virtual tensor field* of ∇. We obtain also the following expressions for them:

$$\begin{cases} C^J_\nabla(X,Y) = \frac{1}{2}[\nabla_{JX}JY - J(\nabla_{JX}Y + \nabla_X JY)] \\ B^J_\nabla(X,Y) = \frac{1}{2}[\nabla_{JX}JY - J(\nabla_{JX}Y - \nabla_X JY)]. \end{cases} \tag{3.3}$$

We notice that they satisfy the following properties:

$$\begin{cases} C^J_\nabla(JX,Y) = C^J_\nabla(X,JY) = -\frac{1}{2}J(\nabla_{JX}JY); \quad C^J_\nabla(JX,JY) = 0 \\ B^J_\nabla(JX,Y) = -B^J_\nabla(X,JY) = \frac{1}{2}J(\nabla_{JX}JY); \quad B^J_\nabla(JX,JY) = 0 \\ C^J_\nabla(JX,Y) = -B^J_\nabla(JX,Y) \end{cases} \tag{3.4}$$

and the skew-symmetry (3.4_2) means that $B^J_\nabla(J\cdot,\cdot)$ is a vectorial 2-form. Another important property is that these tensor fields are invariant with respect to J-conjugation of linear connections:

$$C^J_{\nabla^{(J)}} = C^J_\nabla; \quad B^J_{\nabla^{(J)}} = B^J_\nabla. \tag{3.5}$$

With respect to the invariance of these associated tensor fields under projective changes we get that only C^J is invariant:

3.1. Proposition. *Let ∇ and ∇' be two linear projectively equivalent connections:*

$$\nabla' = \nabla + \eta \otimes I + I \otimes \eta \tag{3.6}$$

for η a 1-form. Then $C^J_{\nabla'} = C^J_\nabla$ and $B^J_{\nabla'} = B^J_\nabla + J \otimes (\eta \circ J)$ while the tangent conjugate connection $\nabla'^{(J)}$ of ∇' satisfies:

$$\nabla'^{(J)} = \nabla^{(J)} + \eta \otimes I + I \otimes \eta - J \otimes (\eta \circ J) \tag{3.7}$$

and so it is not invariant under projective equivalence.

Proof Follows form a direct computation. □

4. Invariant distributions

Let $\mathcal{D} \subset TM$ be a fixed distribution considered as a vector subbundle of TM. As usually, we denote by $\Gamma(\mathcal{D})$ its $C^\infty(M)$-module of sections.

4.1. Definition. i) $\mathcal{D}$ is called *J-invariant* if $X \in \Gamma(\mathcal{D})$ implies $JX \in \Gamma(\mathcal{D})$.
ii) The linear connection ∇ *restricts to* $\mathcal{D}$ if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$ for any $X \in \Gamma(TM)$.

4.2. Example. The distribution $\mathcal{D}_J = \ker J = ImJ$ is J-invariant.

If ∇ restricts to $\mathcal{D}$ then it may be considered as a connection in the vector bundle $\mathcal{D}$. From this fact, a connection which restricts to $\mathcal{D}$ is called sometimes *adapted to* $\mathcal{D}$.

4.3. Proposition. *If the distribution $\mathcal{D}$ is J-invariant and the linear connection ∇ restricts to $\mathcal{D}$ then $\nabla^{(J)}$ also restricts to $\mathcal{D}$.*

Proof Fix $Y \in \Gamma(\mathcal{D})$. Then $JY \in \Gamma(\mathcal{D})$ and for any $X \in \Gamma(TM)$ we have $\nabla_X Y$, $\nabla_X JY \in \Gamma(\mathcal{D})$. Therefore, $J(\nabla_X JY) \in \Gamma(\mathcal{D})$ and so $\nabla^{(J)}_X Y = \nabla_X Y - J(\nabla_X JY) \in \Gamma(\mathcal{D})$. □

4.4. Example. Returning to Example 4.2 we have that $\nabla_X = \nabla^{(J)}_X$ on $\mathcal{D}_J = \ker J = ImJ$.

A more general notion like restricting to a distribution is that of geodesically invariance [4, p. 118]. The distribution $\mathcal{D}$ is ∇-*geodesically invariant* if for every geodesic $\gamma : [a, b] \to M$ of ∇ with $\dot{\gamma}(a) \in \mathcal{D}_{\gamma(a)}$ it follows $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for any $t \in [a, b]$. The cited book gives a necessary and sufficient condition for a distribution $\mathcal{D}$ to be ∇-geodesically invariant: for any X, $Y \in \Gamma(\mathcal{D})$, *the symmetric product* $\langle X : Y \rangle_\nabla := \nabla_X Y + \nabla_Y X$ to belong to $\Gamma(\mathcal{D})$ or equivalently, for any $X \in \Gamma(\mathcal{D})$ to have $\nabla_X X \in \Gamma(\mathcal{D})$.

A direct computation gives:

$$\langle \cdot : \cdot \rangle_{\nabla^{(J)}} = \langle \cdot : \cdot \rangle_\nabla - J \circ d^\nabla J \tag{4.1}$$

and then the ∇-geodesically invariance and $\nabla^{(J)}$-geodesically invariance for $\mathcal{D}$ coincides if and only if $J \circ d^\nabla J$ is zero on $\mathcal{D} \times \mathcal{D}$. In particular, $\mathcal{D}_J$ is ∇-geodesically invariant if and only if is $\nabla^{(J)}$-geodesically invariant.

5. Affine combination of tangent conjugate connections

In what follows we shall see what happens to the tangent conjugate connection for families of almost tangent structures. Let J_1, J_2 be two almost tangent structures; conditions for their simultaneous integrability are given in [13]-[14]. Then for any a, $b \in \mathbb{R}$ the tensor field $J_{ab} := aJ_1 + bJ_2$ is an almost tangent structure if and only if $J_1 J_2 = -J_2 J_1$. Then its tangent conjugate connection is given by:

$$\nabla^{(J_{ab})}_X Y = a^2 \nabla^{(J_1)}_X Y + b^2 \nabla^{(J_2)}_X Y + (1 - a^2 - b^2)\nabla_X Y - ab[J_1(\nabla_X J_2 Y) + J_2(\nabla_X J_1 Y)]. \tag{5.1}$$

5.1. Proposition. *Let ∇ be a linear connection and J_1 and J_2 two anti-commuting almost tangent structures. If (∇, J_1, J_2) is a mixed-recurrent structure i.e. $\nabla J_i = \eta \otimes J_j$ for $i \neq j$ then ∇ is the average of the two tangent conjugate connections:*

$$\nabla = \frac{1}{2}[\nabla^{(J_1)} + \nabla^{(J_2)}] \tag{5.2}$$

and $\nabla^{(J_{ab})}$ is an affine combination of them:

$$\nabla^{(J_{ab})} = \frac{1 + a^2 - b^2}{2}\nabla^{(J_1)} + \frac{1 - a^2 + b^2}{2}\nabla^{(J_2)}. \tag{5.3}$$

Proof Applying J_i to $\nabla_X J_i Y - J_i(\nabla_X Y) = \eta(X) J_j Y$ with $i \neq j$ and the anti-commuting hypothesis we obtain:

$$J_1(\nabla_X J_1 Y) = -J_2(\nabla_X J_2 Y). \tag{5.4}$$

Summing the expression of the tangent conjugate connections we get (5.2) and from a previous computation, the relation (5.3). □

6. Exponential tangent conjugate connections

For θ a real number we define the *exponential tangent conjugate connection* of ∇ as:

$$\nabla^{(J,\theta)} := \nabla - \exp(-\theta J) \circ \nabla \circ \exp(\theta J) \tag{6.1}$$

where $\exp(\pm\theta J) := \cos(\theta) \cdot I \pm \sin(\theta) \cdot J$. Explicitly we get:

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta)\nabla J + \sin^2(\theta) J \circ \nabla J = 2\sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta)\nabla J - \sin^2(\theta)\nabla^{(J)} \tag{6.2}$$

and then:

$$\nabla^{(J,\theta)} J = \sin^2(\theta)\nabla J + \frac{1}{2}\sin(2\theta) J \circ \nabla J. \tag{6.3}$$

It follows:

6.1. Proposition. *Let ∇ be a symmetric linear connection.*
i) If J is ∇-recurrent with η the 1-form of recurrence then:

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \frac{1}{2}\sin(2\theta) \cdot \eta \otimes J \tag{6.4}$$

and $\nabla^{(J,\theta)}$ is a quarter-symmetric connection.
ii) If $\nabla J = \eta \otimes I$ then:

$$\nabla^{(J,\theta)} = \sin^2(\theta)\nabla - \sin(\theta) \cdot \eta \otimes \exp(-\theta J) \tag{6.5}$$

and:

$$T_{\nabla^{(J,\theta)}} = \sin(\theta) \otimes \exp(-\theta J) \wedge \eta. \tag{6.6}$$

Proof i) Follows from the fact that the hypothesis implies $J \circ \nabla J = 0$. The quarter-symmetry elements are $F = J$ and $\pi = \sin(\theta)\cos(\theta) \cdot \eta$.
ii) From $\cos(\theta) \cdot \eta \otimes I - \sin(\theta) \cdot \eta \otimes J = \eta \otimes \exp(-\theta J)$ we get:
$T_{\nabla^{(J,\theta)}} = -\sin(\theta) \cdot [\eta \otimes \exp(-\theta J) - \exp(-\theta J) \otimes \eta]$. □

7. Generalized tangent conjugate connections

In this section we present a natural generalization of the tangent conjugate connection.

7.1. Definition. A *generalized tangent conjugate connection* of ∇ is:

$$\nabla^{(J,C)} = \nabla^{(J)} + C \tag{7.1}$$

with $C \in T^1_2(M)$ an arbitrary $(1,2)$-tensor field.

Let us search for tensor fields C such that the duality $(\nabla^{(J,C)})^{(J,C)} = 2\nabla^{(J,C)} - \nabla$ holds as is given by Proposition 2.1. It results that we are interested in finding solutions C to the equation:

$$J(C(X, JY)) = 2C(X, Y) \tag{7.2}$$

for all $X, Y \in \Gamma(TM)$ and let us remark that: i) $C_0 = 0$ is a particular solution of (7.2); ii) applying J to (7.2) gives that $ImC \subseteq \ker J = ImJ$. Then returning to (7.2) it follows from the left-hand-side that C_0 is the unique solution of (7.2).

Also, we have:

$$\nabla^{(J,C)} J = \nabla^{(J)} J + C(\cdot, J\cdot) - J \circ C \tag{7.3}$$

and then:
i) $\nabla^{(J,C)} J = \nabla J$ as in i) of Proposition 2.1 if and only if: $C(\cdot, J\cdot) = J \circ C(\cdot, \cdot)$,
ii) $\nabla^{(J,C)}$ is a J-linear connection if and only if:

$$\nabla J + C(\cdot, J\cdot) = J \circ C. \tag{7.4}$$

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