TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. xx, pp. 0-0, xx 20xx
DOI: 10.11650/tjm.19.2015.5297
This paper is available online at http://journal.taiwanmathsoc.org.tw

SLANT AND LEGENDRE CURVES IN BERGER $su(2)$: THE LANCRET INVARIANT AND QUANTUM SPHERICAL CURVES

C. Călin and M. Crasmareanu*

Abstract. Slant and Legendre curves are considered on Berger $su(2)$ and are characterized through the scalar product between the normal at the curve and the vertical vector field; in the helix case they have a proper (non-harmonic) mean curvature vector field. The general expression of these curves is obtained as well as their curvature and torsion. For the slant non-Legendre case we derive a Lancret-type invariant. By using the exponential map we obtain remarkable classes of curves on $S^3(1)$; in the helix case, and taking into account a B.-Y. Chen characterization of Legendre curves, we get a 1-parameter family of curves in relationship with the spectrum of the quantum harmonic oscillator. These curves, called by us *quantum spherical curves*, and their mates, provided by integer multiples of π, belong to antipodal Hopf fibres.

1. INTRODUCTION

An important notion of classical differential geometry of curves is that of *curve of constant slope*, also called *cylindrical helix*. This is a curve in the Euclidean space $\mathbb{E}^3$ for which the tangent vector field has a constant angle with a fixed direction called the *axis*. The second name corresponds to the fact that there exists a cylinder on which the curve moves in such a way that it cuts each ruling at a constant angle. The classical characterization of these curves is the Bertrand-Lancret-de Saint Venant Theorem ([2]): the curve γ in $\mathbb{E}^3$ is of constant slope if and only if the ratio of the torsion τ and the curvature κ is constant. More precisely, for a cylindrical helix we have the constant ratio $\frac{\cos\theta}{|\sin\theta|} = \frac{\tau}{\kappa}$ and then, inspired by the title of [2], we define *the Lancret invariant* as $Lancret(\gamma) = \frac{\cos\theta}{|\sin\theta|}$. By computing κ and τ in terms of θ we get the result above and therefore the expression of Lancret invariant in the 3-dimensional Euclidean geometry is:

Received September 15, 2014, accepted December 8, 2014.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification*: 22E60, 53C40, 53A55, 53C25.
Key words and phrases: Hopf bundle, Berger sphere, Slant curve, Legendre curve, Lancret invariant, Quantum spherical curve and its mate, Proper mean curvature vector field.
*Corresponding author.

$$Lancret(\gamma) = \frac{\tau}{\kappa}. \tag{1.1}$$

An interesting generalization of this class of curves is that of *slant curve* in almost contact metric geometry. This concept was introduced in [8] with the constant angle θ between the tangent vector field and the Reeb vector field. The particular case of $\theta = \frac{\pi}{2}$ (or $\theta = \frac{3\pi}{2}$) is very important since we recover the *Legendre curves* of [1]. For the same general expression $\frac{\cos\theta}{|\sin\theta|}$ the Theorem 3.1. of [8, p. 362] gives the form of Lancret invariant in 3-dimensional Sasakian geometry:

$$Lancret(\gamma) = \frac{\tau \pm 1}{\kappa}. \tag{1.2}$$

Although the Bibliography in Legendre curves is very rich (see the references of [6]), slant curves are studied until now only for the Sasakian geometry in [8], for the contact pseudo-Hermitian geometry in [9], for trans-Sasakian geometry in [12], for the f-Kenmotsu geometry in [6] and for almost paracontact geometry in [18]. So, the purpose of this paper is to begin a study of slant curves in another important class of 3-dimensional geometries, namely Berger $su(2)$. These spaces are obtained from the Riemannian Hopf fibration $S^3 \to S^2$ by varying the length of the S^1-fibers with a factor and a slant curve will be one that has constant angle with these fibers.

Our work is structured as follows. The second section is a very brief review of Berger spheres and Frenet curves in general Riemannian geometry. The next section is devoted to the study of slant (particularly Legendre) curves on these manifolds. So, we obtain a characterization of slant curves similar to Proposition 3.1. of [8, p. 362]. We obtain a complete description of slant curves and then we compute the curvature and torsion which yield the corresponding Lancret invariant for the non-Legendre case. As example we treat a class of helices (depending of a real parameter ω called *angular velocity*) which are Legendre curves for all Berger $su(2)$.

In the last part of third section we move our study on S^3 by using the exponential map. On this way we obtain remarkable classes of curves e.g. starting with helices γ and taking into account a B.-Y. Chen characterization of Legendre curves in S^3 we get a class of curves $\exp(\gamma)$ such that the positive angular velocities of γ's are the inverse of energy eigenvalues for the quantum harmonic oscillator. Hence, we call *quantum spherical curves* this class of curves and remark that they are periodical and then, in a direct connection with the periodic curves from [14]. These curves and their mates (provided by inverse of integer multiplies of π) are mapped by the Hopf bundle projection in antipodal points on $S^2(\frac{1}{2})$. It follows a kind of "mirror symmetry" in $\mathbb{R} \times \mathbb{C}$ with the "mirror" $\mathbb{C}$ placed in the origin of $\mathbb{R}$. Moreover, we extend the B.-Y. Chen characterization from Legendre to arbitrary slant curves in $S^3(c)$.

The last section is devoted to another main result namely that a slant curve of helix type has a proper (non-harmonic) curvature vector field. We obtain the general expression of the corresponding eigenvalue function of the Laplacian in terms of angular velocity ω.

2. Berger $su(2)$'s and Frenet Curves in Riemannian Geometry

The starting point of this paper is provided by the Hopf bundle $\pi : S^3(1) \to S^2(\frac{1}{2}) \subset \mathbb{R} \oplus \mathbb{C}$ where, by using the complex numbers, the projection is ([16, p. 4]):

$$\pi(z, w) = \left(\frac{1}{2}(|w|^2 - |z|^2), z\bar{w}\right). \tag{2.1}$$

Identifying $S^3(1)$ with $SU(2)$ we get the parallelization of S^3 given by the Pauli vector fields ([16, p. 7]):

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.2}$$

Then $S^3(1)$ is also a Lie group with the Lie algebra $su(2)$ spanned by X_i, $1 \leq i \leq 3$. For $\varepsilon \in (0, 1]$ we define on $su(2)$ the Riemannian metric g_ε by requiring the following vector fields to be orthonormal:

$$E_1 = \frac{X_1}{\varepsilon}, \quad E_2 = X_2, \quad E_3 = X_3. \tag{2.3}$$

We call the pair $(su(2), g_\varepsilon)$ as *Berger* $su(2)$ and the curvature can be computed with [16, p. 81]:

$$\begin{cases} \nabla_{E_1} E_2 = \frac{2-\varepsilon^2}{\varepsilon} E_3, \nabla_{E_1} E_3 = -\frac{2-\varepsilon^2}{\varepsilon} E_2 \\ \nabla_{E_2} E_1 = -\varepsilon E_3, \nabla_{E_3} E_1 = \varepsilon E_2, \nabla_{E_2} E_3 = \varepsilon E_1 = -\nabla_{E_3} E_2. \end{cases} \tag{2.4}$$

Here ∇ is the Levi-Civita connection of g_ε. The metric g_1 is the canonical one of $su(2)$ so we call sometimes $(su(2), g_1)$ as *the Euclidean* $su(2)$.

Next we recall the notion of Frenet curve after [3, p. 164]: let n and m be integers with $1 \leq m \leq n$. The curve $\gamma : I \subseteq \mathbb{R} \to (M_n, g)$ parametrized by the arc length s is called m*-Frenet curve* on M if there exists m orthonormal vector fields $E_1 = \gamma', E_2, \dots, E_m$ along γ such that there exists positive smooth functions $\kappa_1, \dots, \kappa_{m-1}$ of s such that:

$$\nabla_{\gamma'} E_1 = \kappa_1 E_2, \nabla_{\gamma'} E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \dots, \nabla_{\gamma'} E_m = \kappa_{m-1} E_{m-1}. \tag{2.5}$$

The function κ_j is called the j-th curvature of γ while γ is:

(a) *a geodesic* if $m = 1$; then we get the well-known equation $\nabla_{\gamma'} \gamma' = 0$.

(b) *a circle* if $m = 2$ and κ_1 is a constant: then we have $\nabla_{\gamma'} E_1 = \kappa_1 E_2, \ \ \nabla_{\gamma'} E_2 = -\kappa_1 E_1$.

(c) *a helix* of order m if $\kappa_1, \dots, \kappa_{m-1}$ are constants.

The Frenet curve γ is called *non-geodesic* if $\kappa_1 > 0$ everywhere on I and in dimension 3 it is called *a generalized helix* if $\frac{\kappa_2}{\kappa_1} = const.$

3. Slant and Legendre Curves in $(su(2), g_\varepsilon)$: Lancret Invariant and a Kind of Mirror Symmetry

Let $\gamma : I \to (su(2), g_\varepsilon)$ be a 3-Frenet curve for which we denote the Frenet frame as usual $(T = \gamma', N, B)$ and the Frenet equations:

$$\nabla_T T = \kappa N, \ \nabla_T N = -\kappa T + \tau B, \ \nabla_T B = -\tau N, \tag{3.1}$$

where κ is the curvature and τ the torsion.

Definition 3.1. The *structural angle* of γ is the function $\theta : I \to [0, 2\pi)$ given by:

$$\cos\theta(s) = g_\varepsilon(T(s), E_1). \tag{3.2}$$

The curve γ is *a slant curve*, or more precisely θ*-slant curve*, if θ is a constant function, [8, p. 361]. In the particular case of $\theta = \frac{\pi}{2}$ the curve γ is called *a Legendre curve*, [1]. For a slant non-Legendre curve γ its *Lancret invariant* is:

$$Lancret(\gamma) = \frac{\cos\theta}{|\sin\theta|}. \tag{3.3}$$

From $\nabla_{E_i} E_i = 0$ it results that E_i are geodesic vector fields i.e. the integral curves of E_i are geodesics. In particular the 0-slant and π-slant curves are geodesics. In the following we suppose that γ is non-geodesic i.e. $\kappa > 0$ and $\theta \in (0, \pi)$ and it follows a characterization of these curves:

Proposition 3.2. *The Frenet curve γ is a θ-slant curve in $(su(2), g_\varepsilon)$ if and only if its normal vector field N is g_ε-orthogonal to $E_1|_\gamma$.*

Proof. By taking the covariant derivative in the relation (2.2) along γ and using the fact that E_1 is a Killing vector field:

$$0 = -\theta'(s)\sin\theta(s) = g_\varepsilon(\kappa(s)N(s), E_1)$$

we derive the conclusion. ∎

Remarks 3.3. In [15, p. 155] it is introduced the following notion: a non-geodesic curve is called *a slant helix* if the principal normal lines of γ make a constant angle with a fixed direction. Therefore, a slant curve is a slant helix with E_1 as fixed direction.

By using the definition of $su(2)$ ([16, p. 6]) the curve γ has the expression:

$$\gamma(s) = \begin{pmatrix} iA(s) & B(s)+iC(s) \\ -B(s)+iC(s) & -iA(s) \end{pmatrix} := (A(s), B(s), C(s)). \tag{3.4}$$

The main result of this Section is as follows:

Theorem 3.4. *Suppose γ has the form $\gamma(s) = (A(s), B(s), C(s)) \in su(2)$. Then γ is a θ-slant curve if and only if there exists a real number A_0 and a smooth parametrization $(\cos u(s), \sin u(s))$ of S^1 such that:*

$$\begin{cases} A(s) = \dfrac{s\cos\theta}{\varepsilon} + A_0 \\ B(s) = \sin\theta \displaystyle\int_0^s \cos u(t)dt \\ C(s) = \sin\theta \displaystyle\int_0^s \sin u(t)dt. \end{cases} \tag{3.5}$$

Supposing u as being a strictly increasing function we have:

$$\begin{cases} \kappa(s) = \sin\theta\left(u'(s) + 2\cos\theta\dfrac{1-\varepsilon^2}{\varepsilon}\right) \\ \tau(s) = \varepsilon + \cos\theta\left(u'(s) + 2\cos\theta\dfrac{1-\varepsilon^2}{\varepsilon}\right). \end{cases} \tag{3.6}$$

Its Lancret invariant is:

$$Lancret(\gamma) = \frac{\tau - \varepsilon}{\kappa}. \tag{3.7}$$

Therefore, γ is a helix if and only if $u(s) = \omega s$ with $\omega > 0$.

Proof. Denote $a = A'$, $b = B'$, $c = C'$. The derivative of γ is:

$$\gamma'(s) = \begin{pmatrix} a(s)i & b(s)+ic(s) \\ -b(s)+ic(s) & -a(s)i \end{pmatrix}$$

equivalently:

$$\gamma'(s) = a(s)X_1 + b(s)X_2 + c(s)X_3 = \varepsilon a(s)E_1 + b(s)E_2 + c(s)E_3 \tag{3.8}$$

and the equation (3.2) yields (3.5_1). The unit speed of $\gamma(s)$ means the existence of a unit-length parametrization $(\cos u(s), \sin u(s))$ of S^1 such that $(3.5_{2,3})$ hold. Also:

$$T(s) = \cos\theta E_1 + \sin\theta\cos u(s)E_2 + \sin\theta\sin u(s)E_3 \tag{3.9}$$

The relation (3.6_1) follows from (3.8) taking into account the definition of the curvature. More precisely:

$$\begin{aligned}\kappa(s)N(s) &= \left[b'(s)-2c(s)\cos\theta\frac{1-\varepsilon^2}{\varepsilon}\right]E_2+\left[c'(s)+2b(s)\cos\theta\frac{1-\varepsilon^2}{\varepsilon}\right]E_3 \\ &= \sin\theta\left(\frac{b'}{c}-2\cos\theta\frac{1-\varepsilon^2}{\varepsilon}\right)\left(\frac{c}{\sin\theta}E_2-\frac{b}{\sin\theta}E_3\right).\end{aligned}\tag{3.10}$$

Differentiating the above relation along γ and using the Frenet equations it follows:

$$\tau B=\left(\kappa\frac{\cos\theta}{\sin\theta}+\varepsilon\right)(\sin\theta E_1-\cos\theta\cos uE_2-\cos\theta\sin uE_3)\tag{3.11}$$

which yields (3.6_2) and (3.7). ∎

Remarks 3.5. (i) We have now the Frenet frame:

$$N(s)=-\sin uE_2+\cos uE_3,\ B(s)=\sin\theta E_1-\cos\theta\cos uE_2-\cos\theta\sin uE_3,\tag{3.12}$$

$$\begin{aligned}E_1|_\gamma &= \cos\theta T+\sin\theta B,\\ \nabla_{\gamma'}E_1 &= (\varepsilon c(s))E_2+(-\varepsilon b(s))E_3=(\kappa\cos\theta-\tau\sin\theta)N.\end{aligned}\tag{3.13}$$

Then the norm of $\nabla_{\gamma'}E_1$ is independent of γ: $\|\nabla_{\gamma'}E_1\|=\varepsilon\sin\theta$.

(ii) For a helix $\gamma=\gamma_{A_0,\omega}$ we can call $\omega>0$ as *the angular velocity* of γ and say that γ is a *ω-helix*.

(iii) A slant curve is a Bertrand one i.e. there exist $x,y\in\mathbb{R}\setminus\{0\}$ such that $x\kappa+y\tau=1$, namely $x=-\frac{\varepsilon\cos\theta}{\sin\theta}$ and $y=\frac{1}{\varepsilon}$.

Examples 3.6. Let $A\in\mathbb{R}$ and a smooth (increasing) real function u.

(i) (Legendre) The curve $\gamma_{A,u}(s)=\left(Ai,\int_0^s e^{iu(s)}\right)$ is an "universal" (i.e. not depending of ε) Legendre curve in $(su(2),g_\varepsilon)$ with:

$$\kappa_{\varepsilon,A,u}=u'(s),\quad \tau_{\varepsilon,A,u}=\varepsilon\le 1.\tag{3.14}$$

(ii) With $\varepsilon=1$ we get the slant curve $\gamma(s)=(s\cos\theta+A,\sin\theta\int_0^s e^{iu(t)}dt)$ in the Euclidean $su(2)$.

(iii) For $\theta\in(\frac{\pi}{2},\pi)$ and $\omega=-\left(\frac{\varepsilon}{\cos\theta}+2\cos\theta\frac{1-\varepsilon}{\varepsilon}\right)$ the ω-helix γ becomes a circle with $\kappa=-\varepsilon\tan\theta$.

(iv) For $u(t)=\arccos\frac{1}{\cosh t}$ we get: $\cos u(t)=\frac{1}{\cosh t}$, $\sin u(t)=\frac{\sinh t}{\cosh t}$; then $B(s)=2\sin\theta\arctan\exp(t)|_0^s=2\sin\theta(\arctan\exp(s)-\frac{\pi}{4})$ and $C(s)=\sin\theta\ln\cosh t|_0^s=\sin\theta\ln\cosh s$.

(v) For $u(t) = t^2$ we derive that $B(s) = \sin\theta C(s)$ and $C(s) = \sin\theta S(s)$ where $S(s)$ and $C(s)$ (of right hand-side) are the Fresnel transcendental functions. It results that the projection of $\gamma \in \mathbb{R}^3$ on its last two coordinates is a conformal deformation (with the constant factor $\sin\theta$) of the Euler-Cornu spiral. ■

A curve γ in a Lie group G endowed with a bi-invariant metric $<,>$ is called *general helix* in [10, p. 1598] if γ makes a constant angle with a left-invariant vector field. The corresponding Lancret invariant is [10, p. 1599]:

$$Lancret_{(G,<,>)}(\gamma) = \frac{\tau - \tau_{(G,<>)}}{\kappa} \tag{3.15}$$

where:

$$\tau_{(G,<>)} = \frac{1}{2} < [T,N], B > . \tag{3.16}$$

For our framework:

$$[T,N] = (2\varepsilon \sin\theta)E_1 - \left(\frac{2}{\varepsilon}\cos\theta\cos u\right)E_2 - \left(\frac{2}{\varepsilon}\cos\theta\sin u\right)E_3 \tag{3.17}$$

and then:

$$\tau_{(SU(2),g_\varepsilon)} = \varepsilon\sin^2\theta + \frac{\cos^2\theta}{\varepsilon} = \varepsilon + \cos^2\theta\frac{1-\varepsilon^2}{\varepsilon}. \tag{3.18}$$

For $\varepsilon = 1$ we reobtain $\tau_{S^3} = 1$ of [10, p. 1599] and then our Lancret invariant (3.7) coincides with that of the cited paper and also with (1.2) from Introduction; recall that the Euclidean $S^3(1)$ is a Sasakian manifold. For an arbitrary ε the expression (3.18) coincides with ε only for $\theta = \frac{\pi}{2}$, a case without Lancret invariant. Recall now the exponential on $SU(2)$; if $X \in su(2)$ is different to 0 then:

$$\exp(X) = \cos\|X\| + \frac{X}{\|X\|}\sin\|X\|. \tag{3.19}$$

If γ is a ω-helix slant curve with $A_0 = 0$ we have:

$$\|\gamma(s)\| = \sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2} + \frac{\sin^2\theta}{\omega^2}} \tag{3.20}$$

and then, in a quaternion-type expression, it results the curve in (S^3, g_ε):

$$\begin{aligned} \exp(\gamma)(s) = {} & \cos\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2} + \frac{\sin^2\theta}{\omega^2}} \\ & + \frac{\sin\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2} + \frac{\sin^2\theta}{\omega^2}}}{\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2} + \frac{\sin^2\theta}{\omega^2}}}\left(\frac{s\cos\theta}{\varepsilon}\bar{i} + \sin\theta\frac{\sin(\omega s)}{\omega}\bar{j} - \sin\theta\frac{\cos(\omega s)}{\omega}\bar{k}\right). \end{aligned} \tag{3.21}$$

In [7, p. 76] it is proved that an unit-speed curve $z : I \to \mathbb{C}^2$ is a Legendre curve in the Euclidean $S^3(c)$ (with $c > 0$) if and only if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that:

$$z'' = i\lambda z' - cz. \tag{3.22}$$

A straightforward computation gives that the curve $\exp(\gamma)$ of (3.21) with $\theta = \frac{\pi}{2}$, although is not an unit-speed curve, satisfies (3.22) for $c = 1$ only when:

$$\begin{cases} \omega = \omega_l := \dfrac{2}{(2l+1)\pi} \\ \lambda = \dfrac{\omega^2 - 1}{\omega} = constant \end{cases} \tag{3.23}$$

for arbitrary integers l. We call these curves as being *quantum spherical curves* since the inverse of its angular velocity are particular cases of the energy eigenvalues of the quantum harmonic oscillator, [11, p. 93]:

$$E_l = \hbar\omega\left(l + \frac{1}{2}\right) \tag{3.24}$$

where ω is the frequency of the oscillator and now, l is a positive integer. With $\hbar = 1$ and $\omega = \pi$ we get the $1/\omega_l$'s of (3.23) for $l \geq 0$. Let us remark that a quantization problem regarding all periodic magnetic curves of arbitrary strength on a Sasakian space form $M^3(c)$ is considered in [14].

Hence the expression of a quantum spherical curve is:

$$\exp(\gamma_l)(s) = (0, 0, \sin(\omega_l s), -\cos(\omega_l s)) \in S^3(1) \tag{3.25}$$

with a constant $\lambda = \lambda_l$:

$$\lambda_l = \frac{2}{(2l+1)\pi} - \frac{(2l+1)\pi}{2}. \tag{3.26}$$

This curve is periodical with the principal period $T_l = (2l+1)\pi^2$ and for a positive l we have $\lambda_l < 0$. The curvature and torsion of its source curve γ_l with $l \in \mathbb{N}$ are:

$$\kappa = \omega_l = \frac{2}{(2l+1)\pi}, \quad \tau = 1. \tag{3.27}$$

From a symmetry argument we can add the mate $\exp(\gamma_l^c)$ of the quantum spherical curve (3.25) by having as source curve the ω_l^c-helix γ_l^c provided by:

$$\omega_l^c = \frac{1}{l\pi}, \quad l \in \mathbb{N}^* = \{1, 2, ...\}. \tag{3.28}$$

Returning to the general curve $\exp(\gamma)$ of (3.21) its image through the Hopf projection (2.1) is:

$$\pi \circ \exp(\gamma)(s) = (\pi^1(s), \pi^2(s))$$

with:

$$(3.29)\quad \begin{cases} \pi^1(s) = \dfrac{\sin^2\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}\left[\dfrac{\sin^2\theta}{2\omega^2}-\dfrac{s^2\cos^2\theta}{2\varepsilon^2}\right] \\ \qquad -\dfrac{1}{2}\cos^2\sqrt{\dfrac{s^2\cos^2\theta}{\varepsilon^2}+\dfrac{\sin^2\theta}{\omega^2}}, \\ \dfrac{\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}{\sin\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}\dfrac{\omega}{\sin\theta}\pi^2(s) = \cos\sqrt{\dfrac{s^2\cos^2\theta}{\varepsilon^2}+\dfrac{\sin^2\theta}{\omega^2}}\sin(\omega s) \\ \qquad -\dfrac{\sin\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}{\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}\dfrac{s\cos\theta\cos(\omega s)}{\varepsilon} \\ \qquad +i\left(\dfrac{s\cos\theta\sin(\omega s)\sin\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}{\varepsilon\omega\sqrt{\frac{s^2\cos^2\theta}{\varepsilon^2}+\frac{\sin^2\theta}{\omega^2}}}\right. \\ \qquad \left. +\cos(\omega s)\cos\sqrt{\dfrac{s^2\cos^2\theta}{\varepsilon^2}+\dfrac{\sin^2\theta}{\omega^2}}\right). \end{cases}$$

For the Legendre case we have:

$$(3.30)\quad \begin{aligned} &\pi \circ \exp(\gamma)(s) = \\ &\left(\frac{1}{2}(\sin^2\frac{1}{\omega}-\cos^2\frac{1}{\omega}), \sin\frac{1}{\omega}\cos\frac{1}{\omega}(\sin(\omega s)+i\cos(\omega s))\right) \in S^2\left(\frac{1}{2}\right) \end{aligned}$$

and in the particular case of quantum spherical curves and their mates we get a unique point:

$$(3.31)\quad \pi \circ \exp(\gamma_l)(s) = \left(\frac{1}{2}, 0\right) \in \mathbb{R}\oplus\mathbb{C}, \quad \pi \circ \exp(\gamma_l^c)(s) = \left(-\frac{1}{2}, 0\right) \in \mathbb{R}\oplus\mathbb{C}$$

and thus we have a point reflection across the origin $(0,0) \in \mathbb{R}\times\mathbb{C}$. It follows a kind of "geometrical mirror symmetry" in $\mathbb{R}\times\mathbb{C}$ with the "mirror" $\mathbb{C}$ placed in the origin of $\mathbb{R}$. From a geometrical point of view, we can say that a quantum spherical curve and its mate belongs to *antipodal Hopf fibres*: $\mathbf{H}^{quantum} = \mathbf{H}_{1/2} = \pi^{-1}\left(\frac{1}{2}\right)$, $\mathbf{H}^{quantum,c} = \mathbf{H}_{-1/2} = \pi^{-1}\left(-\frac{1}{2}\right)$; regarding them in $S^3(1)$ these are symmetrical with respect to the "Equator" $S^2(1) \times \{0\} \subset S^3(1)$. In fact, by Lemma 1 of [17, p. 380], $\mathbf{H}_{\pm 1/2}$ is isometric to $\{0\} \times S^1$=the flat torus $\mathbb{R}^2/\Lambda$, where Λ is the lattice generated by $(0,0)$ and $(2\pi, 0)$.

A final remark for this Section is that the equation (3.22) of B.-Y. Chen admits the following generalization to the slant curves in $S^3(c)$:

$$z''(s) = \lambda(s)(iz'(s)) - c(1 + \lambda(s)\cos\theta)z(s) \tag{3.32}$$

where λ is a real-valued function being $\lambda =< z'', iz' >$ with respect to the Kähler metric $<, >$ of $\mathbb{C}^2$.

4. The Mean Curvature Vector Field

Let h be the second fundamental form of γ in $(su(2), g_\varepsilon)$ and H its mean curvature field. We know that:

$$H = \mathrm{trace}(h) = h(T,T) = \nabla_T T. \tag{4.1}$$

Then γ is called *a curve with proper mean curvature vector field* if there exists $\rho \in C^\infty(\gamma)$ such that:

$$\Delta H = \rho H. \tag{4.2}$$

In particular, if $\rho = 0$ then γ is known as *a curve with harmonic mean curvature vector field*. Here the Laplace operator Δ acts on the vector valued function H and it is given by:

$$\Delta H = -\nabla_T \nabla_T \nabla_T T. \tag{4.3}$$

Making use of Frenet equations, we can rewrite (4.2) as:

$$-3\kappa' k T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B = -\lambda\kappa N. \tag{4.4}$$

It follows that both κ and τ are constants, and the function λ becomes a constant too, namely:

$$\rho = \kappa^2 + \tau^2. \tag{4.5}$$

(see also Theorem 1.1 in [13]). As consequence, it follows that there are no θ-slant, non-geodesic curves in a 3-dimensional manifold with harmonic mean curvature vector field.

For our framework we state the following:

Proposition 4.1. *A non-geodesic θ-slant curve γ in $(su(2), g_\varepsilon)$ has a proper mean curvature vector field if and only if γ is a ω-helix and then:*

$$\rho = \left(\omega + 2\cos\theta\frac{1-\varepsilon^2}{\varepsilon}\right)^2 + 2\varepsilon\cos\theta\left(\omega + 2\cos\theta\frac{1-\varepsilon^2}{\varepsilon}\right) + \varepsilon^2. \tag{4.6}$$

In particular, a ω-helix Legendre curve has:

$$\rho = \omega^2 + \varepsilon^2 \in (\omega^2, 1+\omega^2]. \tag{4.7}$$

For example, the source γ_l of a quantum spherical curve $\exp(\gamma_l)$ and its mate γ_l^c have:

$$\rho_l = 1 + \frac{4}{(2l+1)^2\pi^2}, \quad \rho_l^c = 1 + \frac{1}{l^2\pi^2}. \tag{4.8}$$

Acknowledgments

The authors cordially thank to referee(s) for a carefully analysis of this work and some useful corrections.

References

1. Ch. Baikoussis and D. E. Blair, On Legendre curves in contact 3-manifolds, *Geom. Dedicata*, **49(2)** (1994), 135-142.
2. M. Barros, General helices and a theorem of Lancret, *Proc. Amer. Math. Soc.*, **125(5)**, (1997), 1503-1509.
3. D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Second edition, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010.
4. C. Călin and M. Crasmareanu, Slant curves and particles in 3-dimensional warped products and their Lancret invariants, *Bull. Austral. Math. Soc.*, **88(1)** (2013), 128-142.
5. C. Călin and M. Crasmareanu, Slant curves in 3-dimensional normal almost contact metric geometry, *Mediterr. J. Math.*, **10(2)** (2013), 1067-1077.
6. C. Călin, M. Crasmareanu and M.-I. Munteanu, Slant curves in 3-dimensional f-Kenmotsu manifolds, *J. Math. Anal. Appl.*, **394(1)** (2012), 400-407.
7. B.-Y. Chen, Interaction of Legendre curves and Lagrangian submanifolds, *Israel J. Math.*, **99** (1997), 69-108.
8. J. T. Cho, J.-I. Inoguchi and J.-E. Lee, On slant curves in Sasakian 3-manifolds, *Bull. Austral. Math. Soc.*, **74(3)** (2006), 359-367.
9. J. T. Cho and J.-E. Lee, Slant curves in contact pseudo-Hermitian 3-manifolds, *Bull. Austral. Math. Soc.*, **78(3)** (2008), 383-396.
10. Ü. Çiftçi, A generalization of Lancret's theorem, *J. Geom. Phys.*, **59(2)** (2009), 1597-1603.

11. D. Chruściński and A. Jamiołkowski, *Geometric phases in classical and quantum mechanics*, Progress in Mathematical Physics, 36, Birkhäuser Boston, Inc., Boston, MA, 2004.

12. Ş. Güvenç and C. Özgür, On slant curves in trans-Sasakian manifolds, *Rev. Un. Mat. Argentina*, **55(2)** (2014), 81-100.

13. J.-I. Inoguchi, Submanifolds with harmonic mean curvature vector field in contact 3-manifolds, *Colloq. Math.*, **100(2)** (2004), 163-179.

14. J.-I. Inoguchi and M.-I. Munteanu, *Periodic magnetic curves in elliptic Sasakian space forms*, arXiv:1310.2899v1 [math.DG] 10 Oct 2013.

15. S. Izumiya and N. Takeuchi, New special curves and developable surfaces, *Turkish J. Math.*, **28(2)** (2004), 153-163.

16. P. Petersen, *Riemannian geometry*, Second edition, Graduate Texts in Mathematics, 171, Springer, New York, 2006.

17. U. Pinkall, Hopf tori in S^3, *Invent. Math.*, **81(2)** (1985), 379-386.

18. J. Welyczko, Slant curves in 3-dimensional normal almost paracontact metric manifolds, *Mediterr. J. Math.*, **11(3)** (2014), 965-978.

Constantin Călin
Department of Mathematics
Technical University "Gh. Asachi"
Iaşi 700049
Romania
E-mail: c0nstc@yahoo.com

Mircea Crasmareanu
Faculty of Mathematics
University "Al. I. Cuza"
Iaşi 700506
Romania
E-mail: mcrasm@uaic.ro
http://www.math.uaic.ro/∼mcrasm