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Unitary vector fields are Fermi–Walker transported along Rytov–Legendre curves

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Fix ξ a unitary vector field on a Riemannian manifold M and γ a non-geodesic Frenet curve on M satisfying the Rytov law of polarization optics. We prove in these conditions that γ is a Legendre curve for ξ if and only if the γ-Fermi–Walker covariant derivative of ξ vanishes. The cases when γ is circle or helix as well as ξ is (conformal) Killing vector filed or potential vector field of a Ricci soliton are analyzed and an example involving a three-dimensional warped metric is provided. We discuss also K-(para)contact, particularly (para)Sasakian, manifolds and hypersurfaces in complex space forms.

Keywords: Reeb vector field; Legendre curve; Rytov curve; Fermi–Walker transport; circle.

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One of the fruitful notions of classical differential geometry of curves is that of *curve of constant slope*. Also called *cylindrical helix*, this is a curve in the Euclidean space $\mathbb{E}^3$ for which the tangent vector field has a constant angle with a fixed direction called the *axis* of the curve; the second name is due to the fact that there exists a cylinder on which the curve moves in such a way as to cut each ruling at a constant angle. The well-known characterization of this curve is the Bertrant–Lancret–de Saint Venant Theorem (see [2]): the curve γ in $\mathbb{E}^3$ is of constant slope if and only if the ratio of torsion and curvature is constant.

A very interesting generalization of this object is that of *slant curve* in almost contact metric geometry and was introduced in [8] (see also [9]) with the slant angle

θ between the tangent to curve and the structural (Reeb) vector field and in the three-dimensional case. In the particular case of $\theta = \frac{\pi}{2}$ (or $\theta = \frac{3\pi}{2}$) we recover the *Legendre curves* of [1]. Let us pointed out that the bibliography in Legendre curves (in different ambient manifolds) is by now very rich (see [3, 5, 7, 13–24]).

Fix (M, g) an m-dimensional Riemannian manifold and let ξ be a g-unitary vector field on it; let us point out that all results hold also for a semi-Riemannian metric g and space-like vector field $g(\xi, \xi) = +1$. Consider also η the g-dual of ξ. The following example is classical.

Example 1. Supposing $m = 2n + 1$ the pair (M, η) is a *contact manifold* if $V = \eta \wedge (d\eta)^n$ is a volume form on M. On a given contact manifold there exists a unique vector field ξ such that:

$$i_\xi \eta = 1, \quad i_\xi d\eta = 0 \tag{1}$$

called *the Reeb vector field* of (M, η). An *adapted metric* on (M, η) is a Riemannian metric for which ξ is unitary; then the triple (M, η, g) will be called *contact metric manifold* or *Riemannian contact manifold*.

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Now we recall the definition of *Frenet curve* in an m-dimensional (semi-)-Riemannian manifold after [4, p. 164]; let r be an integer with $1 \leq r \leq m$. A given curve $\gamma : J \subseteq \mathbb{R} \to M$ parametrized by the arc length s is called r-*Frenet curve* on M if there exist r orthonormal vector fields $(E_1 = \gamma', \ldots, E_r)$ along γ so that there exist positive smooth functions $k_1, \ldots, k_{r-1}$ of s with:

$$\begin{cases} \nabla_{\gamma'} E_1 = k_1 E_2, \\ \nabla_{\gamma'} E_2 = -k_1 E_1 + k_2 E_3, \\ \qquad \vdots \\ \nabla_{\gamma'} E_r = -k_{r-1} E_{r-1}. \end{cases} \tag{2}$$

Here ∇ is the Levi-Civita connection of g and for the semi-Riemannian case we will consider only space-like curves. The function k_j is called *the* jth *curvature* of γ and the Frenet curve γ is called:

(1) a *geodesic* if $r = 1$; then we get the well-known equation $\nabla_{\gamma'} \gamma' = 0$,
(2) a *circle* if $r = 2$ and k_1 is a constant; then, we have: $\nabla_{\gamma'} E_1 = k_1 E_2$ together with $\nabla_{\gamma'} E_2 = -k_1 E_1$,
(3) a *helix of order* r if $k_1, \ldots, k_{r-1}$ are constants.

Returning to our framework (M, g, ξ, γ) with γ a r-Frenet curve we introduce the following.

Definition 2. (i) The *structural angle* of γ is the function $\theta : J \to [0, 2\pi)$ given by:

$$\cos \theta(s) = g(E_1(s), \xi) = \eta(E_1(s)). \tag{3}$$

(ii) γ is a *slant curve* (or more precisely θ-*slant curve*) if θ is a constant function, [8, p. 361]. In the particular case of $\theta \equiv \frac{\pi}{2}$ (or $\frac{3\pi}{2}$), γ is called *Legendre curve* [1].

(iii) γ is called ξ-*Rytov curve* (after [10, p. 244]) if $\nabla_{E_1}\xi$ is parallel with E_1; extending also the words of the cited book we say that ξ is *not-gyrotropic* with respect to γ.

The last tool we need is the Fermi–Walker transport used recently in [15–18]. Let $\mathcal{X}_\gamma$ be the set of vector fields along γ. To the data (M, g, γ) we associate the γ-*Fermi–Walker derivative* (see [10, p. 245] or [14, p. 207]): $\nabla^{\mathrm{FW}}_\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma$:

$$\nabla^{\mathrm{FW}}_\gamma(X) = \nabla_{E_1}X + g(X, \nabla_{E_1}E_1)E_1 - g(X, E_1)\nabla_{E_1}E_1. \tag{4}$$

From the first Frenet equation we have:

$$\nabla^{\mathrm{FW}}_\gamma(X) = \nabla_{E_1}X + k_1[g(X, E_2)E_1 - g(X, E_1)E_2] \tag{5}$$

and then:

$$\nabla^{\mathrm{FW}}_\gamma(E_1) = 0, \quad \nabla^{\mathrm{FW}}_\gamma(E_2) = k_2E_3, \quad \nabla^{\mathrm{FW}}_\gamma(E_s) = \nabla_{E_1}E_s \tag{6}$$

for $3 \leq s \leq r$. As in the usual Levi-Civita case to which the Fermi–Walker derivative reduces if γ is a geodesic for g, $X \in \mathcal{X}_\gamma$ is called γ-*Fermi–Walker parallel* if $\nabla^{\mathrm{FW}}_\gamma(X) = 0$.

In the following we suppose that γ is non-geodesic i.e. $k_1 > 0$ which means $r \geq 2$. The main result of this paper is the following characterization of Legendre curves (satisfying the Rytov condition) similar to that of geodesics through Levi-Civita parallelism of tangent vector field.

Theorem 3. *The non-geodesic* ξ-*Rytov curve* γ *is a Legendre curve in the manifold* (M, g, ξ) *if and only if* ξ *is* γ-*Fermi–Walker parallel. It follows*:

$$\nabla_{E_1}\xi = -k_1\eta(E_2)E_1. \tag{7}$$

Proof. The relation (4) becomes:

$$\nabla^{\mathrm{FW}}_\gamma(\xi) = \nabla_{E_1}\xi + \eta(\nabla_{E_1}E_1)E_1 - \eta(E_1)\nabla_{E_1}E_1. \tag{8}$$

From the Rytov condition we have:

$$\nabla_{E_1}\xi = g(\nabla_{E_1}\xi, E_1)E_1 = [E_1(\eta(E_1)) - \eta(\nabla_{E_1}E_1)]E_1 \tag{9}$$

and thus we get:

$$\nabla^{\mathrm{FW}}_\gamma(\xi) = E_1(\eta(E_1))E_1 - k_1\eta(E_1)E_2. \tag{10}$$

Then $\nabla^{\mathrm{FW}}_\gamma(\xi) = 0$ if and only if $\eta(E_1) = 0$. Returning with this relation in (9) we obtain (7). □

Remarks and Examples 4.

(i) Trying to extend this result we can consider a *generalized Rytov curve*, namely a curve for which $\nabla_{E_1}\xi$ is orthogonal on E_2. Analyzing a relation similar to (9) we derive that a generalized Rytov curve for which ξ is Fermi–Walker parallel reduces to a Rytov curve.

(ii) In the framework provided by Theorem 3 the sectional curvature of the plane spanned by ξ and E_1 is:

$$\begin{aligned} K(\xi, E_1) &= g(R(\xi, E_1)E_1, \xi) \\ &= [\xi(k_1) + k_1^2\eta(E_2)]\eta(E_2) + k_1 g(\nabla_\xi E_2, \xi) \\ &\quad + g(\nabla_{E_1}[E_1, \xi] + \nabla_{[E_1,\xi]}E_1, \xi). \end{aligned} \tag{11}$$

But the last term of previous equation is exactly *the symmetric product* of [6, p. 118] with respect to the Levi-Civita connection:

$$\langle E_1, [E_1, \xi]\rangle_\nabla := \nabla_{E_1}[E_1, \xi] + \nabla_{[E_1,\xi]}E_1. \tag{12}$$

Let us recall the geometrical meaning of this product after the cited book. We say that a distribution D on M is *geodesically invariant* under a linear connection ∇ of M if, as a submanifold of the tangent bundle TM, D is invariant under the geodesic spray associated with ∇. Then the distribution D is geodesically invariant if and only if the symmetric product of any D-valued vector fields is again a D-valued vector field.

So, let us suppose that the vector fields E_1 and $[E_1, \xi]$ are linearly independent and then we consider D spanned by E_1 and $[E_1, \xi]$. If this distribution is geodesically invariant by g it results the existence of two smooth functions $\alpha, \beta \in C^\infty(M)$ such that:

$$\langle E_1, [E_1, \xi]\rangle_\nabla = \alpha E_1 + \beta[E_1, \xi] \tag{13}$$

and then we get the following.

Proposition 5. *Let γ be a non-geodesic ξ-Rytov curve and Legendre curve for ξ. Suppose also that the distribution spanned by E_1 and $[E_1, \xi]$ is geodesically invariant for the Levi-Civita connection of g. Then the sectional curvature of the plane spanned by ξ and E_1 is:*

$$K(\xi, E_1) = [\xi(k_1) + k_1^2\eta(E_2)]\eta(E_2) + k_1 g(\nabla_\xi E_2, \xi) + \beta\eta([E_1, \xi]). \tag{14}$$

(iii) Suppose that ξ is Killing and in its Killing equation:

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0,$$

let $X = Y = E_1$. Then, with (7), it results $2k_1\eta(E_2) = 0$ and we get that $\eta(E_2) = 0$ and hence: (i) ξ is parallel along γ; (ii) $\xi|_\gamma \in \text{span}\{E_3, \ldots, E_r\}$. For $r = 3$ this implies that $\xi|_\gamma = \pm E_3$. Let us point out that if ξ is parallel along the curve γ then the ξ-Rytov condition is satisfied and its Fermi–Walker derivative is:

$$\nabla_\gamma^{\text{FW}}(\xi) = k_1 I \wedge \eta(E_1, E_2), \tag{5Par}$$

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where I is the identity (Kronecker) endomorphism and wedge is the exterior product.

A particular case is when (M, g) is a Lie group with a bi-invariant metric and ξ is a (unitary) left-invariant vector field. Then ξ is a Killing vector field and moreover:

$$\nabla_{E_1}\xi = \frac{1}{2}[E_1, \xi]$$

which yields, via the discussion above, that (7) reduces to $[E_1, \xi] = 0$ which means that E_1 is a Lie symmetry of ξ. Plugging this relation in (11) we get that: $K(\xi, E_1) = k_1 g(\nabla_\xi E_2, \xi) = k_1\eta(\nabla_\xi E_2)$.

More generally than Killing condition suppose that (g, ξ) is a Ricci soliton on M; then there exists a scalar λ such that:

$$\mathcal{L}_\xi g + 2\mathrm{Ric}_g + 2\lambda g = 0, \tag{15}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field ξ and Ric_g is the Ricci tensor field of g. Applying this relation on the pair (E_1, E_1) and using (7) it follows:

$$k_1\eta(E_2) = \mathrm{Ric}_g(E_1, E_1) + \lambda. \tag{16}$$

In particular, if (M, g) is a Ricci-flat manifold i.e. $\mathrm{Ric}_g = 0$ (or ξ is conformal Killing vector field with $-\lambda$ the conformal factor) then $k_1\eta(E_2)$ is the constant (function) λ. It follows that this Ricci soliton is expanding if $\eta(E_2) > 0$, steady if $\xi|_\gamma \in \mathrm{span}\{E_3, \ldots, E_r\}$ and shrinking if $\eta(E_2) < 0$. For $r = 3$ the steady case implies that $\xi|_\gamma = \pm E_3$.

(iv) Suppose $m = 3$ and γ a ξ-Rytov Legendre curve which is r-Frenet with $r \geq 2$. Applying ∇_{E_1} to $\xi|_\gamma = \eta(E_2)E_2 + \eta(E_3)E_3$ and using the Frenet equations (2) we obtain, for $\eta(E_2)\cdot\eta(E_3) \neq 0$:

$$k_2 = -\frac{E_1(\eta(E_3))}{\eta(E_2)} = \frac{E_1(\eta(E_2))}{\eta(E_3)}. \tag{17}$$

If $\eta(E_2) = 0$ then $\xi|_\gamma = \pm E_3$ and $k_2 = 0$. This yields the following remark, namely (v).

(v) If γ is a circle it results that γ is E_2-Rytov curve (by restricting the definition to a vector field along the curve) and also Legendre for E_2. In conclusion, E_2 is γ-Fermi Walker parallel along a circle. The same result holds for a helix with $k_2 = \cdots = k_{r-1} = 0$.

In this case, the formula (11) becomes:

$$K(E_1, E_2) = k_1^2 + g(\langle E_1, [E_1, E_2]\rangle_\nabla, E_2). \tag{18}$$

Now, we can provide an example. Let $M = \mathbb{R}^3$ with the warped metric:

$$g = e^{2z}(dx^2 + dy^2) + dz^2 \tag{19}$$

and the (vertical) vector field $\xi = \frac{\partial}{\partial z}$ with $\eta = dz$. The curve $\gamma : \mathbb{R} \to M$:

$$\gamma(s) = (\sin s, -\cos s, 0) \tag{20}$$

is unitary Legendre curve for ξ and circle for (M, g) with $k_1 = 1$. We have:

$$E_1 = (\cos s, \sin s, 0), \quad E_2 = (0, 0, -1) = -\xi. \tag{21}$$

It follows that:

(1) γ is a vertical-Rytov curve,
(2) ξ is γ-Fermi–Walker parallel along γ.

A motivation for the above choice of γ is the fact that in a real space form a unit speed curve is a circle if and only if it is a plane curve i.e. it lies in a 2-dimensional totally geodesic submanifold. The above manifold is a model of hyperbolic geometry having the constant curvature $K = -1$; see also [7].

(vi) Recall, after [11, p. 58], that ξ on an affine pair (M, ∇) is *a concircular vector field* if: $\nabla\xi = \rho I$ with ρ a smooth function on M and I the Kronecker tensor field. It follows that every curve on M is ξ-Rytov. An important remark of [11, p. 61] is that in the Riemannian case (i.e. ∇ is the Levi-Civita of a metric g) a concircular vector field is a gradient.

Suppose now that $\xi = \nabla f$ with f a function on M. The Legendre condition for the curve γ with respect to ξ means that f is constant along γ or, in other words, $f|_\gamma$ is a first integral for E_1. Let us point also that the unit norm of ξ means that f is a *distance function* according to [20].

For example, in the Euclidean plane $\mathbb{E}^2$ with the global coordinates (x, y) the distance function:

$$f(x, y) = \sqrt{x^2 + y^2} \tag{22}$$

yields the radial vector field on $\mathbb{R}^2 \backslash \{0\}$:

$$\xi = \nabla f = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y} \tag{23}$$

and the unitary Legendre curve for ξ is the unit circle $\mathbb{S}^1$.

(vii) Returning to Example 1 suppose now that the contact metric manifold is endowed with the tensor field ϕ of $(1, 1)$-type satisfying (see [4]):

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad g(\phi\cdot, \phi\cdot) = g - \eta \otimes \eta. \tag{24}$$

A new tensor field of the same type is (see [4, p. 84]): $h = \frac{1}{2}\mathcal{L}_\xi \phi$. So, after Lemma 6.2 of the same citation we have:

$$\nabla_X \xi = -\phi X - \phi h X. \tag{25}$$

It results that γ is a ξ-Rytov curve if and only if E_1 is an eigenvector of the tensor field $\phi + \phi h$. In conditions of Theorem 3 the relation (7) gives the eigenvalue $k_1 \eta(E_2)$.

Proposition 6. *Let γ be a Legendre curve in a K-contact (particularly Sasakian) manifold (M, g, ξ, ϕ). Then γ is not ξ-Rytov.*

Proof. Suppose that γ is ξ-Rytov and we can apply Theorem 3. The K-contact condition means $h = 0$ conform [4, p. 87] and with the discussion above we have the eigenvalue $k_1\eta(E_2)$ for ϕ corresponding to the eigenvector E_1. But the only real eigenvalue of ϕ is zero and its eigenvector is ξ. It results that $\xi|_\gamma = \pm E_1$ which is a contradiction with the Legendre condition. □

Combining this result with remark (v) it results that in a K-contact (particularly Sasakian) manifold there does not exist a circle with $E_2 = \xi|_\gamma$. Hence we study two other cases:

(A) (M, g, ξ, ϕ) is cosymplectic i.e. $\nabla_X \xi = 0$ for all vector fields X. The condition (7) yields $\eta(E_2) = 0$.

(B) (M, g, ξ, ϕ) is f-Kenmotsu, $f \in C^\infty(M)$ conform [7], which means that $\nabla_X \xi = f(X - \eta(X)\xi)$. Again the condition (7) yields that $k_1\eta(E_2) = -f|_\gamma$. The example of remark (v) is of this type with $f \equiv 1$.

As example let M be a real hypersurface of a complex space form $M_{2n+2}(c)$ and fix N a unit normal vector field on M. Let g be the Riemannian metric of M induced from the Fubini–Study metric of $M_{2n+2}(c)$ and denote by J the almost complex structure of the ambient manifold and by A the shape operator of M. Define $\xi = -JN$ and then for any vector field $X \in \mathfrak{X}(M)$ the decomposition holds: $JX = \varphi X + \eta(X)N$. The data (φ, ξ, η, g) is an almost contact metric structure on M and from the Kählerian nature of $M_{2n+2}(c)$ by making use of the Gauss and Weingarten formulas, we obtain for the Levi-Civita connection ∇ of g:

$$(\nabla_X \varphi)(Y) = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX. \tag{26}$$

Then the condition (7) means that E_1 is an eigenvector of $\varphi \circ A$ with the eigenvalue $-k_1\eta(E_2)$.

Let us study two cases:

(1) the shape operator has the form $A = \alpha I$. From [12, Lemma 16.1, p. 103] it follows that α is a constant and then $c = 0$. Then, the condition (7) reads:

$$\alpha\varphi E_1 = -k_1\eta(E_2)E_1 \tag{27}$$

and if $\alpha = 0$ then $\eta(E_2) = 0$ which gives the consequences of remark (iii). For $\alpha \neq 0$ it results that $-\frac{k_1\eta(E_2)}{\alpha}$ is an eigenvalue of φ. It results again that $\eta(E_2) = 0$ and $\xi|_\gamma = \pm E_1$ which is a contradiction with the Legendre condition.

(2) The shape operator A has two distinct eigenvalues; then these eigenvalues are constant [12, p. 126], and $A = \sigma I + (\rho - \sigma)\xi \oplus \eta$ as [12, (19.17), p. 129], ρ and σ being different to zero. From $\varphi(\xi) = 0$ it results the same relation (27) but with σ instead α.

(viii) Suppose now that the data (M, g, ξ, η, ϕ) is an almost paracontact metric manifold i.e. (see [25]):

$$\phi^2 = I - \eta \otimes \xi, \quad \phi(\xi) = 0, \quad g(\phi\cdot, \phi\cdot) = -g + \eta \otimes \eta. \tag{28}$$

Let us remark that here the metric g has the signature $(n+1, n)$ and ξ is unitary space-like vector field. From [25, Lemma 2.5, p. 41] we have in the paracontact metric manifolds (conform [25, Definition 2.1, p. 39]):

$$\nabla_X \xi = -\phi X + \phi h X \tag{29}$$

and then the ξ-Rytov condition means that E_1 is an eigenvalue of $\phi - \phi h$ with the eigenvalue $k_1 \eta(E_2)$. It results again the impossibility of circles with $E_2 = \pm\xi|_\gamma$.

Proposition 7. *Let γ be a space-like Frenet curve of order $r = 2$ with $E_2 = \pm\xi|_\gamma$ in a K-paracontact (particularly paraSasakian) manifold (M, g, ξ, ϕ). Then γ is not a circle on M.*

Proof. Suppose that γ is a circle. From $E_2 = \pm\xi$ and remark (v) it results that γ is Legendre and we can apply Theorem 3. The K-paracontact condition means $h = 0$ conform [4, p. 87] and again from hypothesis we have $\eta(E_2) = \pm 1$. With the discussion above we have the eigenvalue k_1 for ϕ. But the nonzero eigenvalues of ϕ are ± 1 and we get the curvature: $k_1 = 1$. Returning in (29) with $X = E_1$ it follows:

$$\nabla_{E_1} E_2 = -\phi(E_1) = -E_2 = -\xi \tag{30}$$

and then applying ϕ it results: $E_1 = \phi(\xi) = 0$ which is impossible. □

(ix) A linear connection naturally associated to the pair (g, ξ) is *the Weyl connection* [21, p. 1089]:

$$\nabla^\xi_X Y = \nabla_X Y - \frac{1}{2}\eta(X)Y - \frac{1}{2}\eta(Y)X + \frac{1}{2}g(X,Y)\xi. \tag{31}$$

Then:

$$\nabla^\xi_{E_1} \xi = \nabla_{E_1} \xi - \frac{1}{2} E_1 \tag{32}$$

which means that the Rytov condition can be also expressed in terms of ∇^ξ instead of ∇ with the same argument. In the Legendre case we get from (7):

$$\nabla^\xi_{E_1} \xi = -\left[k_1 \eta(E_2) + \frac{1}{2}\right] E_1. \tag{33}$$

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