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Results in Mathematics

Almost Analyticity on Almost (Para) Complex Lie Algebroids

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Abstract. The goal of this paper is to generalize the notion of almost analyticity in the almost (para)complex Lie algebroids framework. We use a global formalism which yields to generalizations of the main results of previous known almost (para)complex manifolds case.

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1. Introduction

The notion of almost analyticity of tensor fields was introduced in the almost complex geometry a long time ago, and hence, it was treated in local coordinates, especially by Japanese geometers [27–30]. A global approach for differential forms appeared in [20] and [26]; for example, in the former paper a differential is introduced in the algebra of a pair of almost analytic forms, and a corresponding Poincaré type lemma is proved. Other significant developments, concerning the study of almost analyticity both in almost complex and almost paracomplex geometry setting, can be found for instance in [12,13,24,25] and other papers of these authors. In [7] almost analytic forms, with respect to a quadratic endomorphism, were studied in a unifying context of almost (para)complex geometry, and also an associated cohomology of these forms in relation with the Frölicher–Nijenhuis theory has been introduced.

The Lie algebroids, [16,17], are generalizations of Lie algebras and integrable distributions. In fact a Lie algebroid is an anchored vector bundle with a Lie bracket on the module of sections [23]. Initially defined as infinitesimal

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part of Lie groupoids, in the last decades, the Lie algebroids have an important place in some different categories in differential geometry. The notion of almost complex Lie algebroid over almost complex manifolds was introduced in [5] as a natural extension of the notion of an almost complex manifold to that of an almost complex Lie algebroid. More generally, in [11], the notion of almost complex Lie algebroid over a smooth manifold is considered, and some problems concerning the geometry of almost complex Lie algebroids over smooth manifolds are studied in relation with corresponding notions from the geometry of almost complex manifolds. In the same manner we can consider the almost paracomplex Lie algebroids over smooth manifolds, see [2].

The aim of the present paper is to extend the notion of almost analyticity for differential forms sections and tensor sections on almost (para)complex Lie algebroids over smooth manifolds. This generalization is possible taking into account the Cartan calculus in the Lie algebroid framework, see for instance [18]. The paper is organized as follows. In the first section we briefly recall some basic facts about Lie algebroids as: exterior derivative, cohomology, linear connections and Bianchi identities. In the second section we consider the Lie algebroids of even rank endowed with an almost (para)complex structure J_E, and we define the notion of J_E-analyticity for smooth functions on the base M, for r-forms sections and more generally for tensor sections of type $(0, r)$ on E, with $r \geq 1$. An exterior derivative d_{J_E} for almost J_E-analytic forms sections on E is introduced in relation with the Frölicher–Nijenhuis theory. We prove that for an almost J_E-analytic form section on E its closeness with respect to the Frölicher–Nijenhuis derivative d_{J_E} is characterized by the usual (i.e. exterior derivative d_E) closeness. Also, an associated d_{J_E}-cohomology of almost J_E-analytic forms sections and some deformations of these form sections with pairs of almost J_E-analytic functions are considered. In the third section we consider the case of (para)anti-Hermitian Lie algebroids. When the Lie algebroid is para-Hermitian and invariantly oriented, we obtain a characterization of almost J_E-analyticity in terms of harmonicity on E. Also, some important results concerning almost analyticity from the almost (para)complex manifolds case, see [12,13,25] are generalized in the Lie algebroids framework. In particular, we obtain that for a (para)anti-Hermitian Lie algebroid the Riemann curvature tensor section and the scalar curvature are almost J_E-analytic. Finally we consider the complete, vertical and horizontal lifts to the prolongation of a Lie algebroid over its bundle projection to illustrate some examples of our study.

2. Generalities on Lie Algebroids

In the first section we briefly recall some basic facts about Lie algebroids as: exterior derivative, cohomology, linear connections and Bianchi identities. Fore more about Lie algebroids see for instance [4,8,9,16–19,23].

Definition 2.1. We say that $p : E \to M$ is an *anchored* vector bundle if there exists a vector bundle morphism $\rho : E \to TM$. The morphism ρ will be called *the anchor map.*

Let $p : E \to M$ be an anchored vector bundle with the anchor $\rho : E \to TM$ and the induced morphism $\rho_E : \Gamma(E) \to \mathcal{X}(M)$. Assume there exists defined a bracket $[\cdot,\cdot]_E$ on the space $\Gamma(E)$ that provides a structure of real Lie algebra on $\Gamma(E)$.

Definition 2.2. The triple $(E, \rho_E, [\cdot,\cdot]_E)$ is called a *Lie algebroid* if:

(i) $\rho_E : (\Gamma(E, [\cdot,\cdot]_E) \to (\mathcal{X}(M), [\cdot,\cdot])$ is a Lie algebra homomorphism, i.e. $\rho_E([s_1, s_2]_E) = [\rho_E(s_1), \rho_E(s_2)]$;

(ii) $[s_1, fs_2]_E = f[s_1, s_2]_E + \rho_E(s_1)(f)s_2$,

for every $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

A Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ is said to be *transitive* if ρ_E is surjective. The space $C^\infty(M)$ is a $\Gamma(E)$-module relative to the representation $\Gamma(E) \times C^\infty(M) \to C^\infty(M),\ (s, f) \mapsto \rho_E(s)f$.

According to the well-known Chevalley-Eilenberg cohomology theory [6], we can introduce a cohomology complex associated to a Lie algebroid. A r-linear mapping $\omega : \Gamma(E) \times \ldots \times \Gamma(E) \to C^\infty(M)$ is called a $C^\infty(M)$-valued r-cochain. Let $\mathcal{C}^r(E)$ denotes the vector space of these cochains. The operator $d_E : \mathcal{C}^r(E) \to \mathcal{C}^{r+1}(E)$ given by:

$$d_E\omega(s_0, \ldots, s_r) = \sum_{i=0}^{r} (-1)^i \rho_E(s_i)(\omega(s_0, \ldots, \widehat{s_i}, \ldots, s_r)) + \sum_{i<j=1}^{r} (-1)^{i+j} \omega([s_i, s_j]_E, s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_r) \tag{2.1}$$

for $\omega \in \mathcal{C}^r(E)$ and $s_0, \ldots, s_r \in \Gamma(E)$, defines a coboundary since $d_E \circ d_E = 0$. Hence, $(\mathcal{C}^r(E), d_E)$, $r \geq 1$ is a differential complex and the corresponding cohomology spaces are called the cohomology groups of E with coefficients in $C^\infty(M)$. For $f \in C^\infty(M)$ we have $d_E f(s) = \rho_E(s)(f)$.

Lemma 2.1. *If $\omega \in \mathcal{C}^r(E)$ is skew-symmetric and $C^\infty(M)$-linear then $d_E\omega$ also is skew-symmetric.*

From now on, the subspace of skew-symmetric and $C^\infty(M)$-linear cochains of the space $\mathcal{C}^r(E)$ will be denoted by $\Omega^r(E)$ and its elements will be called *r-form sections* on E.

Definition 2.3. A *linear connection* on the Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ over M is a map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, $(s_1, s_2) \mapsto \nabla(s_1, s_2) := \nabla_{s_1} s_2 \in \Gamma(E)$ such that:

(1) ∇ is $\mathbb{R}$-bilinear;

(2) $\nabla_{fs_1} s_2 = f\nabla_{s_1} s_2$ for all $f \in C^\infty(M)$ and $s_1, s_2 \in \Gamma(E)$;

(3) $\nabla_{s_1}(fs_2) = (\rho_E(s_1)f)s_2 + f\nabla_{s_1} s_2$ for all $f \in C^\infty(M)$ and $s_1, s_2 \in \Gamma(E)$.

Remark 2.1. A linear connection on a Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ is in fact an E-connection in the vector bundle E. See for instance [9] for the definition of an E-connection in a general vector bundle F.

For every $s_1, s_2 \in \Gamma(E)$ the section $\nabla_{s_1} s_2 \in \Gamma(E)$ is called *the covariant derivative of the section s_2 with respect to section s_1.*

If ∇ is a linear connection on the Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ then the map $T : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ defined by $T(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2]_E$, $\forall\, s_1, s_2 \in \Gamma(E)$, is called the *torsion* of ∇. We have that T defined above is a tensor field of type $(2,1)$ on E which is $C^\infty(M)$—bilinear and skew-symmetric. Also, we consider the map $R : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, $(s_1, s_2, s_3) \mapsto R(s_1, s_2, s_3) = R(s_1, s_2)s_3$, where the section $R(s_1, s_2)s_3$ is defined by $R(s_1, s_2)s_3 = \nabla_{s_1}\nabla_{s_2} s_3 - \nabla_{s_2}\nabla_{s_1} s_3 - \nabla_{[s_1,s_2]_E} s_3$ for every $s_1, s_2, s_3 \in \Gamma(E)$. It is easy to see that the map R is $C^\infty(M)$-linear in every argument and it is skew-symmetric with respect to the first two arguments, that is $R(s_1, s_2, s_3) = -R(s_2, s_1, s_3)$ for every $s_1, s_2, s_3 \in \Gamma(E)$. The map R defined above is called *the curvature* of the linear connection ∇.

If the Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ is endowed with a Riemannian metric g_E then there exists an unique torsion-less linear connection ∇ on E (the Levi-Civita connection, see [1,4]) given by the formula:

$$\begin{aligned} 2g_E(\nabla_{s_1} s_2, s_3) &= \rho_E(s_1)(g_E(s_2, s_3)) + \rho_E(s_2)(g_E(s_1, s_3)) \\ &\quad -\rho_E(s_3)(g_E(s_1, s_2)) + g_E([s_3, s_1]_E, s_2) \\ &\quad +g_E([s_3, s_2]_E, s_1) + g_E([s_1, s_2]_E, s_3). \end{aligned} \tag{2.2}$$

We also notice that for a given linear connection ∇ the following formula holds:

$$\begin{aligned} d_E\omega(s_1, \ldots, s_{r+1}) &= \sum_{i=1}^{r+1} (-1)^{i+1} \left(\nabla_{s_i}\omega\right)(s_1, \ldots, \widehat{s_i}, \ldots, s_{r+1}) \\ &\quad - \sum_{1 \le i < j \le r+1} (-1)^{i+j} \omega(T(s_i, s_j), s_1, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_{r+1}) \end{aligned} \tag{2.3}$$

for every $\omega \in \Omega^r(E)$ and $s_1, \ldots, s_{r+1} \in \Gamma(E)$, where

$$\begin{aligned} \left(\nabla_s\omega\right)(s_1, \ldots, s_r) &= \rho_E(s)(\omega(s_1, \ldots, s_r)) \\ &\quad - \sum_{i=1}^{r} \omega(s_1, \ldots, \nabla_s s_i, \ldots, s_r), \quad \forall\, \omega \in \mathcal{C}^r(E). \end{aligned} \tag{2.4}$$

Also, if we consider the torsion and curvature 2-forms T and R of ∇ then the following identities hold:

$$\begin{aligned} &\sum_{cycl(s_1,s_2,s_3)} [(\nabla_{s_1} R)(s_2, s_3) + R(T(s_1, s_2), s_3)] = 0, \\ &\sum_{cycl(s_1,s_2,s_3)} [(\nabla_{s_1} T)(s_2, s_3) - R(s_1, s_2)s_3 + T(T(s_1, s_2), s_3)] = 0 \end{aligned} \tag{2.5}$$

and these are called the *Bianchi identities* of a Lie algebroid.

3. Almost Analytic Forms on Almost (Para)Complex Lie Algebroids

The almost complex Lie algebroids are studied in [2,5,11,22] from different points of view and the setting of our work is fixed by the following notion:

Definition 3.1. If $\varepsilon = \pm 1$ then an *almost (para)complex structure* J_E on $(E, \rho_E, [\cdot,\cdot]_E)$ with $\operatorname{rank} E = 2m$ is an endomorphism $J_E : \Gamma(E) \to \Gamma(E)$ over the identity, such that $J_E^2 = \varepsilon \mathrm{id}_{\Gamma(\mathrm{E})}$ where $\varepsilon = -1$ for the almost complex case respectively $\varepsilon = 1$ for the almost paracomplex case. A Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E)$ endowed with such a structure will be called an *almost (para)complex Lie algebroid* over the smooth manifold M.

In this section we define the notion of J_E-analyticity for smooth functions on the base M, for r-forms sections and more generally for tensor sections of type $(0,r)$ on E with $r \geq 1$. An exterior derivative d_{J_E} for almost J_E-analytic forms sections on E is introduced in relationship with Frölicher–Nijenhuis theory. We prove that for an almost J_E-analytic form section on E its closeness with respect to the Frölicher–Nijenhuis derivative d_{J_E} is characterized by the usual (i.e. exterior derivative d_E) closeness. Also, an associated d_{J_E}-cohomology of almost J_E-analytic forms sections and some deformations of these forms sections with pairs of almost J_E-analytic functions are considered.

3.1. Almost J_E-Analytic r-Forms Sections

Recall that a given tensor section J_E of $(1,1)$-type defines

(i) an interior product i_{J_E}; for a r-form section ω we have that $i_{J_E}\omega$ is again a r-form section:

$$i_{J_E}\omega(s_1,\ldots,s_r) := \sum_{i=1}^{r} \omega(s_1 \ldots, J_E(s_i), \ldots, s_r),$$
$$r \geq 1 \text{ and } i_{J_E} f = 0, \ \forall f \in C^\infty(M); \tag{3.1}$$

(ii) an exterior J_E-derivative d_{J_E} with:

$$d_{J_E} := i_{J_E} \circ d_E - d_E \circ i_{J_E}. \tag{3.2}$$

Similarly to [13,24,28], we introduce

Definition 3.2. A smooth function f on M is called *almost J_E-analytic* if there exists a smooth function g on M such that:

$$d_E f \circ J_E = d_E g, \tag{3.3}$$

where g is called *a conjugate function* of f. In this case g is also almost J_E-analytic with conjugate function εf.

The set of all almost J_E-analytic functions on M will be denoted by $C^\infty(M, J_E)$.

Definition 3.3. Let $r \geq 2$.

(i) The r-form section ω is called J_E-*symmetric* or *pure* if:

$$\omega(J_E(s_1), \ldots, s_r) = \omega(s_1, \ldots, J_E(s_i), \ldots, s_r) \tag{3.4}$$

for every $s_1, \ldots, s_r \in \Gamma(E)$.

(ii) If $\omega \in \Omega^r(E)$ is pure then its J_E-*conjugate* is $\overline{\omega} \in \Omega^r(E)$ given by:

$$\overline{\omega}(s_1, \ldots, s_r) := \frac{1}{\varepsilon}\omega(J_E(s_1), \ldots, s_r). \tag{3.5}$$

Remark 3.1. By convention for $r = 1$ every 1-form section $\omega \in \Omega^1(E)$ is J_E-symmetric and its J_E-conjugate is given by

$$\overline{\omega} := \omega \circ J_E^{-1} = \frac{1}{\varepsilon}\omega \circ J_E. \tag{3.6}$$

Remark 3.2. A similar J_E-conjugation can be defined for all r-linear mappings $\omega \in \mathcal{C}^r(E)$ and we notice that if $\omega \in \Omega^r(E)$ then $\overline{\omega} \in \Omega^r(E)$ iff ω is J_E-symmetric.

Also, we notice that for every r-form section ω we have

$$\overline{\overline{\omega}} = \frac{1}{\varepsilon}\omega. \tag{3.7}$$

Indeed, by direct calculus it follows

$$\begin{aligned} \overline{\overline{\omega}}(s_1, \ldots, s_r) &= \frac{1}{\varepsilon}\overline{\omega}(J_E(s_1), \ldots, s_r) = \frac{1}{\varepsilon^2}\omega(J_E^2(s_1), \ldots, s_r) \\ &= \frac{\varepsilon}{\varepsilon^2}\omega(s_1, \ldots, s_r) = \frac{1}{\varepsilon}\omega(s_1, \ldots, s_r). \end{aligned}$$

Moreover, the following result hold:

Proposition 3.1. *If ω is J_E-symmetric then $\overline{\omega}$ is J_E-symmetric.*

Proof. We have

$$\overline{\omega}(J_E(s_1), \ldots, s_r) = \frac{1}{\varepsilon}\omega(J_E^2(s_1), \ldots, s_r) = \omega(s_1, \ldots, s_r),$$

and

$$\begin{aligned} \overline{\omega}(s_1, \ldots, J_E(s_i), \ldots, s_r) &= \frac{1}{\varepsilon}\omega(J_E(s_1), \ldots, J_E(s_i), \ldots, s_r) \\ &= \frac{1}{\varepsilon}\omega(s_1, \ldots, J_E^2(s_i), \ldots, s_r) \\ &= \omega(s_1, \ldots, s_r) \end{aligned}$$

which end the proof. □

In the following for every $\omega \in \Omega^r(E)$ we associate $\Omega_{J_E,\omega} \in \Omega^{r+1}(E)$ given by

$$\Omega_{J_E,\omega}(s_1, \ldots, s_{r+1}) := d_E\omega(J_E(s_1), \ldots, s_{r+1}) - \varepsilon d_E\overline{\omega}(s_1, \ldots, s_{r+1}). \tag{3.8}$$

Remark 3.3. The above $(r+1)$-form section $\Omega_{J_E,\omega}$ can be expressed as follows:

$$\Omega_{J_E,\omega} = \varepsilon(\overline{d_E\omega} - d_E\overline{\omega}), \tag{3.9}$$

and this form will be very useful in the sequel.

Definition 3.4. The J_E-symmetric form section $\omega \in \Omega^r(E)$ is called *almost J_E-analytic* if

$$\Omega_{J_E,\omega} = 0. \tag{3.10}$$

The set of all almost J_E-analytic r-forms sections on E with $r \geq 1$ will be denoted by $\Omega^r(E, J_E)$, and we also consider the notation $\Omega^0(E, J_E) = C^\infty(M, J_E)$.

A natural question to ask is when a J_E-symmetric r-form section is almost J_E-analytic. We have

Proposition 3.2. *A J_E-symmetric r-form section ω $(r \geq 1)$ is almost J_E-analytic iff*

$$\begin{aligned}
&\rho_E(J_E(s_1))(\omega(s_2,\dots,s_{r+1})) - \rho_E(s_1)(\omega(J_E(s_2),\dots,s_{r+1})) \\
&\quad = \sum_{j=2}^{r+1}(-1)^{1+j}\omega(J_E([s_1,s_j]_E) - [J_E(s_1),s_j]_E, s_2,\dots,\widehat{s_j},\dots,s_{r+1}).
\end{aligned} \tag{3.11}$$

Proof. It follows by a direct calculation involving the definition of the exterior derivative d_E from (2.1) of the r-forms sections ω and $\overline{\omega}$ from (3.8). □

Remark 3.4. In a more general case of $(0,r)$-tensor sections we can consider the Tachibana operator $\Phi_{J_E} : \mathcal{C}^r(E) \to \mathcal{C}^{r+1}(E)$:

$$\begin{aligned}
(\Phi_{J_E}\omega)(s, s_1, s_2, \dots, s_r) &= \rho_E(J_E(s))(\omega(s_1, s_2, \dots, s_r)) \\
&\quad -\rho_E(s)(\omega(J_E(s_1), s_2, \dots, s_r)) + \omega((\mathcal{L}_{s_1}J_E)s, s_2, \dots, s_r) \\
&\quad + \cdots + \omega(s_1, s_2, \dots, (\mathcal{L}_{s_r}J_E)s),
\end{aligned} \tag{3.12}$$

where $(\mathcal{L}_{s_1}J_E)s_2 = \mathcal{L}_{s_1}(J_E(s_2)) - J_E(\mathcal{L}_{s_1}s_2)$ and $\mathcal{L}_s$ denotes the Lie derivative with respect to s, given by $\mathcal{L}_{s_1}s_2 = [s_1, s_2]_E$. Then, similarly to [13,14,24,25, 27,29], the tensor section ω is called *almost J_E-analytic* if $\Phi_{J_E}\omega = 0$ and for r-forms sections this condition is equivalent to (3.10).

Remark 3.5. A natural characterization of elements of $C^\infty(M, J_E)$ can be given using the Frölicher–Nijenhuis type exterior derivative d_{J_E} as follows: if $f \in C^\infty(M, J_E)$ then we have

$$d_{J_E}f = i_{J_E} \circ d_E f = \varepsilon\overline{d_E f} = d_E f \circ J_E = d_E g, \tag{3.13}$$

and the following relations hold:

$$\overline{d_E f} = \frac{1}{\varepsilon}d_{J_E}f = \frac{1}{\varepsilon}d_E g,\ \overline{d_E g} = \overline{d_{J_E}f} = d_E f. \tag{3.14}$$

Proposition 3.3. *If $f \in C^\infty(M, J_E)$ then $d_E f$ and $d_{J_E} f$ are both almost J_E-analytic.*

Proof. Using (3.9) and the first relation of (3.14) we have

$$\Omega_{J_E, d_E f} = \varepsilon\left(\overline{d_E(d_E f)} - d_E(\overline{d_E f})\right) = -d_E(d_E g) = 0,$$

so we get the first part. Next, we have

$$\Omega_{J_E, d_{J_E} f} = \varepsilon\left(\overline{d_E(d_{J_E} f)} - d_E(\overline{d_{J_E} f})\right) = -\varepsilon d_E(d_E f) = 0.$$

and so the second part follows. □

Also, we have

Proposition 3.4. *Let $f \in C^\infty(M)$. The exact 1-form section $d_E f$ is almost J_E-analytic iff the associated 1-form section $d_E f \circ J_E = d_{J_E} f$ is d_E-closed.*

Proof. Using again (3.9), we easily obtain that $\Omega_{J_E, d_E f} = 0$ iff $-d_E(d_{J_E f}) = 0$. □

Proposition 3.5. *If $\omega \in \Omega^r(E, J_E)$ then its differential is J_E-symmetric while its exterior J_E-differential is:*

$$d_{J_E}\omega = \frac{1}{r+1} i_{J_E} \circ d_E \omega = \varepsilon d_E \overline{\omega}. \tag{3.15}$$

Proof. Since ω is almost J_E-analytic we have $\overline{d_E\omega} = d_E\overline{\omega} \in \Omega^{r+1}(E)$ which says that $d_E\omega$ is J_E-symmetric (see Remark 3.2). Moreover, by direct calculus we have

$$d_{J_E}\omega = i_{J_E} d_E \omega - d_E i_{J_E}\omega = \varepsilon(r+1)\overline{d_E\omega} - d_E(\varepsilon r\overline{\omega}) = \varepsilon\overline{d_E\omega} = \varepsilon d_E\overline{\omega}$$

which ends the proof. □

Proposition 3.6. *The J_E-symmetric r-form section ω is almost J_E-analytic if and only if $\overline{\omega}$ is almost J_E-analytic. If ω is almost J_E-analytic then ω is closed if and only if $\overline{\omega}$ is closed, equivalently ω and $\overline{\omega}$ are d_{J_E}-closed.*

Proof. It is sufficient to prove the implication $\omega =$ almost J_E-analytic $\Rightarrow \overline{\omega} =$ almost J_E-analytic because of (3.7) and by the fact that almost J_E-analyticity is invariant with respect to scalings $\omega \to \lambda\omega$.

Firstly we notice that by Proposition 3.1 $\overline{\omega}$ is J_E-symmetric and the result follows directly using the relation $\overline{\Omega_{J_E,\omega}} = -\Omega_{J_E,\overline{\omega}}$ and formula (3.15). □

Definition 3.5. The Nijenhuis tensor section of J_E is usually defined by

$$\begin{aligned} N_{J_E}(s_1, s_2) = {} & [J_E s_1, J_E s_2]_E - J_E[s_1, J_E s_2]_E - J_E[J_E s_1, s_2]_E \\ & +\varepsilon[s_1, s_2]_E, \quad \forall\, s_1, s_2 \in \Gamma(E), \end{aligned} \tag{3.16}$$

and the almost (para)complex structure J_E of E is called *integrable* if $N_{J_E} = 0$.

Proposition 3.7. *If $\omega \in \Omega^r(E, J_E)$ with $r \geq 2$, then*

$$\omega(N_{J_E}(s_1, s_2), \ldots, s_{r+1}) = \overline{\omega}(N_{J_E}(s_1, s_2), \ldots, s_{r+1}) = 0. \tag{3.17}$$

Proof. Using the characterization of almost J_E-analyticity of ω from (3.11) but with $s_2 \mapsto J_E(s_2)$, we have

$$\begin{aligned}
&\rho_E(J_E(s_1))(\omega(J_E(s_2), \ldots, s_{r+1})) - \varepsilon\rho_E(s_1)(\omega(s_2, \ldots, s_{r+1})) \\
&\quad = -\omega(J_E([s_1, J_E(s_2)]_E) - [J_E(s_1), J_E(s_2)]_E, s_3, \ldots, s_{r+1}) \\
&\qquad + \sum_{j=3}^{r+1}(-1)^{1+j}\omega(J_E([s_1, s_j]_E) - [J_E(s_1), s_j]_E, J_E(s_2), \\
&\qquad\quad s_3, \ldots, \widehat{s_j}, \ldots, s_{r+1}).
\end{aligned} \tag{3.18}$$

On the other hand $\overline{\omega} \in \Omega^r(E, J_E)$, too, and using again (3.11) for $\overline{\omega}$, we have

$$\begin{aligned}
&\rho_E(J_E(s_1))(\omega(J_E(s_2), \ldots, s_{r+1})) - \varepsilon\rho_E(s_1)(\omega(s_2, \ldots, s_{r+1})) \\
&\quad = -\omega(\varepsilon[s_1, s_2]_E - J_E([J_E(s_1), s_2]_E), s_3, \ldots, s_{r+1}) \\
&\qquad + \sum_{j=3}^{r+1}(-1)^{1+j}\omega(J_E([s_1, s_j]_E) - [J_E(s_1), s_j]_E, J_E(s_2), s_3, \ldots, \widehat{s_j}, \ldots, s_{r+1}).
\end{aligned} \tag{3.19}$$

Now, by (3.18) and (3.19) the first equality follows easily. The second equality follows in a similar manner. □

Remark 3.6. It is easy to prove that for $r = 1$ the equalities from (3.17) becomes

$$\omega \circ N_{J_E} = \overline{\omega} \circ N_{J_E} = 0. \tag{3.20}$$

Taking into account that E has an even rank $2m$, it follows that, if $J_E \neq \mathrm{id}_{\Gamma(E)}$ (for almost paracomplex case), there is a local basis of sections of E of type $\{e_1, \ldots, e_m, J_E(e_1), \ldots, J_E(e_m)\}$ and then there exists non-trivial J_E-symmetric r-forms sections only for $r \leq m$. An important result for this choice of dimension is:

Proposition 3.8. *Let* $\operatorname{rank} E = 2m$. *An J_E-symmetric m-form section ω on E is almost J_E-analytic if and only if ω and $\overline{\omega}$ are both d_E-closed.*

Proof. Firstly, we notice that by Proposition 3.6, ω is almost J_E-analytic iff $\overline{\omega}$ is almost J_E-analytic. On the other hand the Proposition 3.5 says that if ω is almost J_E-analytic then $d_E\omega$ is a J_E-symmetric $m+1$-form section. This oblige that $d_E\omega = 0$. The proof of the converse part is directly from Definition 3.4. □

3.2. d_{J_E}-cohomology and deformations

We introduce now an exterior product adapted to our setting:

Definition 3.6. The *exterior J_E-product* is the map $\wedge_{J_E} : \Omega^\bullet(E) \times \Omega^\bullet(E) \to \Omega^\bullet(E)$ given by

$$\theta \wedge_{J_E} \omega := \theta \wedge \omega + \varepsilon \overline{\theta} \wedge \overline{\omega} \tag{3.21}$$

where $\wedge$ is the usual exterior product on $\Omega^\bullet(E)$, see [18].

A long but straightforward computation gives

Proposition 3.9. *Let θ and ω be J_E-symmetric forms sections on E.*

(i) *The form section $\theta \wedge_{J_E} \omega$ is also J_E-symmetric.*
(ii) *The J_E-conjugate of the above form section is*

$$\overline{\theta \wedge_{J_E} \omega} = \theta \wedge \overline{\omega} + \overline{\theta} \wedge \omega. \tag{3.22}$$

As consequence, if θ and ω are almost J_E-analytic forms sections then $\theta \wedge_{J_E} \omega$ is also an almost J_E-analytic form section.

Proposition 3.10. *Let $\omega \in \Omega^r(E, J_E)$ and $\theta \in \Omega^s(E, J_E)$, $r, s \geq 0$. Then:*

(i) $d_{J_E}\omega \in \Omega^{r+1}(E, J_E)$;
(ii) $d^2_{J_E}\omega = 0$;
(iii) $d_{J_E}(\omega \wedge_{J_E} \theta) = d_{J_E}\omega \wedge_{J_E} \theta + (-1)^r \omega \wedge_{J_E} d_{J_E}\theta$.

Proof. (i) If $\omega \in \Omega^r(E, J_E)$ then by Proposition 3.6 we have that $\overline{\omega} \in \Omega^r(E, J_E)$, that is

$$d_E\overline{\omega} = \overline{d_E\omega} \quad \text{and} \quad d_E\overline{\overline{\omega}} = \overline{d_E\overline{\omega}}. \tag{3.23}$$

Now, by direct calculus we have

$$\begin{aligned}\overline{d_E d_{J_E}\omega} &= \overline{d_E(d_E i_{J_E}\omega - i_{J_E} d_E\omega)} = -\overline{d_E i_{J_E} d_E\omega} \\ &= -\overline{d_E\left(\varepsilon(r+1)\overline{d_E\omega}\right)} = -\varepsilon(r+1)\overline{d_E d_E\overline{\omega}} = 0\end{aligned}$$

and

$$\begin{aligned}d_E\overline{d_{J_E}\omega} &= d_E\overline{d_E i_{J_E}\omega - i_{J_E} d_E\omega} = d_E\overline{d_E(\varepsilon r\overline{\omega}) - \varepsilon(r+1)\overline{d_E\omega}} \\ &= -\varepsilon d_E\overline{d_E\overline{\omega}} = -\varepsilon d_E d_E\overline{\overline{\omega}} = 0\end{aligned}$$

which leads to

$$\Omega_{J_E, d_{J_E}\omega} = \varepsilon\left(\overline{d_E d_{J_E}\omega} - d_E\overline{d_{J_E}\omega}\right) = 0, \tag{3.24}$$

so, $d_{J_E}\omega \in \Omega^{r+1}(E, J_E)$.
(ii) Using (3.15) and (3.23) we have:

$$d_{J_E}(d_{J_E}\omega) = d_{J_E}(\varepsilon d_E\overline{\omega}) = \varepsilon^2 d_E(\overline{d_E\overline{\omega}}) = \varepsilon^2 d_E(d_E\overline{\overline{\omega}}) = 0.$$

(iii) Follows using (3.15), (3.21) and (3.22). □

We notice that $(\Omega^r(E,J_E),\wedge_{J_E})$ is a graded $C^\infty(M,J_E)$-algebra. Since $d_{J_E}^2=0$ we have the differential complex $(\Omega^\bullet(E,J_E),d_{J_E})$ and its cohomology $H^\bullet(E,J_E)$ is called *d_{J_E}-cohomology of almost J_E-analytic forms* on E.

We consider now a deformation of almost J_E-analytic forms sections with pairs of almost J_E-analytic functions:

Definition 3.7. Fix $\omega\in\Omega^r(E,J_E)$ and $\alpha,\beta\in C^\infty(M)$. The (α,β)-*deformation* of ω is the r-form section:

$$\omega_{\alpha,\beta}:=\alpha\omega+\beta\overline{\omega}. \tag{3.25}$$

Since $\omega_{\alpha,\beta}$ is an J_E-symmetric form section it is natural to ask in what conditions regarding these functions the new r-form section is an almost J_E-analytic one. Firstly, we notice that

$$\overline{\omega_{\alpha,\beta}}=\alpha\overline{\omega}+\frac{\beta}{\varepsilon}\omega=\overline{\omega}_{\alpha,\beta}. \tag{3.26}$$

Proposition 3.11. *The J_E-symmetric r-form section $\omega_{\alpha,\beta}$ is almost J_E-analytic if and only if α is almost J_E-analytic with conjugate function β.*

Proof. It follows easy by evaluating $\Omega_{J_E,\omega_{\alpha,\beta}}$. Indeed, by (3.25) and (3.26) it follows

$$\begin{aligned}\Omega_{J_E,\omega_{\alpha,\beta}}&=\varepsilon\left(\overline{d_E\omega_{\alpha,\beta}}-d_E\overline{\omega_{\alpha,\beta}}\right)\\&=\varepsilon\left(\overline{d_E\alpha\wedge\omega}+\overline{d_E\beta\wedge\overline{\omega}}-d_E\alpha\wedge\overline{\omega}-\frac{d_E\beta}{\varepsilon}\wedge\omega\right).\end{aligned} \tag{3.27}$$

Also, we have

$$\begin{aligned}&\overline{d_E\alpha\wedge\omega}(s_1,\ldots,s_{r+1})=\frac{1}{\varepsilon}(d_E\alpha\wedge\omega)(J_E(s_1),s_2,\ldots,s_{r+1})\\&=\frac{1}{\varepsilon}\sum_{\sigma\in S_{1,r}}\operatorname{sgn}(\sigma)d_E\alpha(J_E(s_1))\omega(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})\\&\quad+\frac{1}{\varepsilon}\sum_{\sigma'\in S_{1,r}}\operatorname{sgn}(\sigma')d_E\alpha(s_{\sigma'(1)})\omega(s_{\sigma'(2)},\ldots,J(s_1),\ldots,s_{\sigma'(r+1)})\\&=\sum_{\sigma\in S_{1,r}}\operatorname{sgn}(\sigma)\overline{d_E\alpha}(s_1)\omega(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})\\&\quad+\sum_{\sigma'\in S_{1,r}}\operatorname{sgn}(\sigma')d_E\alpha(s_{\sigma'(1)})\overline{\omega}(s_{\sigma'(2)},\ldots,s_1,\ldots,s_{\sigma'(r+1)}),\end{aligned}$$

where $\sigma=(1\,|\,\sigma(2)\ldots\sigma(r+1))$ is a shuffle with the first element fix equal to 1 and the others are a given by a permutation of the set $\{2,\ldots,r+1\}$ and $\sigma'=(\sigma'(1)\,|\,\sigma'(2)\ldots,1,\ldots,\sigma'(r+1))$ is a shuffle with $\sigma'(1)=k\neq1$ and the rest of the elements are given by a permutation of the set $\{1,\ldots,r+1\}\setminus\{k\}$.

Similarly

$$\overline{d_E\beta\wedge\overline{\omega}}(s_1,\ldots,s_{r+1}) = \sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)\overline{d_E\beta}(s_1)\overline{\omega}(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$
$$+\frac{1}{\varepsilon}\sum_{\sigma'\in S_{1,r}}\mathrm{sgn}(\sigma')d_E\beta(s_{\sigma'(1)})\overline{\omega}(s_{\sigma'(2)},\ldots,J_E(s_1),\ldots,s_{\sigma'(r+1)}),$$
$$=\sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)\overline{d_E\beta}(s_1)\overline{\omega}(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$
$$+\frac{1}{\varepsilon}\sum_{\sigma'\in S_{1,r}}\mathrm{sgn}(\sigma')d_E\beta(s_{\sigma'(1)})\omega(s_{\sigma'(2)},\ldots,s_1,\ldots,s_{\sigma'(r+1)}),$$

and

$$\left(d_E\alpha\wedge\overline{\omega}\right)(s_1,\ldots,s_{r+1}) = \sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)d_E\alpha(s_1)\overline{\omega}(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$
$$+\sum_{\sigma'\in S_{1,r}}\mathrm{sgn}(\sigma')d_E\alpha(s_{\sigma'(1)})\overline{\omega}(s_{\sigma'(2)},\ldots,s_1,\ldots,s_{\sigma'(r+1)}),$$

and also

$$\left(\frac{d_E\beta}{\varepsilon}\wedge\omega\right)(s_1,\ldots,s_{r+1}) = \sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)\frac{d_E\beta}{\varepsilon}(s_1)\omega(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$
$$+\sum_{\sigma'\in S_{1,r}}\mathrm{sgn}(\sigma')\frac{d_E\beta}{\varepsilon}(s_{\sigma'(1)})\omega(s_{\sigma'(2)},\ldots,s_1,\ldots,s_{\sigma'(r+1)})$$

Replacing now in (3.27) we obtain

$$\Omega_{J_E,\omega_{\alpha,\beta}}(s_1,\ldots,s_{r+1})$$
$$=\sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)\varepsilon\left(\overline{d_E\alpha}-\frac{d_E\beta}{\varepsilon}\right)(s_1)\omega(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$
$$+\sum_{\sigma\in S_{1,r}}\mathrm{sgn}(\sigma)\varepsilon\left(\overline{d_E\beta}-d_E\alpha\right)(s_1)\overline{\omega}(s_{\sigma(2)},\ldots,s_{\sigma(r+1)})$$

and the conclusion follows easily by (3.14). □

This result yields the introduction of the set

$$\widetilde{C}^\infty(M,J_E)=\{(\alpha,\beta)\in C^\infty(M,J_E)\times C^\infty(M,J_E);\ d_E\beta=d_E\alpha\circ J_E\}. \tag{3.28}$$

A straightforward computation gives that $\widetilde{C}^\infty(M,J_E)$ is a commutative algebra with respect to the product

$$(\alpha_1,\beta_1)\cdot(\alpha_2,\beta_2):=(\alpha_1\alpha_2+\varepsilon\beta_1\beta_2,\alpha_1\beta_2+\alpha_2\beta_1) \tag{3.29}$$

having as unit the pair of the constant functions $(1,0)\in\widetilde{C}^\infty(M,J_E)$. The inverse of the element $(\alpha,\beta)\in\widetilde{C}^\infty(M,J_E)$ different from $(0,0)$ is the pair

$\left(\frac{\alpha}{\alpha^2-\varepsilon\beta^2}, \frac{-\beta}{\alpha^2-\varepsilon\beta^2}\right)$; for the almost paracomplex case ($\varepsilon = 1$) we exclude the cases $(\alpha, \pm\alpha)$.

Let us introduce the set of pairs of forms sections

$$\widetilde{\Omega}^r(M, J_E) = \{(\omega, \overline{\omega}); \omega \in \Omega^r(M, J_E)\}. \tag{3.30}$$

The Proposition 3.11 says that $\widetilde{\Omega}^r(M, J_E)$ is a $\widetilde{C}^\infty(M, J_E)$-module for all $1 \le r \le m$ and hence the set

$$\widetilde{\Omega}(M, J_E) = \bigoplus_{r=1}^{m} \widetilde{\Omega}^r(M, J_E)$$

is a graduate $\widetilde{C}^\infty(M, J_E)$-algebra. We consider the wedge product

$$(\omega, \overline{\omega})\widetilde{\wedge}(\theta, \overline{\theta}) = (\omega \wedge_{J_E} \theta, \overline{\omega \wedge_{J_E} \theta}) \tag{3.31}$$

and the operator

$$D_E : \widetilde{\Omega}^r(M, J_E) \to \widetilde{\Omega}^{r+1}(M, J_E), \quad D_E(\omega, \overline{\omega}) = (d_E\omega, d_E\overline{\omega}). \tag{3.32}$$

It follows that

(i) D_E is a local and $\mathbb{R}$-linear operator;
(ii) for every $(\omega, \overline{\omega}) \in \widetilde{\Omega}^r(M, J_E)$ and $(\theta, \overline{\theta}) \in \widetilde{\Omega}^s(M, J_E)$ we have
$$D_E\left[(\omega, \overline{\omega})\widetilde{\wedge}(\theta, \overline{\theta})\right] = D_E(\omega, \overline{\omega})\widetilde{\wedge}(\theta, \overline{\theta}) + (-1)^r(\omega, \overline{\omega})\widetilde{\wedge}D_E(\theta, \overline{\theta});$$
(iii) $D_E^2 = (0, 0)$

and an associated cohomology of the differential complex $(\widetilde{\Omega}(E, J_E), D_E)$ can be considered exactly as in [20].

4. Almost Analyticity on (Para)Anti-Hermitian Lie Algebroids

In this section we consider the case of (para)anti-Hermitian Lie algebroids. When the Lie algebroid is invariantly oriented we obtain a characterization of almost J_E-analyticity in terms of harmonicity on E. Also, some important results concerning almost analyticity from almost (para)complex manifolds case, see [12, 13, 25] are generalized in the Lie algebroids framework. In particular, we obtain that for a (para)anti-Hermitian Lie algebroid the Riemann curvature tensor section and the scalar curvature are almost J_E-analytic. Finally we consider the complete, vertical and horizontal lifts to the prolongation of a Lie algebroid over its bundle projection to illustrate some examples of our study.

Definition 4.1. An *anti-Hermitian metric (or Norden metric)* on an almost complex Lie algebroid $(E, \rho_E, [\cdot, \cdot]_E, J_E)$ is a pseudo-Riemannian metric g_E on E which is anti-invariant by J_E, that is

$$g_E(J_E(s_1), J_E(s_2)) = -g_E(s_1, s_2), \quad \forall\, s_1, s_2 \in \Gamma(E), \tag{4.1}$$

or equivalently, g_E is J_E-symmetric

$$g_E(J_E(s_1), s_2) = g_E(s_1, J_E(s_2)), \quad \forall\, s_1, s_2 \in \Gamma(E). \tag{4.2}$$

An almost complex Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E)$ endowed with an anti-Hermitian metric g_E is called an *almost anti-Hermitian Lie algebroid* and we denote it by $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$. If J_E is integrable then E is called an *anti-Hermitian Lie algebroid*.

Definition 4.2. A *para-Hermitian metric* on an almost paracomplex Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E)$ is a pseudo-Riemannian metric g_E on E which is invariant by J_E, that is

$$g_E(J_E(s_1), J_E(s_2)) = g_E(s_1, s_2), \quad \forall\, s_1, s_2 \in \Gamma(E), \tag{4.3}$$

or equivalently, g_E is J_E-symmetric

$$g_E(J_E(s_1), s_2) = g_E(s_1, J_E(s_2)), \quad \forall\, s_1, s_2 \in \Gamma(E). \tag{4.4}$$

An almost paracomplex Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E)$ endowed with a para-Hermitian metric g_E is called an *almost para-Hermitian Lie algebroid* and we denote it by $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$. If J_E is integrable then E is called a *para-Hermitian Lie algebroid*.

In that follows for an unifying setting both for almost complex and almost paracomplex cases, a Riemannian metric g_E which satisfies the relation (4.2) (or (4.4)) with the almost (para)complex structure J_E will be called a *(para)anti-Hermitian* metric on E and $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ will be called an *almost (para)anti-Hermitian Lie algebroid*.

Now, if we consider the almost para-Hermitian Lie algebroid $(E, \rho_E, [\cdot, \cdot]_E, J_E, g_E)$ to be invariantly oriented with the volume form dV_{g_E}, considered for instance in [15] for transitive Lie algebroids (or [3] for general Lie algebroids), the para-Hermitian metric g_E yields the Hodge star operator $\star_E$ on $\Omega^\bullet(E)$, see [3], and the orthonormal basis $B = \{e_1, \ldots, e_m, J_E(e_1), \ldots, J_E(e_m)\}$ when $J_E \neq \mathrm{id}_{\Gamma(E)}$. An argument similar as in [26] leads to:

Proposition 4.1. *If ω is a pure m-form section on an invariantly oriented almost para-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ of* $\operatorname{rank} E = 2m$ *then $\star_E\omega$ is also pure.*

Proof. Recall that $\star_E\omega$ is the unique m-form section on E defined by the equality

$$(\star_E\omega)(s_1, \ldots, s_m) dV_{g_E} = \omega \wedge s_1^* \wedge \ldots \wedge s_m^*, \quad \forall\, s_1, \ldots, s_m \in \Gamma(E), \tag{4.5}$$

where $s_1^*, \ldots, s_m^*$ are their dual with respect to g_E.

Let $\{a_1, \ldots, a_m\} \subset \{1, \ldots, 2m\}$. From (4.5) we have

$$\begin{aligned}(\star_E\omega)(e_{a_1}, \ldots, e_{a_m}) &= \sum_{\sigma \in S_{m,m}} \operatorname{sgn}(\sigma)\omega(e_{\sigma(1)}, \ldots, e_{\sigma(m)}) e_{a_1}^* \wedge \\ &\qquad \cdots \wedge e_{a_m}^*(e_{\sigma(m+1)}, \ldots, e_{\sigma(2m)}) \\ &= \operatorname{sgn}(\sigma)\omega(e_{\sigma(1)}, \ldots, e_{\sigma(m)}),\end{aligned}$$

where $\sigma = (\sigma(1)\dots\sigma(m)\,|\,a_1\dots a_m)$ is the unique shuffle with the last m elements fixed, and the first m elements are given by a permutation of the set $\{\sigma(1),\dots,\sigma(m)\}\subset\{1,\dots,2m\}$ such that $\sigma(1)<\dots<\sigma(m)$. Therefore, with $J_E(e_i)=e_{m+i}$ for all $i=1,\dots,m$,

$$\begin{aligned}
&(\star_E\omega)(e_1,\dots,e_k,J_E(e_{k+1}),\dots,J_E(e_m))\\
&\quad=(\star_E\omega)(e_1,\dots,e_k,e_{m+k+1},\dots,e_{2m})\\
&\quad=\operatorname{sgn}(\sigma)\omega(e_{\sigma(1)},\dots,e_{\sigma(m)})\\
&\quad=(-1)^{mk}\omega(e_{k+1},\dots,e_m,e_{m+1},\dots,e_{m+k})\\
&\quad=(-1)^{mk+(m-k)k}\omega(e_{m+1},\dots,e_{m+k},e_{k+1},\dots,e_m)\\
&\quad=(-1)^{k^2}\omega(J_E(e_1),\dots,J_E(e_k),e_{k+1},\dots,e_m),
\end{aligned}$$

where

$$\begin{aligned}
\sigma&=(\sigma(1)\dots\sigma(m)\,|\,1\dots k(m+k+1)\dots 2m)\\
&=(e_{k+1}\dots e_m e_{m+1}\dots e_{m+k}\,|\,1\dots k(m+k+1)\dots 2m).
\end{aligned}$$

Now, since ω is J_E-symmetric, we have

$$(\star_E\omega)(J_E(e_1),e_2\dots,e_m)-(\star_E\omega)(e_1,e_2,\dots,J_E(e_k),\dots,e_m)=0$$

which end the proof. □

Taking into account the Proposition 3.2 and the above result, by direct calculation we get

Corollary 4.1. *If ω is an almost J_E-analytic m-form section on the invariantly oriented almost para-Hermitian Lie algebroid $(E,\rho_E,[\cdot,\cdot]_E,J_E,g_E)$ of* $\operatorname{rank} E=2m$ *then $\star_E\omega$ is also almost J_E-analytic.*

According to [3], we recall that the exterior coderivative operator $d_E^*:\Omega^r(E)\to\Omega^{r-1}(E)$ on forms sections on an invariantly oriented Lie algebroid E of rankE $=2m$ is defined as $d_E^*\omega=-(\star_E\circ d_E\circ\star_E)\omega$. The r-form section ω is called *harmonic* if $\Delta_E\omega:=(d_Ed_E^*+d_E^*d_E)\omega=0$, and this is equivalent with $d_E\omega=d_E^*\omega=0$.

A large class of almost J_E-analytic form sections is then given by

Proposition 4.2. *A pure m-form section ω on the invariantly oriented almost para-Hermitian Lie algebroid $(E,\rho_E,[\cdot,\cdot]_E,J_E,g_E)$ of* $\operatorname{rank} E=2m$ *is almost J_E-analytic if and only if ω and $\overline{\omega}$ are both harmonic.*

Proof. As well as we have obtained in Proposition 3.8 if $\omega\in\Omega^m(E,J_E)$ then $d_E\omega=d_E\overline{\omega}=0$. Now, the Corollary 4.1 says that $\star_E\omega\in\Omega^m(E,J_E)$ and then $d_E(\star_E\omega)$ is a J_E-symmetric $m+1$-form section which oblige $d_E(\star_E\omega)=0$. Moreover, $\Omega_{J_E,\star_E\omega}=0$ leads to $d_E(\overline{\star_E\omega})=0$. But, by a direct calculation in the formula which gives $\star_E\omega$ we get

$$\begin{aligned}
&\overline{\star_E\omega}(e_1,\dots,e_k,J_E(e_{k+1}),\dots,J_E(e_m))\\
&\quad=-\star_E\overline{\omega}(e_1,\dots,e_k,J_E(e_{k+1}),\dots,J_E(e_m))
\end{aligned}$$

which says that $\overline{\star_E \omega} = -\star_E \overline{\omega}$, and so we obtain $d_E(\star_E \overline{\omega}) = 0$. Hence, ω and $\overline{\omega}$ are both harmonic. Conversely, it follows easy only from $d_E\omega = d_E\overline{\omega} = 0$, by Proposition 3.8. □

Let us consider now the Tachibana operator Φ_{J_E} from (3.12). By (4.2) and (4.4) the pseudo-metric g_E of an (almost) (para)anti-Hermitian Lie algebroid is pure and if

$$\Phi_{J_E} g_E = 0, \tag{4.6}$$

then it is called (almost) J_E-analytic. In this case $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ is called a *J_E-analytic (para)anti-Hermitian Lie algebroid*.

Let ∇ be the Levi-Civita connection given by (2.2). Then as a direct consequence of (4.1) and $\nabla g_E = 0$, we have

Theorem 4.1. *Let $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ be an almost (para)anti-Hermitian Lie algebroid. Then*

$$g_E(s_3, (\nabla_{s_2} J_E)s_1) = g_E((\nabla_{s_2} J_E)s_3, s_1), \quad \forall\, s_1, s_2, s_3 \in \Gamma(E). \tag{4.7}$$

In some aspects, J_E-analytic (para)anti-Hermitian Lie algebroids are similar to Kählerian Lie algebroids [11]. The following theorem is analoque to the next known result: An almost Hermitian Lie algebroid is Kählerian if and only if the almost complex structure J_E is parallel with respect to the Levi-Civita connection, see Theorem 3.1 from [11].

Theorem 4.2. *An almost (para)anti-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ is J_E-analytic if and only if the almost (para)complex structure J_E is parallel with respect to the Levi-Civita connection ∇.*

Proof. It follows using an argument similar to that used in Theorem 2 from [13] for almost complex case (or Theorem 2 from [25] for almost paracomplex case). Although, we present here a sketch of proof for our case.

Let us put $(g_E \circ J_E)(s_1, s_2) = g_E(J_E(s_1), s_2)$. Taking into account that $T_\nabla = 0$, a direct calculus in (3.12) leads to

$$\begin{aligned}
(\Phi_{J_E} g_E)(s, s_1, s_2) = {} & \rho_E(J_E(s))(g_E(s_1, s_2)) - \rho_E(s)(g_E(J_E(s_1), s_2)) \\
& - g_E\left(\nabla_{J_E(s)} s_1, s_2\right) + g_E\left(\nabla_{s_1} J_E(s), s_2\right) \\
& - g_E\left(s_1, \nabla_{J_E(s)} s_2\right) + g_E\left(s_1, \nabla_{s_2} J_E(s)\right) \\
& + g_E\left(J_E\left(\nabla_s s_1\right), s_2\right) - g_E\left(J_E\left(\nabla_{s_1} s\right), s_2\right) \\
& + g_E\left(J_E\left(s_1\right), \nabla_s s_2\right) - g_E\left(s_1, J_E\left(\nabla_{s_2} s\right)\right).
\end{aligned} \tag{4.8}$$

Taking into account $\left(\nabla_{s_1} J_E\right) s_2 = \nabla_{s_1} J_E(s_2) - J_E\left(\nabla_{s_1} s_2\right)$ we find

$$\begin{aligned}
& g_E\left(\nabla_{s_1} J_E(s), s_2\right) - g_E\left(J_E\left(\nabla_{s_1} s\right), s_2\right) + g_E\left(s_1, \nabla_{s_2} J_E(s)\right) \\
& \quad - g_E\left(s_1, J_E\left(\nabla_{s_2} s\right)\right) = g_E\left(\left(\nabla_{s_1} J_E\right) s, s_2\right) + g_E\left(s_1, \left(\nabla_{s_2} J_E\right) s\right).
\end{aligned} \tag{4.9}$$

Replacing (4.9) into (4.8) we obtain

$$\begin{aligned}\left(\Phi_{J_E} g_E\right)(s, s_1, s_2) &= \rho_E(J_E(s))(g_E(s_1, s_2)) - \rho_E(s)(g_E(J_E(s_1), s_2)) \\ &\quad + g_E\left(\left(\nabla_{s_1} J_E\right) s, s_2\right) + g_E\left(s_1, \left(\nabla_{s_2} J_E\right) s\right) \\ &\quad - g_E\left(\nabla_{J_E(s)} s_1, s_2\right) - g_E\left(s_1, \nabla_{J_E(s)} s_2\right) \\ &\quad + g_E\left(J_E\left(\nabla_s s_1\right), s_2\right) + g_E\left(J_E(s_1), \nabla_s s_2\right). \end{aligned} \tag{4.10}$$

But, for the Levi-Civita connection ∇ we have

$$\begin{aligned}&\rho_E(J_E(s))(g_E(s_1, s_2)) - g_E\left(\nabla_{J_E(s)} s_1, s_2\right) - g_E\left(s_1, \nabla_{J_E(s)} s_2\right) \\ &\quad = \left(\nabla_{J_E(s)} g_E\right)(s_1, s_2) = 0 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&-\rho_E(s)(g_E(J_E(s_1), s_2)) + g_E\left(J_E\left(\nabla_s s_1\right), s_2\right) + g_E\left(J_E(s_1), \nabla_s s_2\right) \\ &\quad = -\left(\nabla_s g_E\right)(J_E(s_1), s_2) - g_E\left(\left(\nabla_s J_E\right) s_1, s_2\right) = -g_E\left(\left(\nabla_s J_E\right) s_1, s_2\right). \end{aligned} \tag{4.12}$$

Hence, by taking into account (4.12) and (4.11), the relation (4.10) becomes

$$\begin{aligned}\left(\Phi_{J_E} g_E\right)(s, s_1, s_2) &= -g_E\left(\left(\nabla_s J_E\right) s_1, s_2\right) + g_E\left(\left(\nabla_{s_1} J_E\right) s, s_2\right) \\ &\quad + g_E\left(s_1, \left(\nabla_{s_2} J_E\right) s\right). \end{aligned} \tag{4.13}$$

Similarly, we have

$$\begin{aligned}\left(\Phi_{J_E} g_E\right)(s_2, s_1, s) &= -g_E\left(\left(\nabla_{s_2} J_E\right) s_1, s\right) + g_E\left(\left(\nabla_{s_1} J_E\right) s_2, s\right) \\ &\quad + g_E\left(s_1, \left(\nabla_s J_E\right) s_2\right). \end{aligned} \tag{4.14}$$

The sufficiency follows easily by (4.13) (or (4.14)). Now, using Theorem 4.1, we find

$$\left(\Phi_{J_E} g_E\right)(s, s_1, s_2) + \left(\Phi_{J_E} g_E\right)(s_2, s_1, s) = 2 g_E\left(s, \left(\nabla_{s_1} J_E\right) s_2\right) \tag{4.15}$$

which says that if $\Phi_{J_E} g_E = 0$ then $\nabla J_E = 0$. □

Corollary 4.2. *The almost* (*para*)*complex structure* J_E *of an almost* (*para*)*anti-Hermitian Lie algebroid is integrable if and only if* $\Phi_{J_E} g_E = 0$.

As usual, for an almost (para)anti-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ we consider the associated Norden metric defined by

$$G_E(s_1, s_2) = (g_E \circ J_E)(s_1, s_2) = g_E(J_E(s_1), s_2), \quad \forall\, s_1, s_2 \in \Gamma(E). \tag{4.16}$$

One can easily see that G_E is a Lie algebroid metric, which is also called the *twin* or *dual* metric of g_E and it plays a role similar to the Kähler form section on Hermitian Lie algebroids. Applying the Tachibana operator Φ_{J_E} to the pure Riemannian metric G_E, we have

$$\begin{aligned}(\Phi_{J_E} G_E)(s_1, s_2, s_3) &= (\mathcal{L}_{J_E(s_1)} G_E - \mathcal{L}_{s_1}(G_E \circ J_E))(s_2, s_3) \\ &\quad + G_E(s_2, J_E(\mathcal{L}_{s_1} s_3)) - G_E(J_E(s_2), \mathcal{L}_{s_1} s_3) \\ &= (\Phi_{J_E} g_E)(s_1, J_E(s_2), s_3) + g_E(N_{J_E}(s_1, s_2), s_3). \end{aligned} \tag{4.17}$$

Thus (4.17) implies

Theorem 4.3. *In an almost (para)anti-Hermitian Lie algebroid* $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$*, we have*

$$\Phi_{J_E} G_E = (\Phi_{J_E} g_E) \circ J_E + g_E \circ N_{J_E}.$$

Corollary 4.3. *In a (para)anti-Hermitian Lie algebroid* $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ *we have* $\Phi_{J_E} g_E = 0$ *if and only if* $\Phi_{J_E} G_E = 0$.

From Theorems 4.2 and 4.3 we have

Theorem 4.4. *There do not exist almost (para)anti-Hermitian Lie algebroids with conditions* $\Phi_{J_E} G_E = 0$ *and* $N_{J_E} \neq 0$.

Let us denote by ∇_{g_E} the covariant differentiation of Levi-Civita connection of (para)anti-Hermitian metric g_E. Then, we have

$$\nabla_{g_E} G_E = (\nabla_{g_E} g_E) \circ J_E + g_E \circ (\nabla_{g_E} J_E) = g_E \circ (\nabla_{g_E} J_E)$$

which implies $\nabla_{g_E} G_E = 0$ by virtute of Theorem 4.2. Therefore we have

Theorem 4.5. *Let* $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ *be a (para)anti-Hermitian Lie algebroid. Then the Levi-Civita connection of the (para)anti-Hermitian metric* g_E *coincides with the Levi-Civita connection of twin Norden metric* G_E.

Let us consider now R and S the curvature tensors associated to g_E and G_E, respectively. Then, for (para)anti-Hermitian Lie algebroids we have $R = S$ by means of Theorem 4.5. Taking into account that $\nabla J_E = 0$ we get

$$J_E(R(s_1, s_2)s_3) = R(s_1, s_2)J_E(s_3), \quad \forall\, s_1, s_2, s_3 \in \Gamma(E). \tag{4.18}$$

We have

Theorem 4.6. *For a (para)anti-Hermitian Lie algebroid* $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ *the Riemann curvature tensor section is pure.*

Proof. By means of (4.18) the Riemann curvature tensor section given by $\mathcal{R}(s_1, s_2, s_3, s_4) = g_E(R(s_1, s_2)s_3, s_4)$ is pure with respect to s_3 and s_4. Indeed

$$\begin{aligned}\mathcal{R}(s_1, s_2, J_E(s_3), s_4) &= g_E(R(s_1, s_2)J_E(s_3), s_4) = g_E(J_E(R(s_1, s_2)s_3), s_4)\\ &= g_E(R(s_1, s_2)s_3, J_E(s_4)) = \mathcal{R}(s_1, s_2, s_3, J_E(s_4)).\end{aligned}$$

Moreover, $\mathcal{R}$ is also pure with respect to s_1 and s_2 by the well known relation

$$\mathcal{R}(s_1, s_2, s_3, s_4) = \mathcal{R}(s_3, s_4, s_1, s_2). \tag{4.19}$$

On the other hand, S being the curvature tensor section formed by twin Norden metric G_E, if we put $\mathcal{S}(s_1, s_2, s_3, s_4) = G_E(S(s_1, s_2)s_3, s_4)$, then we have

$$\mathcal{S}(s_1, s_2, s_3, s_4) = \mathcal{S}(s_3, s_4, s_1, s_2). \tag{4.20}$$

Taking into account (3.12), (4.16), (4.18) and $R = S$, we get

$$\begin{aligned}\mathcal{S}(s_1, s_2, s_3, s_4) &= G_E(S(s_1, s_2)s_3, s_4) = g_E(J_E(S(s_1, s_2)s_3), s_4)\\ &= g_E(S(s_1, s_2)s_3, J_E(s_4)) = g_E(R(s_1, s_2)s_3, J_E(s_4))\\ &= \mathcal{R}(s_1, s_2, s_3, J_E(s_4))\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}(s_3, s_4, s_1, s_2) &= G_E(S(s_3, s_4)s_1, s_2) = g_E(J_E(S(s_3, s_4)s_1), s_2)\\ &= g_E(S(s_3, s_4)s_1, J_E(s_2)) = g_E(R(s_3, s_4)s_1, J_E(s_2))\\ &= \mathcal{R}(s_3, s_4, s_1, J_E(s_2)) = \mathcal{R}(s_1, J_E(s_2), s_3, s_4).\end{aligned}$$

Hence, the relation (4.20) becomes

$$\mathcal{R}(s_1, s_2, s_3, J_E(s_4)) = \mathcal{R}(s_1, J_E(s_2), s_3, s_4),$$

which says that $\mathcal{R}(s_1, s_2, s_3, s_4)$ is pure with respect to s_2 and s_4, which end the proof. □

As usual, a torsion-free almost complex connection ∇ on E, see [11], is called *J_E-analytic connection* if $\nabla_{J_E(s_1)}s_2 = J_E(\nabla_{s_1}s_2)$. Also, we can prove that the purity of the curvature tensor section of a connection ∇ is a necessary and sufficient condition for its J_E-analyticity. Therefore, from Theorem 4.6 we have

Corollary 4.4. *For an anti-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ the Levi-Civita connection is J_E-analytic.*

Since the Riemann curvature tensor section $\mathcal{R}$ is pure, we can apply the operator Φ_{J_E} to $\mathcal{R}$. Using $\nabla J_E = 0$, we have

$$(\Phi_{J_E}\mathcal{R})(s, s_1, s_2, s_3, s_4) = (\nabla_{J_E(s)}\mathcal{R})(s_1, s_2, s_3, s_4) - (\nabla_s\mathcal{R})(J_E(s_1), s_2, s_3, s_4). \tag{4.21}$$

Using (4.18) and the second Bianchi identity from (2.5) (with $T = 0$), a similar calculations as in [13] (or [25] for paracomplex case) yields $(\Phi_{J_E}\mathcal{R})$ $(s, s_1, s_2, s_3, s_4) = 0$. Therefore, we have

Theorem 4.7. *In a (para)anti-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$, the Riemann curvature tensor section $\mathcal{R}$ is almost J_E-analytic.*

Let us make now some local considerations. If we consider (x^i), $i = 1, \ldots, n$ a local coordinates system on M and $\{e_a\}$, $a = 1, \ldots, 2m$ a local basis of sections on the bundle E, where $\dim M = n$ and $\operatorname{rank} E = 2m$, then (x^i, y^a), $i = 1, \ldots, n$, $a = 1, \ldots, 2m$ are local coordinates on E. In a such local coordinates system, the anchor ρ_E and the Lie bracket $[\cdot,\cdot]_E$ are expressed by the smooth functions ρ_a^i and C_{bc}^a, namely $\rho_E(e_a) = \rho_a^i \frac{\partial}{\partial x^i}$ and $[e_a, e_b]_E = C_{ab}^c e_c$, $i = 1, \ldots, n$, $a, b, c = 1, \ldots, 2m$. The functions ρ_a^i, $C_{bc}^a \in C^\infty(M)$ are called the *structure functions* of Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ in the given local coordinates system.

Taking into account that $d_E f = \rho^i_a \frac{\partial f}{\partial x^i} e^a$, where $\{e^a\}$, $a = 1, \ldots, 2m$ is the dual basis of $\{e_a\}$, for an almost J_E-analytic function $f \in C^\infty(M, J_E)$ the condition (3.3) can be expressed locally by

$$J^a_b \rho^i_a \frac{\partial f}{\partial x^i} = \rho^i_b \frac{\partial g}{\partial x^i}, \tag{4.22}$$

where $J_E = J^a_b e_a \otimes e^b$ is the almost (para)complex structure on E.

From Theorems 4.6, 4.7 and (4.21) we find that for (para)anti-Hermitian Lie algebroids the covariant derivative of the curvature tensor section $\nabla \mathcal{R}$ is also pure. Let $g_{ab} = g_E(e_a, e_b)$ and R^c_{ab} and R_{abcd} the local components of curvature tensor section R and of Riemann curvature tensor section $\mathcal{R}$, respectively. Now, the covariant derivative of the Ricci tensor section $R_{ab} = R^c_{cab} = g^{dc} R_{dabc}$ is pure in all its indices and hence

$$J^c_d \nabla_c R_{ab} = J^c_a \nabla_d R_{cb}. \tag{4.23}$$

Contracting this equation with contravariant anti-Hermitian metric g^{ab}, we have

$$J^c_d \nabla_c \alpha = g^{ab} J^c_a \nabla_d R_{cb} = \nabla_d (G^{cb} R_{cb}) = \nabla_d \overset{*}{\alpha}, \tag{4.24}$$

where $\alpha = g^{ab} R_{ab}$ and $\overset{*}{\alpha}$ are curvature scalars of anti-Hermitian metric g_E and of twin Norden metric G_E, respectively. Here $G_{ab} = G_E(e_a, e_b)$ and G^{ab} is its inverse.

Taking into account that $\nabla_c \alpha = \rho^i_c \frac{\partial \alpha}{\partial x^i}$, the equation (4.24) yields

Theorem 4.8. *In a (para)anti-Hermitian Lie algebroid, the curvature scalar α is an almost J_E-analytic function.*

Example 4.1. (Complete and vertical lifts to the prolongation of a Lie algebroid). For a Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$ with $\operatorname{rank} E = m$ we can consider the prolongation of E over its vector bundle projection, see [10,19], which is a vector bundle $p_L : \mathcal{L}^p(E) \to E$ of $\operatorname{rank} \mathcal{L}^p(E) = 2m$ which has a Lie algebroid structure over E. More exactly, $\mathcal{L}^p(E)$ is the subset of $E \times TE$ defined by $\mathcal{L}^p(E) = \{(u, z) \mid \rho_E(u) = p_*(z)\}$, where $p_* : TE \to TM$ is the canonical projection. The projection on the second factor $\rho_{\mathcal{L}^p(E)} : \mathcal{L}^p(E) \to TE$, given by $\rho_{\mathcal{L}^p(E)}(u, z) = z$ will be the anchor of the prolongation Lie algebroid $\left(\mathcal{L}^p(E), \rho_{\mathcal{L}^p(E)}, [\cdot,\cdot]_{\mathcal{L}^p(E)}\right)$ over E. For a smooth function $f \in C^\infty(M)$ its *complete* and *vertical lift* to E, f^c and f^v respectively, are given by $f^c(u) = \rho_E(u) f$ and $f^v(u) = (f \circ p)(u)$ for every $u \in E$. According to [19], we can consider the *vertical lift* s^v and the *complete lift* s^c of a section $s \in \Gamma(E)$ as sections of $\mathcal{L}^p(E)$ as follows. The local basis of $\Gamma(\mathcal{L}^p(E))$ is given by $\left\{\mathcal{X}_a(u) = \left(e_a(p(u)), \rho^i_a \frac{\partial}{\partial x^i}|_u\right), \mathcal{V}_a = \left(0, \frac{\partial}{\partial y^a}\right)\right\}$, where $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right\}$, $i = 1, \ldots, n = \dim M$, $a = 1 \ldots, m = \operatorname{rank} E$, is the local basis on TE. Then, the vertical and complete lifts, respectively, of a section $s = s^a e_a \in \Gamma(E)$ are given by $s^v = s^a \mathcal{V}_a$, $s^c = s^a \mathcal{X}_a + \left(\rho_E(e_c)(s^a) - C^a_{bc} s^b\right) y^c \mathcal{V}_a$. In particular, $e^v_a = \mathcal{V}_a$ and $e^c_a = \mathcal{X}_a - C^b_{ac} y^c \mathcal{V}_b$.

Now, if $\operatorname{rank} E = 2m$ and J_E is an almost complex structure on the Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E)$, then J_E^c is an almost complex structure on $\big(\mathcal{L}^p(E), \rho_{\mathcal{L}^p(E)}, [\cdot,\cdot]_{\mathcal{L}^p(E)}\big)$ (because one of the properties of the complete lift is: for $p(T)$ a polynomial, then $p(T^c) = p(T)^c$), and moreover, $J_E^c s^v = (J_E s)^v$ and $J_E^c s^c = (J_E s)^c$.

The complete lift g_E^c to $\mathcal{L}^p(E)$ of almost anti-Hermitian metric g_E on E is defined by

$$g_E^c(s_1^c, s_2^c) = (g_E(s_1, s_2))^c,$$

and it is easy to see that $g_E^c(J_E^c(s_1^c), s_2^c) = g_E^c(s_1^c, J_E^c(s_2^c))$, that is g_E^c is almost anti-Hermitian metric on $\mathcal{L}^p(E)$ too. Furthermore, since $N_{J_E^c} = (N_{J_E})^c$, then J_E^c is integrable if and only if J_E is integrable and then by Corollary 4.2 we have

Theorem 4.9. *The anti-Hermitian Lie algebroid $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ is almost J_E-analytic if and only if the anti-Hermitian Lie algebroid $(\mathcal{L}^p(E), \rho_{\mathcal{L}^p(E)}, [\cdot,\cdot]_{\mathcal{L}^p(E)}, J_E^c, g_E^c)$ is almost J_E^c-analytic.*

Also, we can consider the metric $G_{\mathcal{L}^p(E)} = g_E^c + g_E^v$, where g_E^v is the vertical lift of g_E to $\mathcal{L}^p(E)$ and this is an almost anti-Hermitian metric on $\mathcal{L}^p(E)$ too. As usual we can prove that the Levi-Civita connection of $G_{\mathcal{L}^p(E)}$ coincides with the Levi-Civita connection of g_E^c and, thus we have that $(E, \rho_E, [\cdot,\cdot]_E, J_E, g_E)$ is almost J_E-analytic if and only if $(\mathcal{L}^p(E), \rho_{\mathcal{L}^p(E)}, [\cdot,\cdot]_{\mathcal{L}^p(E)}, J_E^c, G_{\mathcal{L}^p(E)})$ is almost J_E^c-analytic.

*Example 4.2***.** (Horizontal and vertical lifts to the prolongation of a Lie algebroid). Let ∇ a linear connection on the Lie algebroid E (in particular a Riemannian Lie algebroid (E, g_E) and the Levi-Civita connection). Then, the connection ∇ leads to a natural decomposition of $\mathcal{L}^p(E)$ into vertical and horizontal subbundles

$$\mathcal{L}^p(E) = H\mathcal{L}^p(E) \oplus V\mathcal{L}^p(E),$$

where $V\mathcal{L}^p(E) = \operatorname{span}\{\mathcal{V}_a\}$ and $H\mathcal{L}^p(E) = \operatorname{span}\{\mathcal{H}_a = \mathcal{X}_a - \Gamma_{ac}^b y^c \mathcal{V}_b\}$, where $\Gamma_{ac}^b(x)$ are the local coefficients of the linear connection ∇. We notice that, the above decomposition can be obtained also by a nonlinear (Ehresmann) connection, see for instance [21].

The horizontal lift s^h of a section $s = s^a e_a \in \Gamma(E)$ to $\mathcal{L}^p(E)$ is locally given by $s^h = s^a\mathcal{H}_a = s^a(\mathcal{X}_a - \Gamma_{ac}^b y^c \mathcal{V}_b)$. Then, we can define the Sasaki type metric $g_{\mathcal{L}^p(E)}$ on $\mathcal{L}^p(E)$ by

$$g_{\mathcal{L}^p(E)}(s_1^h, s_2^h) = g_{\mathcal{L}^p(E)}(s_1^v, s_2^v) = (g_E(s_1, s_2))^v,\ g_{\mathcal{L}^p(E)}(s_1^h, s_2^v) = 0, \quad (4.25)$$

for every $s_1, s_2 \in \Gamma(E)$. The metric $g_{\mathcal{L}^p(E)}$ has local components

$$g_{\mathcal{L}^p(E)} = \begin{pmatrix} g_{ab} + g_{ef} y^c y^d \Gamma_{cb}^e \Gamma_{da}^f & y^c \Gamma_{cb}^d g_{ad} \\ y^c \Gamma_{ca}^d g_{bd} & g_{ab} \end{pmatrix}$$

with respect to the induced coordinates (x^k, y^a), $k = 1, \dots, n$, $a = 1, \dots, m$ on E, where Γ^a_{bc} are the coefficients of the Levi-Civita connection ∇_{g_E} on E (see for instance formula (12) from [4] for their local expressions).

The diagonal lift on $\mathcal{L}^p(E)$ of a tensor section $F \in \Gamma(E) \otimes \Gamma(E^*)$ is defined by

$$F^d(s^h) = (F(s))^h, \quad F^d(s^v) = -(F(s))^v, \qquad \forall\, s \in \Gamma(E),$$

so that the diagonal lift I^d_E of the identity $I_E \in \Gamma(E) \otimes \Gamma(E^*)$ has the local components

$$I^d_E = \begin{pmatrix} \delta^b_a & 0 \\ -2y^c\Gamma^b_{ac} & -\delta^b_a \end{pmatrix}$$

with respect to the induced coordinates and satisfies $(I^d_E)^2 = I_{\mathcal{L}^p(E)}$. Thus I^d_E is an almost paracomplex structure on $\mathcal{L}^p(E)$ determining the horizontal and vertical distributions.

We put now $A(\widetilde{s}_1, \widetilde{s}_2) = g_{\mathcal{L}^p(E)}(I^d_E(\widetilde{s}_1), \widetilde{s}_2) - g_{\mathcal{L}^p(E)}(\widetilde{s}_1, I^d_E(\widetilde{s}_2))$, for every $\widetilde{s}_1, \widetilde{s}_2 \in \Gamma(\mathcal{L}^p(E))$.

If $A(\widetilde{s}_1, \widetilde{s}_2) = 0$ for all sections $\widetilde{s}_1$ and $\widetilde{s}_2$ which are of the form s^v_1, s^v_2 or s^h_1, s^h_2, then $A = 0$. We have by means of $I^d_E(s^v_1) = -s^v_1$, $I^d_E(s^h_1) = s^h_1$ and (4.25)

$$\begin{aligned} A(s^v_1, s^v_2) &= g_{\mathcal{L}^p(E)}(-s^v_1, s^v_2) - g_{\mathcal{L}^p(E)}(s^v_1, -s^v_2) = 0, \\ A(s^v_1, s^h_2) &= g_{\mathcal{L}^p(E)}(-s^v_1, s^h_2) - g_{\mathcal{L}^p(E)}(s^v_1, s^h_2) = 0, \\ A(s^h_1, s^v_2) &= g_{\mathcal{L}^p(E)}(s^h_1, s^v_2) - g_{\mathcal{L}^p(E)}(s^h_1, -s^v_2) = 0, \\ A(s^h_1, s^h_2) &= g_{\mathcal{L}^p(E)}(s^h_1, s^h_2) - g_{\mathcal{L}^p(E)}(s^h_1, s^h_2) = 0, \end{aligned} \tag{4.26}$$

which says that $g_{\mathcal{L}^p(E)}$ is a para-anti-Hermitian metric on $\mathcal{L}^p(E)$ with respect to I^d_E and thus, $(\mathcal{L}^p(E), g_{\mathcal{L}^p(E)}, I^d_E)$ is an almost para-anti-Hermitian Lie algebroid.

We notice that

$$\begin{aligned} [s^h_1, s^h_2]_{\mathcal{L}^p(E)} &= [s_1, s_2]^h_E + (R(s_1, s_2)u)^v, \quad [s^h_1, s^v_2]_{\mathcal{L}^p(E)} \\ &= (\nabla_{s_1} s_2)^v, \quad [s^v_1, s^v_2]_{\mathcal{L}^p(E)} = 0. \end{aligned}$$

Here the curvature R satisfies $R = -N_h$, where N_h is the Nijenhuis tensor of the horizontal projection, see [21,22].

By direct calculus, we have

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^v, s_2^h, s_3^h) = -2g_{\mathcal{L}^p(E)}((\nabla_{s_2} s_1)^v, s_3^h) + g_{\mathcal{L}^p(E)}(s_2^h, (\nabla_{s_3} s_1)^v) = 0,$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^v, s_2^h, s_3^v) = -2g_{\mathcal{L}^p(E)}\left(s_2^h, [s_3^v, s_1^v]_{\mathcal{L}^p(E)}\right) = 0,$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^v, s_2^v, s_3^h) = -2g_{\mathcal{L}^p(E)}\left([s_2^v, s_1^v]_{\mathcal{L}^p(E)}, s_3^h\right) = 0,$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^v, s_2^v, s_3^v) = 0, \left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^h, s_2^h, s_3^h) = 0,$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^h, s_2^v, s_3^v) = 2\left((\nabla_{s_1} g_E)(s_2, s_3)\right)^v = 0,$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^h, s_2^h, s_3^v) = -2g_{\mathcal{L}^p(E)}((R(s_2, s_1)u)^v, s_3^v),$$

$$\left(\Phi_{I_E^d} g_{\mathcal{L}^p(E)}\right)(s_1^h, s_2^v, s_3^h) = -2g_{\mathcal{L}^p(E)}(s_2^v, (R(s_3, s_1)u)^v).$$

Therefore, we have

Theorem 4.10. *The almost para-anti-Hermitian Lie algebroid $(\mathcal{L}^p(E), g_{\mathcal{L}^p(E)}, I_E^d)$ is almost I_E^d-analytic if and only if the horizontal distribution $H\mathcal{L}^p(E)$ is integrable.*

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