Unitary vector fields are Fermi–Walker transported along Rytov–Legendre curves

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Fix ξ a unitary vector field on a Riemannian manifold M and γ a non-geodesic Frenet curve on M satisfying the Rytov law of polarization optics. We prove in these conditions that γ is a Legendre curve for ξ if and only if the γ-Fermi–Walker covariant derivative of ξ vanishes. The cases when γ is circle or helix as well as ξ is (conformal) Killing vector field or potential vector field of a Ricci soliton are analyzed and an example involving a three-dimensional warped metric is provided. We discuss also K-(para)contact, particularly (para)Sasakian, manifolds and hypersurfaces in complex space forms.

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One of the fruitful notions of classical differential geometry of curves is that of curve of constant slope. Also called cylindrical helix, this is a curve in the Euclidean space \( \mathbb{E}^3 \) for which the tangent vector field has a constant angle with a fixed direction called the axis of the curve; the second name is due to the fact that there exists a cylinder on which the curve moves in such a way as to cut each ruling at a constant angle. The well-known characterization of this curve is the Bertrant–Lancret–de Saint Venant Theorem (see [2]): the curve γ in \( \mathbb{E}^3 \) is of constant slope if and only if the ratio of torsion and curvature is constant.

A very interesting generalization of this object is that of slant curve in almost contact metric geometry and was introduced in [8] (see also [9]) with the slant angle
Definition 2. (i) The structural angle of $\gamma$ is the function $\theta : J \to [0,2\pi)$ given by:

$$\cos \theta(s) = g(E_1(s), \xi) = \eta(E_1(s)).$$

(ii) $\gamma$ is a slant curve (or more precisely $\theta$-slant curve) if $\theta$ is a constant function, [8, p. 361]. In the particular case of $\theta \equiv \frac{\pi}{2}$ (or $\frac{3\pi}{2}$), $\gamma$ is called Legendre curve [1].
(iii) $\gamma$ is called $\xi$-Rytov curve (after [10, p. 244]) if $\nabla_{E_1} \xi$ is parallel with $E_1$; extending also the words of the cited book we say that $\xi$ is not-gyrotropic with respect to $\gamma$.

The last tool we need is the Fermi–Walker transport used recently in [15–18]. Let $\mathcal{X}_\gamma$ be the set of vector fields along $\gamma$. To the data $(M, g, \gamma)$ we associate the $\gamma$-Fermi–Walker derivative (see [10, p. 245] or [14, p. 207]):

$$\nabla_{FW}^\gamma : \mathcal{X}_\gamma \to \mathcal{X}_\gamma :$$

From the first Frenet equation we have:

$$\nabla_{FW}^\gamma (X) = \nabla_{E_1} X + g(X, \nabla_{E_1} E_1) E_1 - g(X, E_1) \nabla_{E_1} E_1.$$  (4)

and then:

$$\nabla_{FW}^\gamma (E_1) = 0, \quad \nabla_{FW}^\gamma (E_2) = k_2 E_3, \quad \nabla_{FW}^\gamma (E_s) = \nabla_{E_1} E_s$$  (5)

for $3 \leq s \leq r$. As in the usual Levi-Civita case to which the Fermi–Walker derivative reduces if $\gamma$ is a geodesic for $g$, $X \in \mathcal{X}_\gamma$ is called $\gamma$-Fermi–Walker parallel if $\nabla_{FW}^\gamma (X) = 0$.

In the following we suppose that $\gamma$ is non-geodesic i.e. $k_1 > 0$ which means $r \geq 2$.

The main result of this paper is the following characterization of Legendre curves (satisfying the Rytov condition) similar to that of geodesics through Levi-Civita parallelism of tangent vector field.

**Theorem 3.** The non-geodesic $\xi$-Rytov curve $\gamma$ is a Legendre curve in the manifold $(M, g, \xi)$ if and only if $\xi$ is $\gamma$-Fermi–Walker parallel. It follows:

$$\nabla_{E_1} \xi = -k_1 \eta(E_2) E_1.$$  (7)

**Proof.** The relation (4) becomes:

$$\nabla_{FW}^\gamma (\xi) = \nabla_{E_1} \xi + \eta(\nabla_{E_1} E_1) E_1 - \eta(E_1) \nabla_{E_1} E_1.$$  (8)

From the Rytov condition we have:

$$\nabla_{E_1} \xi = g(\nabla_{E_1} \xi, E_1) E_1 = [E_1(\eta(E_1)) - \eta(\nabla_{E_1} E_1)] E_1$$  (9)

and thus we get:

$$\nabla_{FW}^\gamma (\xi) = E_1(\eta(E_1)) E_1 - k_1 \eta(E_1) E_2.$$  (10)

Then $\nabla_{FW}^\gamma (\xi) = 0$ if and only if $\eta(E_1) = 0$. Returning with this relation in (9) we obtain (7).

**Remarks and Examples 4.**

(i) Trying to extend this result we can consider a generalized Rytov curve, namely a curve for which $\nabla_{E_1} \xi$ is orthogonal on $E_2$. Analyzing a relation similar to (9) we derive that a generalized Rytov curve for which $\xi$ is Fermi–Walker parallel reduces to a Rytov curve.
(ii) In the framework provided by Theorem 3 the sectional curvature of the plane
spanned by \( \xi \) and \( E_1 \) is:

\[
K(\xi, E_1) = g(R(\xi, E_1)E_1, \xi) = \frac{1}{2}g\left(\left[\frac{\partial}{\partial t}\right] \xi(k_1) + k_1^2 \eta(E_2) \eta(E_2) + k_1 \eta(\nabla_\xi E_2, \xi) + \eta(\nabla_{E_1} E_1, \xi)ight).
\]

But the last term of previous equation is exactly the symmetric product of [6, p. 118]
with respect to the Levi-Civita connection:

\[
\langle E_1, [E_1, \xi]\rangle = \nabla_{E_1} [E_1, \xi] + \nabla_{[E_1, \xi]} E_1.
\]

Let us recall the geometrical meaning of this product after the cited book. We say
that a distribution \( D \) spanned by \( \xi \) and \( E_1 \) is geodesically invariant if and
only if the symmetric product of any \( D \)-valued vector fields is again a \( D \)-valued
vector field.

So, let us suppose that the vector fields \( E_1 \) and \( [E_1, \xi] \) are linearly independent
and then we consider \( D \) spanned by \( E_1 \) and \( [E_1, \xi] \). If this distribution is geodesically
invariant by \( g \) it results the existence of two smooth functions \( \alpha, \beta \in C^\infty(M) \) such
that:

\[
\langle E_1, [E_1, \xi]\rangle = \alpha E_1 + \beta [E_1, \xi]
\]

and then we get the following.

\[\text{Proposition 5.} \] Let \( \gamma \) be a non-geodesic \( \xi \)-Rytov curve and Legendre curve for \( \xi \).
Suppose also that the distribution spanned by \( E_1 \) and \( [E_1, \xi] \) is geodesically invariant
for the Levi-Civita connection of \( g \). Then the sectional curvature of the plane
spanned by \( \xi \) and \( E_1 \) is:

\[
K(\xi, E_1) = \frac{1}{2}g\left(\left[\frac{\partial}{\partial t}\right] \xi(k_1) + k_1^2 \eta(E_2) \eta(E_2) + k_1 \eta(\nabla_\xi E_2, \xi) + \beta \eta([E_1, \xi])\right).
\]

(iii) Suppose that \( \xi \) is Killing and in its Killing equation:

\[
g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0,
\]

let \( X = Y = E_1 \). Then, with (7), it results \( 2k_1 \eta(E_2) = 0 \) and we get that \( \eta(E_2) = 0 \)
and hence: (i) \( \xi \) is parallel along \( \gamma \); (ii) \( \xi |_\gamma \in \text{span}\{E_3, \ldots, E_r\} \). For \( r = 3 \) this
implies that \( \xi |_\gamma = \pm E_3 \). Let us point out that if \( \xi \) is parallel along the curve \( \gamma \) then
the \( \xi \)-Rytov condition is satisfied and its Fermi–Walker derivative is:

\[
\nabla^\text{FW}_\xi \xi = k_1 I \land \eta(E_1, E_2),
\]

where \( I \) is the identity (Kronecker) endomorphism and wedge is the exterior
product.
A particular case is when \((M, g)\) is a Lie group with a bi-invariant metric and \(\xi\) is a (unitary) left-invariant vector field. Then \(\xi\) is a Killing vector field and moreover:

\[
\nabla_{E_1}\xi = \frac{1}{2}[E_1, \xi]
\]

which yields, via the discussion above, that (7) reduces to \([E_1, \xi] = 0\) which means that \(E_1\) is a Lie symmetry of \(\xi\). Plugging this relation in (11) we get that: \(K(\xi, E_1) = k_1 g(\nabla_{E_2}\xi, E_1) = k_1 \eta(\nabla_{E_2}\xi).\)

More generally than Killing condition suppose that \((g, \xi)\) is a Ricci soliton on \(M\); then there exists a scalar \(\lambda\) such that:

\[
\mathcal{L}_\xi g + 2\text{Ric}_g + 2\lambda g = 0,
\]

where \(\mathcal{L}_\xi\) is the Lie derivative with respect to the vector field \(\xi\) and \(\text{Ric}_g\) is the Ricci tensor field of \(g\). Applying this relation on the pair \((E_1, E_1)\) and using (7) it follows:

\[
k_1 \eta(E_2) = \text{Ric}_g(E_1, E_1) + \lambda.
\]

In particular, if \((M, g)\) is a Ricci-flat manifold i.e. \(\text{Ric}_g = 0\) (or \(\xi\) is conformal Killing vector field with \(-\lambda\) the conformal factor) then \(k_1 \eta(E_2)\) is the constant (function) \(\lambda\). It follows that this Ricci soliton is expanding if \(\eta(E_2) > 0\), steady if \(\xi|_\gamma \in \text{span}\{E_3, \ldots, E_r\}\) and shrinking if \(\eta(E_2) < 0\). For \(r = 3\) the steady case implies that \(\xi|_\gamma = \pm E_3\).

(iv) Suppose \(m = 3\) and \(\gamma\) a \(\xi\)-Rytov Legendre curve which is \(r\)-Frenet with \(r \geq 2\). Applying \(\nabla_{E_1}\) to \(\xi|_\gamma = \eta(E_2)E_2 + \eta(E_3)E_3\) and using the Frenet equations (2) we obtain, for \(\eta(E_2) \cdot \eta(E_3) \neq 0\):

\[
k_2 = -\frac{E_1(\eta(E_3))}{\eta(E_2)} = \frac{E_1(\eta(E_2))}{\eta(E_3)}.
\]

If \(\eta(E_2) = 0\) then \(\xi|_\gamma = \pm E_3\) and \(k_2 = 0\). This yields the following remark, namely (v).

(v) If \(\gamma\) is a circle it results that \(\gamma\) is \(E_2\)-Rytov curve (by restricting the definition to a vector field along the curve) and also Legendre for \(E_2\). In conclusion, \(E_2\) is \(\gamma\)-Fermi Walker parallel along a circle. The same result holds for a helix with \(k_2 = \cdots = k_{r-1} = 0\).

In this case, the formula (11) becomes:

\[
K(E_1, E_2) = k_1^2 + g([E_1, [E_1, E_2]]\nabla, E_2).
\]

Now, we can provide an example. Let \(M = \mathbb{R}^3\) with the warped metric:

\[
g = e^{2z}(dx^2 + dy^2) + dz^2
\]

and the (vertical) vector field \(\xi = \frac{\partial}{\partial z}\) with \(\eta = dz\). The curve \(\gamma : \mathbb{R} \to M\):

\[
\gamma(s) = (\sin s, -\cos s, 0)
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is unitary Legendre curve for $\xi$ and circle for $(M, g)$ with $k_1 = 1$. We have:

$$E_1 = (\cos s, \sin s, 0), \quad E_2 = (0, 0, -1) = -\xi.$$  \hfill (21)

It follows that:

1. $\gamma$ is a vertical-Rytov curve,
2. $\xi$ is $\gamma$-Fermi–Walker parallel along $\gamma$.

A motivation for the above choice of $\gamma$ is the fact that in a real space form a unit speed curve is a circle if and only if it is a plane curve i.e. it lies in a 2-dimensional totally geodesic submanifold. The above manifold is a model of hyperbolic geometry having the constant curvature $K = -1$; see also [7].

(vi) Recall, after [11, p. 58], that $\xi$ on an affine pair $(M, \nabla)$ is a concircular vector field if: $\nabla \xi = \rho I$ with $\rho$ a smooth function on $M$ and $I$ the Kronecker tensor field. It follows that every curve on $M$ is $\xi$-Rytov. An important remark of [11, p. 61] is that in the Riemannian case (i.e. $\nabla$ is the Levi-Civita of a metric $g$) a concircular vector field is a gradient.

Suppose now that $\xi = \nabla f$ with $f$ a function on $M$. The Legendre condition for the curve $\gamma$ with respect to $\xi$ means that $f$ is constant along $\gamma$ or, in other words, $f|_{\gamma}$ is a first integral for $E_1$. Let us point also that the unit norm of $\xi$ means that $f$ is a distance function according to [20].

For example, in the Euclidean plane $E^2$ with the global coordinates $(x, y)$ the distance function:

$$f(x, y) = \sqrt{x^2 + y^2}$$  \hfill (22)

yields the radial vector field on $\mathbb{R}^2 \setminus \{0\}$:

$$\xi = \nabla f = \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y}$$  \hfill (23)

and the unitary Legendre curve for $\xi$ is the unit circle $S^1$.

(vii) Returning to Example 1 suppose now that the contact metric manifold is endowed with the tensor field $\phi$ of $(1, 1)$-type satisfying (see [4]):

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad g(\phi, \phi) = g - \eta \otimes \eta.$$  \hfill (24)

A new tensor field of the same type is (see [4, p. 84]): $h = \frac{1}{2} L_\xi \phi$. So, after Lemma 6.2 of the same citation we have:

$$\nabla X \xi = -\phi X - \phi h X.$$  \hfill (25)

It results that $\gamma$ is a $\xi$-Rytov curve if and only if $E_1$ is an eigenvector of the tensor field $\phi + \phi h$. In conditions of Theorem 3 the relation (7) gives the eigenvalue $k_1 \eta(E_2)$.

**Proposition 6.** Let $\gamma$ be a Legendre curve in a $K$-contact (particularly Sasakian) manifold $(M, g, \xi, \phi)$. Then $\gamma$ is not $\xi$-Rytov.
Proof. Suppose that $\gamma$ is $\xi$-Rytov and we can apply Theorem 3. The $K$-contact condition means $h = 0$ conform [4, p. 87] and with the discussion above we have the eigenvalue $k_1 \eta(E_2)$ for $\phi$ corresponding to the eigenvector $E_1$. But the only real eigenvalue of $\phi$ is zero and its eigenvector is $\xi$. It results that $\xi|_\gamma = \pm E_1$ which is a contradiction with the Legendre condition. \qed

Combining this result with remark (v) it results that in a $K$-contact (particularly Sasakian) manifold there does not exist a circle with $E_2 = \xi|_\gamma$. Hence we study two other cases:

(A) $(M, g, \xi, \phi)$ is cosymplectic i.e. $\nabla_X \xi = 0$ for all vector fields $X$. The condition (7) yields $\eta(E_2) = 0$.

(B) $(M, g, \xi, \phi)$ is $f$-Kenmotsu, $f \in C^\infty(M)$ conform [7], which means that $\nabla_X \xi = f(X - \eta(X)\xi)$. Again the condition (7) yields that $k_1 \eta(E_2) = -f|_\gamma$.

The example of remark (v) is of this type with $f \equiv 1$.

As example let $M$ be a real hypersurface of a complex space form $M_{2n+2}(c)$ and fix $N$ a unit normal vector field on $M$. Let $g$ be the Riemannian metric of $M$ induced from the Fubini–Study metric of $M_{2n+2}(c)$ and denote by $J$ the almost complex structure of the ambient manifold and by $A$ the shape operator of $M$. Define $\xi = -JN$ and then for any vector field $X \in \mathfrak{x}(M)$ the decomposition holds: $JX = \varphi X + \eta(X)N$. The data $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$ and from the Kählerian nature of $M_{2n+2}(c)$ by making use of the Gauss and Weingarten formulas, we obtain for the Levi-Civita connection $\nabla$ of $g$:

$$
(\nabla_X \varphi)(Y) = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX.
$$

Then the condition (7) means that $E_1$ is an eigenvector of $\varphi \circ A$ with the eigenvalue $-k_1 \eta(E_2)$.

Let us study two cases:

(1) the shape operator has the form $A = \alpha I$. From [12, Lemma 16.1, p. 103] it follows that $\alpha$ is a constant and then $c = 0$. Then, the condition (7) reads:

$$
\alpha \varphi E_1 = -k_1 \eta(E_2) E_1
$$

and if $\alpha = 0$ then $\eta(E_2) = 0$ which gives the consequences of remark (iii). For $\alpha \neq 0$ it results that $-k_1 \eta(E_2)$ is an eigenvalue of $\varphi$. It results again that $\eta(E_2) = 0$ and $\xi|_\gamma = \pm E_1$ which is a contradiction with the Legendre condition.

(2) The shape operator $A$ has two distinct eigenvalues; then these eigenvalues are constant [12, p. 126], and $A = \sigma I + (\rho - \sigma) \xi \otimes \eta$ as [12, (19.17), p. 129], $\rho$ and $\sigma$ being different to zero. From $\varphi(\xi) = 0$ it results the same relation (27) but with $\sigma$ instead $\alpha$.

(viii) Suppose now that the data $(M, g, \xi, \eta, \phi)$ is an almost paracontact metric manifold i.e. (see [25]):

$$
\phi^2 = I - \eta \otimes \xi, \quad \phi(\xi) = 0, \quad g(\phi\cdot, \phi\cdot) = -g + \eta \otimes \eta.
$$
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Let us remark that here the metric $g$ has the signature $(n+1,n)$ and $\xi$ is unitary space-like vector field. From [25, Lemma 2.5, p. 41] we have in the paracontact metric manifolds (conform [25, Definition 2.1, p. 39]):

$$\nabla X\xi = -\phi X + \phi h X$$

and then the $\xi$-Rytov condition means that $E_1$ is an eigenvalue of $\phi - \phi h$ with the eigenvalue $k_1\eta(E_2)$. It results again the impossibility of circles with $E_2 = \pm \xi|_\gamma$.

Proposition 7. Let $\gamma$ be a space-like Frenet curve of order $r = 2$ with $E_2 = \pm \xi|_\gamma$ in a K-paracontact (particularly paraSasakian) manifold $(M,g,\xi,\phi)$. Then $\gamma$ is not a circle on $M$.

Proof. Suppose that $\gamma$ is a circle. From $E_2 = \pm \xi$ and remark (v) it results that $\gamma$ is Legendre and we can apply Theorem 3. The K-paracontact condition means $h = 0$ conform [4, p. 87] and again from hypothesis we have $\eta(E_2) = \pm 1$. With the discussion above we have the eigenvalue $k_1$ for $\phi$. But the nonzero eigenvalues of $\phi$ are $\pm 1$ and we get the curvature: $k_1 = 1$. Returning in (29) with $X = E_1$ it follows:

$$\nabla_{E_1}E_2 = -\phi(E_1) = -E_2 = -\xi$$

and then applying $\phi$ it results: $E_1 = \phi(\xi) = 0$ which is impossible. \qed

(ix) A linear connection naturally associated to the pair $(g,\xi)$ is the Weyl connection [21, p. 1089]:

$$\nabla^\xi_X Y = \nabla_X Y - \frac{1}{2}\eta(X)Y - \frac{1}{2}\eta(Y)X + \frac{1}{2}g(X,Y)\xi.$$  \hspace{1cm} (31)

Then:

$$\nabla^\xi_{E_1}\xi = \nabla_{E_1}\xi - \frac{1}{2}E_1$$

which means that the Rytov condition can be also expressed in terms of $\nabla^\xi$ instead of $\nabla$ with the same argument. In the Legendre case we get from (7):

$$\nabla^\xi_{E_1}\xi = -\left[k_1 \eta(E_2) + \frac{1}{2}\right]E_1.$$  \hspace{1cm} (33)

References


Unitary vector fields are Fermi–Walker transported


