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LAST MULTIPLIERS ON WEIGHTED MANIFOLDS AND THE WEIGHTED LIOUVILLE EQUATION

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Dedicated to Professor Constantin Udrişte
on the occasion of his 75th birthday
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Studiem noţiunea de ultim multiplicator ca soluţie independentă de timp pentru ecuaţia Liouville de transport pe varietăţi ponderate (Riemanniene). Extindem pe această cale o serie de rezultate precedente într-un cadru generalizat.

We study the notion of last multipliers as time-independent solutions of the Liouville equation of transport in weighted (Riemannian) manifolds. On this way, several results from previous papers are generalized in a larger framework.

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MSC2000: 58A15; 58A30; 34A26; 34C40.

1. Introduction

In January 1838, Joseph Liouville (1809-1882) published a note ([9]) on the time-dependence of the Jacobian of the "transformation" exerted by the solution of an ODE on its initial condition. In modern language if $A = A(x)$ is the vector field corresponding to the given ODE and $m = m(t, x)$ is a smooth function (depending also of the time t) then the main equation of the cited paper is:

$$\frac{dm}{dt} + m \cdot divA = 0 \tag{LE}$$

called, by then, the *Liouville equation*. The notion of *last multiplier* was introduced by Carl Gustav Jacob Jacobi (1804-1851) in "Vorlesugen über Dynamik", edited by R. F. A. Clebsch in Berlin in 1866. So, sometimes is used under the name of *Jacobi* (*last*) *multiplier*. Since then, this tool for understanding ODE was intensively studied by mathematicians in the usual Euclidean space $\mathbb{R}^n$, conform the bibliography of [1]. In [2] we have obtained that, placed in a general oriented manifold, the last multipliers are the autonomous solutions of (LE). Moreover, in the series of papers

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[1]-[4] we consider these notions in some important frameworks as Riemannian, Poisson and Lie algebroids geometries. Let us remark that a Sturm-Liouville operator was studied in Riemannian manifolds by Prof. dr. C. Udrişte and I. Ţevy in [15].

The aim of the present note is to discuss some results of this useful theory extended to a new framework namely *weighted manifolds*. Our study is based on the excellent book [7] where this concept is considered from the point of view of geometrical analysis, more precisely the heat kernel is computed. Let us remark that a relationship between the heat equation and the general method of multipliers is well-known; see the examples from [13, p. 364].

The content of the paper is as follows. The first section is a review of definition of last multipliers and previous important results. The next section starts the new framework given by a *weighted oriented manifold* and presents the associated Liouville equation and last multipliers. The last section is devoted to the weighted Riemannian manifolds and assuming a Helmholtz type decomposition, some examples are given.

2. General facts about last multipliers

Let M be a real, smooth, n-dimensional manifold, $C^{\infty}(M)$ the algebra of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Lambda^{k}(M)$ the $C^{\infty}(M)$-module of k-differential forms, $0 \leq k \leq n$. Suppose that M is orientable with the fixed volume form $V \in \Lambda^{n}(M)$ and for a fixed $A \in \mathcal{X}(M)$ let us consider the $(n-1)$-form $\Omega = i_A V \in \Lambda^{n-1}(M)$.

Definition 2.1([5, p. 107], [11, p. 428]) The function $m \in C^{\infty}(M)$ is called a *last multiplier* of A if $m\Omega$ is closed:

$$d(m\Omega) := (dm) \wedge \Omega + md\Omega = 0. \tag{2.1}$$

Let $LM(A)$ and $FInt(A)$ be respectively the set of last multipliers and first integrals for A.

In dimension 2 the notions of last multiplier and integrating factor are identical and Sophus Lie gave a method to associate a last multiplier to every symmetry vector field of A (Theorem 1.1 in [8, p. 752]). The Lie method is extended to any dimension in [11].

Characterizations of $LM(A)$ can be obtained in terms of Witten's differential [16] and Marsden's differential [10] but we present here only the last since the former appears in [2, p. 458]. If $f \in C^{\infty}(M)$ the Marsden deformation of the differential is $d^f : \Lambda^*(M) \to \Lambda^{*+1}(M)$ defined by:

$$d^f(\omega) = \frac{1}{f} d(f\omega) \tag{2.2}$$

and whence m is a last multiplier if and only if Ω is d^m-closed.

The following characterization of last multipliers will be useful:

Lemma 2.2([11, p. 428]) *$m \in C^{\infty}(M)$ belongs to $LM(A)$ if and only if:*

$$A(m) + m \cdot divA = 0 \tag{2.3}$$

where $divA$ is the divergence of A with respect to volume form V.

Remarks 2.3 (i) The equation (2.3) is the time-independent version of *the Liouville equation* studied in [2] on manifolds. An important feature of equation (2.3) is that it does not always admit solutions conform [6, p. 269].

(ii) A first result given by (2.3) is the case of solenoidal i.e. divergence-free vector fields: $LM(A) = FInt(A)$. The importance of this result is shown by the fact that three remarkable classes of solenoidal vector fields are provided by: Killing vector fields in Riemannian geometry, Hamiltonian vector fields in symplectic geometry and Reeb vector fields in contact geometry (in particulary K-almost contact geometry). Also, there are many equations of mathematical physics which are modeled by a solenoidal vector field.

(iii) For the general case, namely A is not solenoidal, the ratio of two last multipliers is a first integral and conversely, the product between a first integral and a last multiplier is a last multiplier. Since $FInt(A)$ is a subalgebra in $C^{\infty}(M)$ it results that $LM(A)$ is a $FInt(A)$-module.

(iv) Recalling the formulae:

$$div(fX) = X(f) + fdivX \tag{2.4}$$

it follows that $m \in LM(A)$ if and only if the vector field mA is solenoidal i.e. $div(mA) = 0$. Then $LM(A)$ is a linear subspace in $C^{\infty}(M)$.

(v) To the vector field A we can associate an *adjoint* A^*, acting on functions in the following manner, [14]:

$$A^*(m) = -A(m) - mdivA.$$

Then, another simple characterization is: $LM(A) = Fint(A^*)$. □

An important structure generated by a last multiplier is given by:

Proposition 2.4([2, p. 459]) *Let $m \in C^{\infty}(M)$ be fixed. The set of vector fields admitting m as last multiplier is a Lie subalgebra in $\mathcal{X}(M)$.*

3. Last multipliers on weighted oriented manifolds

We extend the framework of previous section in the following manner:

Definition 3.1 i) A *weighted oriented manifold* is a triple (M, V, Υ) with (M, V) as above and $\Upsilon \in C^{\infty}_{+}(M)$ i.e. Υ is a smooth and strictly positive function on M.
ii) Following the expression (1.2) we define *the weighted divergence* of $X \in \mathcal{X}(M)$ as:

$$div_{\mu}X = \frac{1}{\Upsilon}div(\Upsilon X). \tag{3.1}$$

iii) $m \in C^{\infty}(M)$ is a Υ-*weighted last multiplier* for A if is a solution of *the weighted Liouville equation*:

$$A(m) + m div_{\mu} A = 0. \tag{3.2}$$

Let $\Upsilon LM(A)$ be the set of these functions and with a subscript "+" we will denote the subsets of strictly positive functions.

Remarks 3.2 i) The weighted Liouville equation can be read as follows: m is an "eigenvector" of A considered as derivation over the real algebra $C^{\infty}(M)$ with $-div_{\mu} A$ as "eigenvalue".
ii) If $m \in C^{\infty}_{+}(M)$ then (1.4) yields the following expression of (3.2):

$$A(\ln(m\Upsilon)) + divA = 0 \tag{3.3}$$

which means that $\Upsilon LM_{+}(A) = \frac{1}{\Upsilon} LM_{+}(A)$. □

For the general case, two situations when $\Upsilon LM(A)$ is completely determined are provided by the following result:

Proposition 3.3 i) *If* $\Upsilon \in FI_{+}(A)$ *then*: $\Upsilon LM(A) = LM(A)$.
ii) *If A is divergence-free then*: $\Upsilon LM(A) = \frac{1}{\Upsilon} FInt(A)$.

Proof The equation (3.2) has the form:

$$A(m) + \frac{m}{\Upsilon} A(\Upsilon) + m divA = 0 \tag{3.4}$$

and then both implications above are immediately. □

The next result is a natural extension of the Proposition 2.4:

Proposition 3.4 *Let $m \in C^{\infty}_{+}(M)$ be fixed. Then the set of vector fields X with $m \in \Upsilon LM_{+}(X)$ is a Lie subalgebra in $\mathcal{X}(M)$.*

Proof Obviously, the result can be obtained from Proposition 2.4 and the Remark 3.2 but we prefer to present a direct proof based on the identity:

$$div\,[X, Y] = X\,(divY) - Y\,(divX)\,. \tag{3.5}$$

Let X, Y with the above property. Then:

$$[X,Y](\ln(m\Upsilon)) + div([X,Y]) = X(Y(\ln(m\Upsilon))) - Y(X(\ln(m\Upsilon))) + X(divY) - Y(divX) = 0$$

which gives the conclusion. □

4. **Last multipliers on weighted Riemannian manifolds**

A more interesting framework is provided by [7, p. 67]:

Definition 4.1 A *weighted manifold* is a triple $(M, g =<,>, \Upsilon)$ with (M, g) a Riemannian manifold.

On any weighted manifold there exists an induced volume form $V = V_g$. Let $\omega \in \Lambda^1(M)$ be the g-dual of A and δ the co-derivative operator $\delta : \Lambda^*(M) \to \Lambda^{*-1}(M)$. Then:

$$div_{V_g} A = -\delta\omega, \quad A\,(f) = g^{-1}\,(df, \omega)\,. \tag{4.1}$$

and the condition (3.3) means:

$$g^{-1}\left(d(\ln(m\Upsilon),\omega\right)=\delta\omega. \tag{4.2}$$

It follows that $m\in\Upsilon LM_{+}(A)$ if and only if ω belongs to the kernel of the differential operator: $g^{-1}\left(d\ln(m\Upsilon),\cdot\right)-\delta:\Lambda^{1}\left(M\right)\rightarrow\Lambda^{0}=C^{\infty}\left(M\right)$.

For the general case of m an important fact is given by the *product rule for divergence* ([7, p. 69]):

$$div_{\mu}(fX)=g(\nabla f,X)+f div_{\mu}X \tag{4.3}$$

where ∇f is the g-gradient of f and then the weighted Liouville equation (3.2) reads:

$$div_{\mu}(mA)=0 \tag{4.4}$$

which means that $\Upsilon LM(A)$ is a is a "measure of how far away" is A from being μ-divergence-free.

An important tool in the Riemannian case is *the weighted Laplacian* ([7, p. 68]):

$$\Delta_{\mu}=div_{\mu}\circ\nabla. \tag{4.5}$$

Now, assume that the vector field A admits a Helmholtz type decomposition:

$$A=X+\nabla u \tag{4.6}$$

where X is a solenoidal vector field and $u\in C^{\infty}\left(M\right)$; for example if M is compact such decompositions always exist. From $\nabla u\left(m\right)=<\nabla u,\nabla m>$ it follows that (4.2) becomes:

$$X(m)+<\nabla u,\nabla m>+m[X(\ln\Gamma)+\Delta_{\mu}u]=0. \tag{4.7}$$

Example 4.1 u is a Υ-last multiplier of $A=X+\nabla u$ if and only if:

$$X\left(u\right)=-u[X(\ln\Upsilon)+\Delta_{\mu}u]-\|\nabla u\|_{g}^{2}. \tag{4.8}$$

Suppose that M is a cylinder $M=I\times N$ with $I\subseteq\mathbb{R}$ and N a $(n-1)$-manifold; then for $X=-\frac{1}{2}\frac{\partial}{\partial t}\in\mathcal{X}\left(I\right)$ which is divergence-free with respect to $V=dt\wedge V_{N}$ with V_{N} a volume form on N, the previous relation yields:

$$u_{t}=2\left[u(-\frac{1}{2}(\ln\Upsilon)_{t}+\Delta u)+\|\nabla u\|_{g}^{2}\right]. \tag{4.9}$$

By the product rule for the weighted Laplacian ([12, p. 55]):

$$<\nabla f,\nabla g>=\frac{1}{2}\left(\Delta_{\mu}\left(fg\right)-f\cdot\Delta_{\mu}g-g\cdot\Delta_{\mu}f\right) \tag{4.10}$$

the previous equation becomes:

$$u_{t}=-u(\ln\Upsilon)_{t}+\Delta_{\mu}\left(u^{2}\right) \tag{4.11}$$

In particular, if $\Upsilon\in C_{+}^{\infty}(N)$ we get:

$$u_{t}=\Delta_{\mu}\left(u^{2}\right) \tag{4.12}$$

which is a weighted version of the nonlinear parabolic equation of porous medium type.

Example 4.2 Returning to (4.6) suppose that $X = 0$. The condition (4.7) reads:

$$m \cdot \Delta_{\mu} u + < \nabla u, \nabla m >= 0 \tag{4.13}$$

which is equivalent, via (4.10) to:

$$\Delta_{\mu}(um) + m \cdot \Delta_{\mu} u = u \cdot \Delta_{\mu} m. \tag{4.14}$$

which yields:

Proposition 4.3 *Let $u, m \in C^{\infty}(M)$ such that $u \in \Upsilon LM(\nabla m)$ and $m \in \Upsilon LM(\nabla u)$. Then $u \cdot m$ is a Υ-harmonic function on M. $u \in \Upsilon LM(\nabla u)$ if and only if u^2 is a Υ-harmonic function on M.*

Proof Adding to (4.14) a similar relation with u replaced by m gives the conclusion. $\square$

Example 4.4. The gradient of distance function with respect to a 2D rotationally symmetric metric

Let M be a 2D manifold with local coordinates (t, θ) endowed with a *rotationally symmetric* metric $g = dt^2 + \varphi^2(t)d\theta^2$ conform [12, p. 11]. Let the smooth function $u(t, \theta) = t$ which appear as a distance function with respect to the given metric. Then $\nabla u = \frac{\partial}{\partial t}$ and $\Delta_{\mu} u = \frac{1}{\Upsilon\varphi}\frac{\partial \Upsilon\varphi}{\partial t}$; the equation (3.13) is:

$$m \cdot \frac{(\Upsilon\varphi)_t}{\Upsilon\varphi} + \frac{\partial m}{\partial t} = 0 \tag{4.15}$$

with the solutions: $m = \frac{cf(\theta)}{\Upsilon\varphi}$ for $c \in \mathbb{R}$. Therefore $\frac{1}{\varphi}LM(\frac{\partial}{\partial t}) = \mathbb{R} \cdot C^{\infty}([0, 2\pi])$.

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