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Weighted Riemannian 1-manifolds for classical orthogonal polynomials and their heat kernel

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Abstract Through the eigenvalue problem we associate to the classical orthogonal polynomials two classes of weighted Riemannian 1-manifolds having the coordinate x. For the first class the eigenvalues contains x and the metric is fixed as being the Euclidean one while for the second class the eigenvalues are independent of this variable and the metric and weight function are founded. The Hermite polynomials is the only case which generates the same manifold. The geometry of second class of weighted manifolds is studied from several points of view: geodesics, distance and exponential map, harmonic functions and their energy density, volume, zeta function, heat kernel. A partial heat equation is studied for these metrics and for the Poincaré ball model of hyperbolic geometry.

Keywords Weighted Riemannian manifold · Classical orthogonal polynomial · Spectrum · Zeta function · Volume · Heat kernel · Sturm–Liouville problem

Mathematics Subject Classification 33C45 · 53C20 · 58C05 · 58C40 · 47J10

1 Introduction

In [1] there are obtained Riemannian metrics in dimension 1 of the type $g(x) = \varphi(x) f(x)$ where f is a special function of Mathematical Physics and φ is an eigenfunction for a problem of equality between g and the Hessian of f with respect

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to a general connection. Although the expression of g for some remarkable examples (Bessel, Hermite, Legendre, Laguerre, Chebyshev) is derived, there are no other informations regarding these metrics.

The aim of this work is to associate classes of weighted Riemannian structures (also known as smooth measure metric spaces or Riemannian measure metric spaces) to the classical orthogonal polynomials. More precisely, we search for the pairs (g, Υ) as considered in [6] such that a classical orthogonal polynomial p_n is an eigenfunction of the weighted Laplacian Δ_μ associated to the initial pair. In fact we search two types of eigenvalue problem: the first have the metric fixed as being the Euclidean one while in the second we search for both (metric g, weight Υ). If follows that in the first class of weighted Riemannian manifolds $(M \subseteq \mathbb{R}, g)$ the eigenvalues Λ_n depend on the coordinate x (i.e. we get a continuous spectrum) while in the second class we derive a discrete spectrum given by Λ_n with $n \in \mathbb{N}$. For these metrics we obtain, using the Pearson differential equation for p_n, that the spectrum is exactly the spectrum of p_n and we study their geometry by computing the Christoffel symbol, the geodesics, the distance function and the exponential map (and then the injectivity radius), the harmonic functions (in fact, totally geodesic functions) with their (constant) energy density. As it is usual in spectral geometry, we associate also their zeta function and compute it for some examples, in terms of Riemannian zeta function.

There exists only one common case, namely the Hermite polynomials which are associated to the usual Euclidean 1-geometry. In fact, this Example appears completely solved in [6] and this fact was the starting point of this study. We add to our metrics other two considered by Nomizu and Sasaki [11] for which we obtain again the geodesics, the distance function and the exponential map, the harmonic functions and the volume. For all five metrics we compute the (weighted) volume, in some cases by using the Gamma function of Euler. The distance associated to all five Riemannian metrics is of Mazur–Ulam type according to [9, p. 166] and also of Busemann type.

Having the eigenvalues as well as the eigenfunctions we express the heat kernel of these weighted Riemannian manifolds and their compact forms is based on historical formulae of Mehler and Hille–Hardy from [12]; let us note that in [6] appears only the case of Hermite polynomials and that heat kernels expressed as an integral of a product between a volume function and an exponential term are given in [3]. Due to the analogy of methods we study a particular case of the Sturm–Liouville problem which is expressed in terms of weighted Riemannian geometry. As example, the Dirichlet problem is discussed in the Euclidean setting and its heat kernel is expressed in a form suitable for computation in terms of θ_3 function.

We add to our study a result of John Roe from [14] regarding a partial heat equation on manifolds containing the Euclidean origin of $\mathbb{R}^m$. More precisely, Roe's Lemma is considered for Jacobi metric, for the Nomizu–Sasaki metrics and for the Poincaré ball model of hyperbolic geometry. On this way, we determine a function a_1 which is a "measure of how far away" is the metric g from being Euclidean from the heat equation point of view. The picture of this function a_1 is plotted with Matlab for these four metrics.

2 Weighted Riemannian manifolds and their Laplacians

A very interesting geometrical framework was introduced in [6, p. 67]:

Definition 2.1 A *weighted manifold* is a triple $(M^m, g =<,>, \Upsilon)$ with (M, g) a m-dimensional Riemannian manifold and $\Upsilon \in C^\infty_+(M)$ i.e. Υ is a smooth and strictly positive function on M. Then, we consider the new measure on M: $d\mu = \Upsilon dv_g$ where dv_g is the usual Riemannian measure associated to the metric g.

On any weighted manifold there exists an induced divergence of vector fields $X \in \mathcal{X}(M)$:

$$div_\mu X = \frac{1}{\Upsilon} div(\Upsilon X) \tag{2.1}$$

and the corresponding *the weighted Laplacian* ([6, p. 68]):

$$\Delta_\mu = div_\mu \circ \nabla \tag{2.2}$$

where, as usual, ∇ is the gradient operator $\nabla : C^\infty(M) \to \mathcal{X}(M)$.

In the following we restrict to $m = 1$ and denote by x the (local) coordinate on M; thus $g \in C^\infty_+(M)$ and the expression $g(x)dx^2$ of the metric can be thought of as a quadratic differential. If $\Delta = div \circ \nabla$ is the usual Laplacian associated to the metric g then it results the following relationship between these Laplace operators:

$$\Delta_\mu = \Delta + \frac{\Upsilon'}{\Upsilon g}. \tag{2.3}$$

Denote by "*can*" the Euclidean 1-dimensional metric: $g(x) = 1$ for any $x \in M$. The formula (3.46) of [6, p.68] for the one-dimensional weighted Laplacian on $f \in C^\infty(M, can)$ is:

$$\Delta_\mu f = f'' + \frac{\Upsilon'}{\Upsilon} f' \tag{2.4}$$

where $'$ means $\frac{d}{dx}$. Remark that the expression of the unique Christoffel symbol of the general (M, g) is:

$$\Gamma_g = \frac{g'}{2g}. \tag{2.5}$$

At the end of this section we note that our framework appears several times with another name, *smooth* (or *Riemannian*) *measure metric space*; from a large list of references we select as example the paper [8]. Also, the weighted Laplacian is sometimes called *Witten* Laplacian and in the general dimension m has the expression:

$$\Delta_\mu f = \Delta f + g(\nabla \ln \Upsilon, \nabla f) \tag{2.6}$$

where ∇ is the gradient operator with respect to the metric g and Δ the Laplacian of g.

3 Classical orthogonal polynomials as (x-dependent) eigenfunctions of weighted Euclidean Laplacians

Fix now the real interval $M = (a, b)$ (hence x is a global coordinate) and the function $\sigma \in C^{\infty}_{+}(M)$:

$$\sigma(x) = \begin{cases} (x-a)(b-x), & a, b \in \mathbb{R} \\ x-a, & a \in \mathbb{R}, b = +\infty \\ b-x, & a = -\infty, b \in \mathbb{R} \\ 1, & a = -\infty, b = +\infty. \end{cases} \tag{3.1}$$

Recall after [10, p.39] that *the classical orthogonal polynomials with weight* $\rho \in C^{\infty}_{+}(M)$ are the orthogonal polynomials $\{p_n; n \in \mathbb{N}\}$ with weight ρ satisfying *the Pearson differential equation*:

$$[\sigma\rho]' = \tau\rho \tag{3.2}$$

where τ is a polynomial of first order.

The aim of this Section is to obtain the classical orthogonal polynomials as eigenfunctions of the Euclidean weighted Laplacian (1.4). In [10, p.46] it is proved that the classical orthogonal polynomials satisfy:

$$[\sigma\rho p'_n]' = \lambda_n \rho p_n \tag{3.3}$$

with the eigenvalues:

$$\lambda_n = n\left[\tau' + \frac{n-1}{2}\sigma''\right]. \tag{3.4}$$

Comparing 2.3 with the equation of eigenfunctions for Δ_μ:

$$\Delta_\mu f = \frac{(\Upsilon f')}{\Upsilon} = \Lambda_n f \tag{3.5}$$

we get the claimed relationship for:

$$\Upsilon(x) = \rho(x)\sigma(x), \quad \Lambda_n = \Lambda(x) = \frac{\lambda_n}{\sigma(x)}. \tag{3.6}$$

Example 3.1 (i) *Hermite polynomials*: $M = \mathbb{R}$, $\rho(x) = e^{-x^2}$, $\sigma(x) = 1$, $\tau(x) = -2x$. With notation $p_n = h_n$ we reobtain the Exercise 3.10 of [6, p.69]:

$$\Upsilon(x) = e^{-x^2}, \quad \Lambda_n = -2n. \tag{3.7}$$

(ii) *Laguerre polynomials* with parameter $\alpha > -1$: $M = (0, +\infty)$, $\rho(x) = e^{-x}x^{\alpha}$, $\sigma(x) = x$, $\tau(x) = -x + \alpha + 1$. It results that $p_n = L_n^{\alpha}$ are eigenfunctions for Δ_μ with:

$$\Upsilon(x) = e^{-x}x^{1+\alpha}, \quad \Lambda_n = \Lambda_n(x) = -\frac{n}{x}. \tag{3.8}$$

(iii) *Jacobi polynomials* with parameters $\alpha > -1$, $\beta > -1$: $M = (-1, 1)$, $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\sigma(x) = 1 - x^2$, $\tau(x) = -(\alpha+\beta+2)x + \beta - \alpha$. It results that $p_n = P_n^{(\alpha,\beta)}$ are eigenfunctions of Δ_μ with:

$$\Upsilon(x) = (1-x)^{1+\alpha}(1+x)^{1+\beta}, \quad \Lambda_n = \Lambda_n(x) = -\frac{n(n+\alpha+\beta+1)}{1-x^2}. \tag{3.9}$$

Particular cases 2.2:

(iii1) *Legendre polynomials* P_n correspond to $\alpha = \beta = 0$. We get:

$$\Upsilon(x) = 1 - x^2, \quad \Lambda_n(x) = -\frac{n(n+1)}{1-x^2}. \tag{3.10}$$

(iii2) *Chebyshev polynomials of first type* T_n correspond to $\alpha = \beta = -\frac{1}{2}$: We have:

$$\Upsilon(x) = \sqrt{1-x^2}, \quad \Lambda_n(x) = -\frac{n^2}{1-x^2}. \tag{3.11}$$

(iii3) *Chebyshev polynomials of second type* U_n corresponds to $\alpha = \beta = \frac{1}{2}$. We have:

$$\Upsilon(x) = (1-x^2)^{\frac{3}{2}}, \quad \Lambda_n(x) = -\frac{n(n+2)}{1-x^2}. \tag{3.12}$$

(iii4) *Gegenbauer* (or *ultraspherical*) *polynomials* C_n^{λ} correspond to $\alpha = \beta = \lambda - \frac{1}{2}$. We get:

$$\Upsilon(x) = (1-x^2)^{\lambda-\frac{1}{2}}, \quad \Lambda_n(x) = -\frac{n(n+2\lambda)}{1-x^2}. \tag{3.13}$$

4 Weighted Riemannian manifolds associated to classical orthogonal manifolds

In this Section we study again the eigenvalue problem but searching now for x-independent eigenvalues and arbitrary metric g on M. A straightforward computation yields the general expression of the weighted Laplacian:

$$\Delta_\mu f = \frac{1}{g}f'' + \left(\frac{\Upsilon'}{\Upsilon g} - \frac{g'}{2g^2}\right)f'. \tag{4.1}$$

The differential equation 2.3 becomes:

$$\sigma p_n'' + \tau p_n' = \lambda_n p_n \tag{4.2}$$

and comparing with the eigenvalue problem:

$$\Delta_\mu f = -\Lambda_n^g f \tag{4.3}$$

we get the general answer:

$$g(x) = \frac{1}{\sigma(x)}, \quad \Lambda_n^g = -\lambda_n, \quad \Upsilon(x) = \rho(x)\sqrt{\sigma(x)}. \tag{4.4}$$

Let us remark that we choose now the minus sign in the right hand side in order to obtain a positive spectrum like in the usual spectral geometry.

These relations yield:

Definition 4.1 (i) *The weighted Hermite manifold* is: $(\mathbb{R}, can, \Upsilon(x) = e^{-x^2})$; the associated spectrum is $S(M, g, \Upsilon) = 2\mathbb{N}$.

(ii) *The weighted α-Laguerre manifold* is: $((0, +\infty), g(x) = \frac{1}{x}, \Upsilon_\alpha(x) = e^{-x}x^{\alpha+\frac{1}{2}})$ with the spectrum $S(M, g, \Upsilon) = \mathbb{N}$.

(iii) *The weighted (α, β)-Jacobi manifold* is: $((-1, 1), g(x) = \frac{1}{1-x^2}, \Upsilon_{\alpha,\beta}(x) = (1-x)^{\alpha+\frac{1}{2}}(1+x)^{\beta+\frac{1}{2}})$ with the spectrum $S(M, g, \Upsilon) = \{n(n+\alpha+\beta+1); n \in \mathbb{N}\}$.

Recall after [6, p.281] that the spectrum of the sphere S^k is $\{n(n+k-1); \quad n \in \mathbb{N}\}$. It follows that for $\alpha + \beta = k - 2$ we have:

$$S(M, g, \Upsilon_{\alpha,\beta}) = S(S^k) \tag{4.5}$$

and some remarkable Jacobi manifolds are as follows:

Example 4.2 (iii1) *weighted Legendre manifold* corresponds to:

$$\Upsilon(x) = \sqrt{1-x^2}, \quad S(M, g, \Upsilon) = \{n(n+1); \quad n \in \mathbb{N}\} = S(S^2). \tag{4.6}$$

(iii2) *weighted Chebyshev manifold of first type* corresponds to:

$$\Upsilon(x) = 1, \quad S(M, g, \Upsilon) = \{n^2; \quad n \in \mathbb{N}\} = S(S^1). \tag{4.7}$$

(iii3) *weighted Chebyshev manifold of second order* corresponds to:

$$\Upsilon(x) = 1 - x^2, \quad S(M, g, \Upsilon) = \{n(n+2); \quad n \in \mathbb{N}\} = S(S^3). \tag{4.8}$$

(iii4) *Gegenbauer* (or *ultraspherical*) *manifold* corresponds to:

$$\Upsilon(x) = (1-x^2)^\lambda, \quad S(M, g, \Upsilon_\lambda) = \{n(n+2\lambda); \quad n \in \mathbb{N}\}. \tag{4.9}$$

For $\lambda = \frac{k-1}{2}$ we have $S(M, g, \Upsilon_\Lambda) = S(S^k)$; in particular for $k = 0$, i.e. $\lambda = -\frac{1}{2}$, the measure $d\mu = \frac{dx}{1-x^2}$ is half of the invariant measure μ_A associated to the Artin A-code in [7, p. 120].

Remark 4.3 (i) It is known after [15] that for a compact (M, g), if $S(M, g) = S(S^k)$ then (M, g) is isometric to S^k up to $k \leq 6$. Our manifolds above are not compact and we work in the weighted framework which is different to the usual Riemannian geometry.

(ii) The measure $d\mu = e^{-x^2}dx$ of the weighted Hermite manifold is exactly the Example (d) of [16, p. 387] of manifolds satisfying the $CD(K = 2, +\infty)$ curvature-dimension condition. With the approach of Example 21.3 of [16, p. 547] it results that the probability measure $\frac{1}{\sqrt{\pi}}d\mu$ satisfies the logarithmic Sobolev inequality with constant $K = 2$.

Now we can compute the associated *zeta function*:

$$Z(M, g, \Upsilon)(s) = \sum_{n \geq 1} \Lambda_n^{-s} \tag{4.10}$$

defined for a complex number s with the real part Res sufficiently great. By recalling the classical Riemann zeta function:

$$\zeta(s) = \sum_{n \geq 1} n^{-s} \tag{4.11}$$

we get the following:

Example 4.4 (a) $Z(Hermite)(s) = \frac{1}{2^s}\zeta(s)$; (b) $Z(Laguerre) = \zeta$ for $Res > 1$; (c) $Z(Chebyshev_1)(s) = \zeta(2s)$ for $Res > \frac{1}{2}$. The zeta functions for higher-dimensional spheres are computed in [4].

The rest of this Section is devoted to a study of these metrics. The Christoffel symbol of these manifolds is: (i) $\Gamma_{can} = 0$, (ii) $\Gamma_{Laguerre} = -\frac{1}{2x}$, (iii) $\Gamma_{Jacobi} = \frac{x}{1-x^2}$. The geodesics of these metrics are as follows: (i) $x(t) = ct + d$, ii) $x(t) = (ct + d)^2$, (iii) $x(t) = \sin(ct + d)$. Here, c and d are arbitrary real constants. Since these geodesics are defined on the whole $\mathbb{R}$ it follows that all these metrics are complete. Examples of incomplete metrics and linear connections in dimension 1 are included in [12]. More precisely, in this paper in addition to the usual Euclidean metric other two remarkable Riemannian metrics are provided on the real line by requiring that their Levi-Civita connections to be globally defined ([12, p. 209]):

(iv) $g(x) = e^{2x}$ with the corresponding flat coordinate system confined to $(-1, +\infty)$ and Christoffel symbol $\Gamma(x) = 1$. It follows the geodesics $x(t) = \ln(ct+d)$ for $t \in (-c, +\infty)$.

(v) $g(x) = \frac{1}{(1+x^2)^2}$ with the corresponding flat coordinate system confined to $(-\frac{\pi}{2}, \frac{\pi}{2})$ and Christoffel symbol $\Gamma(x) = -\frac{2x}{1+x^2}$. We obtain the geodesics $x(t) = \tan(ct + d)$.

By using the expression of geodesic we can get the distance function and the exponential map on these manifolds:

(ii) (Laguerre) for the points $x_0, x_1 \in M$ the unique geodesic joining x_0 to x_1 is $\gamma : [0, 1] \to M$ given by $\gamma(t) = (x_1 - x_0)\sqrt{t} + x_0$ and then:

$$d^{Laguerre}(x_0, x_1) = L(\gamma) = \int_0^1 \sqrt{g(\gamma(t))}|\gamma'(t)|dt$$
$$= \int_0^1 \frac{|\gamma'(t)|}{\sqrt{\gamma(t)}}dt = 2|\sqrt{x_1} - \sqrt{x_0}|. \tag{4.12}$$

Let $x_{p,v}(\cdot)$ be the geodesic determined by $x(0) = p$ and $x'(0) = v$. Since in the Laguerre case, we have: $x_{p,v}(t) = \left(\frac{v}{2\sqrt{p}}t + \sqrt{p}\right)^2$ it results the exponential map $\exp_p(v) = \frac{1}{4p}(v + 2p)^2$ and thus the injectivity radius is $inj(p) = 2p$.

(iii) (Jacobi) for the points $x_0, x_1 \in M$ the unique geodesic joining x_0 to x_1 is $\gamma : [0, 1] \to M$ given by $\gamma(t) = \sin[(\arcsin x_1 - \arcsin x_0)t + \arcsin x_0]$ and then:

$$d^{Jacobi}(x_0, x_1) = L(\gamma) = \int_0^1 \sqrt{g(\gamma(t))}|\gamma'(t)|dt$$
$$= \int_0^1 \frac{|\gamma'(t)|}{\sqrt{1 - \gamma^2(t)}}dt = |\arcsin x_1 - \arcsin x_0|. \tag{4.13}$$

Therefore, the diameter of the Jacobi manifold is $diam(Jacobi) = |\frac{\pi}{2} - \frac{-\pi}{2}| = \pi$. Since $x_{p,v}(t) = \sin\left(v\sqrt{1 - \arcsin^2 p}t + \arcsin p\right)$ it follows $\exp_p(v) = \sin\left(v\sqrt{1 - \arcsin^2 p} + \arcsin p\right)$ and the injectivity radius $inj(p) = \frac{2\pi - \arcsin p}{\sqrt{1 - arcsin^2 p}}$.

(iv) (Nomizu–Sasaki first metric) for the points $x_0, x_1 \in M$ the unique geodesic joining x_0 to x_1 is $\gamma : [0, 1] \to M$ given by $\gamma(t) = \ln[(e^{x_1} - e^{x_0})t + e^{x_0}]$ and then:

$$d^{NS1}(x_0, x_1) = L(\gamma) = \int_0^1 \sqrt{g(\gamma(t))}|\gamma'(t)|dt$$
$$= \int_0^1 |\gamma'(t)|e^{\gamma(t)}dt = |e^{x_1} - e^{x_0}|. \tag{4.14}$$

Since $x_{p,v}(t) = p + \ln(vt + 1)$ we obtain $\exp_p(v) = p + \ln(v + 1)$ and the injectivity radius $inj(p) = 1$ for any point $p \in M = \mathbb{R}$.

(v) (Nomizu–Sasaki second metric) for the points $x_0, x_1 \in M$ the unique geodesic joining x_0 to x_1 is $\gamma : [0, 1] \to M$ given by $\gamma(t) = \tan[(\arctan x_1 - \arctan x_0)t + \arctan x_0]$ and then:

$$d^{NS2}(x_0, x_1) = L(\gamma) = \int_0^1 \sqrt{g(\gamma(t))}|\gamma'(t)|dt$$
$$= \int_0^1 \frac{|\gamma'(t)|}{1 + \gamma^2(t)}dt = |\arctan x_1 - \arctan x_0|. \tag{4.15}$$

Again, the diameter is $diam(NS2) = |\frac{\pi}{2} - \frac{-\pi}{2}| = \pi$. Since $x_{p,v}(t) = \tan(v(1 + \arctan^2 p)t + \arctan p)$ we have $\exp_p(v) = \tan(v(1 + \arctan^2 p) + \arctan p)$ and the injectivity radius is $+\infty$.

Let $I \subseteq \mathbb{R}$ be a real interval and $f : I \to \mathbb{R}$. The function $d_f : I \times I \to \mathbb{R}_+$, $d_f(x, y) = |f(x) - f(y)|$ is a distance on I if and only if f is injective. Let us remark that all above functions f are injective on their corresponding M. The group of isometries for d_f is:

$$Isom(I, d_f) = \{F_c^{\pm}(x) = f^{-1}(c \pm f(x)); c \in (-\varepsilon(f), \varepsilon(f))\} \tag{4.16}$$

where $\varepsilon(f)$ is determined by requiring f to be bijective. The group $Isom(I, d_f)$ is isomorphic with the additive group $((-\varepsilon(f), \varepsilon(f)), +)$. These functions are: (i) $f_{Hermite}(x) = x$, (ii) $f_{Laguerre}(x) = 2\sqrt{x}$, (iii) $f_{Jacobi}(x) = \arcsin x$, (iv) $f_{NS1}(x) = e^x$, (v) $f_{NS2}(x) = \arctan x$. The open ball of radius r in these metric spaces are $B_f(x, r) = f^{-1}(f(x) - r, f(x) + r)$ and its volume is then:

$$V_\mu(B_f(x, r)) = \int_{f^{-1}(f(x)-r)}^{f^{-1}(f(x)+r)} \Upsilon(x)dx. \tag{4.17}$$

With the notations and terminology of [9, p. 166] we have that these metric spaces are *Mazur–Ulam spaces* with the midpoint between x and y given by: $x\sharp y = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$. They are also *Busemann convex metric spaces* ([5, p. 17]) since we have equality in the Busemann condition i.e. $d_f(x\sharp z, y\sharp z) = \frac{1}{2}d_f(x, y)$ for all $x, y, z \in M$.

In the following we determine the harmonic functions of these metrics. From the expression of the usual Laplacian:

$$\Delta f = \frac{1}{g}\left(f'' - \frac{g'}{2g}f'\right) = \frac{f'}{g}\left(\frac{f''}{f'} - \frac{g'}{2g}\right) \tag{4.18}$$

we get that a non-constant f is harmonic with respect to the metric g has the expression:

$$f_c(x) = c\int_{x_0}^{x} \sqrt{g(t)}dt \tag{4.19}$$

where c is a real constant and x_0 is a fixed point of M.

Example 4.5 (i) *weighted Hermite geometry* = Euclidean geometry on $\mathbb{R}$: $f_c(x) = c(x - x_0)$. We reobtain that the linear functions $f_{c,d}(x) = cx + d$ ($d \in \mathbb{R}$) are Euclidean harmonic.

(ii) *weighted Laguerre geometry*: $f_c(x) = 2c(\sqrt{x} - \sqrt{x_0})$. The general expression of harmonic functions is $f_{c,d}(x) = 2c\sqrt{x} + d$.

(iii) *weighted Jacobi geometry*: $f_c(x) = c(\arcsin x - \arcsin x_0)$. It follows the general expression of harmonic functions $f_{c,d}(x) = c \arcsin x + d$.

(iv) *Nomizu–Sasaki first metric*: $f_c(x) = c(e^x - e^{x_0})$. The general expression is then $f_{c,d}(x) = ce^x + d$.
(v) *Nomizu–Sasaki second metric*: $f_c(x) = c(\arctan x - \arctan x_0)$. We have the general expression of harmonic functions $f_{c,d}(x) = c \arctan x + d$.

In fact, since we work in dimension 1 it follows that a harmonic map is a totally geodesic one. Recall that the energy density of the function $f : (M, g) \to \mathbb{R}$ is the function:

$$e(f) = \frac{1}{2}|df|_g^2 = \frac{1}{2}g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \tag{4.20}$$

and the characterization of harmonic functions as critical points of the integrals of this density. Thus we are interested in the value of the energy density for these totally geodesic functions:

$$e(f)(x) = \frac{\left(f'(x)\right)^2}{2g(x)} \tag{4.21}$$

and then for all five metrics we obtain the constant energy density $e(f_{c,d}) = \frac{c^2}{2}$.

We finish this Section with the weighted volume of these metrics:

$$V(M, g, \mu) = \int_M d\mu = \int_M \Upsilon(x)\sqrt{g(x)}dx = \int_M \rho(x)dx. \tag{4.22}$$

In [10, p.129] is used the notation:

$$d_n^2 = \int_M p_n^2(x)\rho(x)dx \tag{4.23}$$

and hence the following results are derived ([10, p.51]):

(i) (Hermite) $d_n^2 = 2^n n! \sqrt{\pi}$,
(ii) (Laguerre) $d_n^2 = \frac{\Gamma(n+\alpha+1)}{n!}$,
(iii) (Jacobi) $d_n^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$.

Here, Γ is the Euler Gamma function. We get:

$$V(M, g, \mu) = d_0^2 \tag{4.24}$$

and then:

(i) $V(Hermite) = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.
(ii) $V(Laguerre) = \int_0^{+\infty} e^{-x} x^\alpha = \Gamma(\alpha + 1) = \Pi(\alpha)$ with $\Pi(\cdot)$ the Pi function.
(iii) $V(Jacobi) = \int_{-1}^{1}(1 - x)^\alpha(1 + x)^\beta dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$. In particular:

$$V(Jacobi \quad \alpha = \beta) = 2^{2\alpha+1}\frac{(\Gamma(\alpha + 1))^2}{\Gamma(2\alpha + 1)} \tag{4.25}$$

and then:

(iii1) $V(Legendre) = 2$, (iii2) $V(Chebyshev1) = \pi$; hence the Chebyshev1 manifold can be though as a half-circle of S^1, (iii3) $V(Chebyshev2) = \frac{\pi}{2}$.

(iv) $V(Nomizu - Sasaki1) = \int_{-\infty}^{\infty} e^x dx = +\infty$.

(v) $V(Nomizu - Sasaki2) = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

5 Heat kernel for classical orthogonal polynomials and reduced Sturm–Liouville problems

After Exercise 10.14 of [6] the heat kernel of a weighted manifold (M, g, Υ) is:

$$p_t(x, y) = \sum_{n \geq 0} e^{-\Lambda_n t} \varphi_n(x) \varphi_n(y) \tag{5.1}$$

where $\{\varphi_n\}$ is an orthonormal basis in $L^2(M, g, d\mu)$ that consists of the eigenfunctions. By using the expression of d_n^2 form the previous section we obtain the following heat kernels:

Proposition 5.1 *The heat kernel of the weighted manifolds associated to classical orthogonal polynomials are:*

(i) *Hermite* [6, p. 284], $(Mehler formula, 1866)$:

$$\begin{aligned} p_t(x, y) &= \frac{1}{\sqrt{\pi}} \sum_{n \geq 0} e^{-2nt} \frac{h_n(x) h_n(y)}{2^n n!} \\ &= \frac{1}{\sqrt{2\pi \sinh(2t)}} \exp\left(\frac{2xye^{2t} - (x^2 + y^2)}{e^{4t} - 1} + t \right). \end{aligned} \tag{5.2}$$

(ii) *Laguerre, (Hille, 1926 and Hardy, 1932):*

$$\begin{aligned} p_t^{\alpha}(x, y) &= \sum_{n \geq 0} e^{-nt} \frac{n! L_n^{\alpha}(x) L_n^{\alpha}(y)}{\Gamma(n + \alpha + 1)} \\ &= \frac{e^t}{e^t - 1} \exp\left(\frac{x + y}{1 - e^t} \right) \left(\frac{xy}{e^t} \right)^{-\frac{\alpha}{2}} I_{\alpha}\left(\frac{2\sqrt{xye^t}}{e^t - 1} \right). \end{aligned} \tag{5.3}$$

(iii) *Jacobi:*

$$\begin{aligned} p_t^{\alpha,\beta}(x, y) = \frac{1}{2^{\alpha+\beta+1}} \sum_{n \geq 0} e^{-n(n+\alpha+\beta+1)t} \\ \frac{n!(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1) P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y)}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}. \end{aligned} \tag{5.4}$$

Here, I_{α} is the modified Bessel function of the first kind, conform [12]*. This paper discuss also the Jacobi heat kernel as well as the historical aspects included above.*

After a long but tedious computation we get several examples:

Particular cases 5.2 (ii1) Laguerre-0:

$$p_t^0(x,y)=\sum_{n\geq 0}e^{-nt}L_n^0(x)L_n^0(y)=\sum_{n\geq 0}\frac{e^{x+y-nt}}{(n!)^2}\frac{d^n}{dx^n}(e^{-x})\frac{d^n}{dy^n}(e^{-y})$$
$$=\frac{e^t}{e^t-1}\exp\left(\frac{x+y}{1-e^t}\right)I_0\left(\frac{2\sqrt{xye^t}}{e^t-1}\right). \tag{5.5}$$

(iii1) Legendre:

$$2p_t(x,y)=\sum_{n\geq 0}e^{-n(n+1)t}(2n+1)P_n^{(0,0)}(x)P_n^{(0,0)}(y)$$
$$=\sum_{n\geq 0}e^{-n(n+1)t}\frac{2n+1}{(2^n n!)^2}\frac{d^n}{dx^n}(1-x^2)^n\frac{d^n}{dy^n}(1-y^2)^n. \tag{5.6}$$

(iii2) Chebyshev of first type:

$$p_t(x,y)=\frac{2}{\pi}\sum_{n\geq 0}e^{-n^2t}\left[\frac{(n!)^2}{(2n)!}\right]^2 2^{4n}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)P_n^{(-\frac{1}{2},-\frac{1}{2})}(y). \tag{5.7}$$

Let us remark that the heat kernel of the unit circle is expressed in [2, p.49,100] by means of theta-function θ_3.

(iii3) Chebyshev of second type:

$$p_t(x,y)=\frac{2}{\pi}\sum_{n\geq 0}e^{-n(n+2)t}\left[\frac{n!(n+1)!}{(2n+1)!}\right]^2 2^{4n}P_n^{(\frac{1}{2},\frac{1}{2})}(x)P_n^{(\frac{1}{2},\frac{1}{2})}(y). \tag{5.8}$$

We study now a particular case of the Sturm–Liouville problem. Recall that the usual Sturm–Liouville operator $LS: C^2(M)\to C^0(M)$ has the expression:

$$LS(f)=-\frac{d}{dx}\left(p(x)\frac{df}{dx}\right)+q(x)f \tag{5.9}$$

with the hypothesis $p, p'\in C(M)$, $p(x)\geq const>0$ and $q(x)\geq 0$ for all $x\in M$. Let us call it *reduced* and denotes LS_r if $q=0$.

The reduced Sturm–Liouville problem LS_r: $-pf''-p'f'=\lambda f$ is an eigenvalue problem for the weighted 1-manifold (M,g,Υ) where:

$$g(x)=\frac{1}{p(x)},\quad \Upsilon(x)=\frac{1}{\sqrt{p(x)}}. \tag{5.10}$$

Example 5.3 Suppose that $a, b \in \mathbb{R}$ and $p = 1$ i.e. we work in the Euclidean setting. With the Dirichlet condition $f(a) = f(b) = 0$ it is well-known that the eigenvalues are indexed by $n \geq 1$:

$$\lambda_n = \frac{n^2\pi^2}{(b-a)^2} \tag{5.11}$$

and the orthonormal system of eigenfunctions is:

$$u_n(x) = \sqrt{\frac{2}{b-a}} \sin \frac{n\pi(x-a)}{b-a}. \tag{5.12}$$

For example, if M is the Jacobi (closed) interval $[-1, 1]$ we get the heat kernel:

$$p_t^{Dirichlet}(x, y) = \sum_{n\geq 1} e^{-\frac{n^2\pi^2 t}{4}} \sin \frac{n\pi(x+1)}{2} \sin \frac{n\pi(y+1)}{2}. \tag{5.13}$$

Another formula is:

$$2p_t^{Dirichlet}(x, y) = \sum_{n\geq 1} e^{-\frac{n^2\pi^2 t}{4}} \cos \frac{n\pi(y-x)}{2} - \sum_{n\geq 1} e^{-\frac{n^2\pi^2 t}{4}} \cos \frac{n\pi(y+x)}{2} \tag{5.14}$$

and a suitable expression in terms of θ_3 function can be obtained following the approach of [2, p.101].

Let us finish this paper with an approximate solution of the heat equation given by the following:

Lemma 5.4 ([14, p.71]) *Let $g_{ij}(x)$ be a Riemannian metric on a neighbourhood of the origin in $\mathbb{R}^m$, and define a function $f(x, t)$ by:*

$$f(x, t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} exp\left(-\frac{d^2(0, x)}{4t}\right). \tag{5.15}$$

Then:

$$f_t - g^{ij} f_{ij} = \left(\frac{a_1(x)}{t} + a_2(x)\right) f \tag{5.16}$$

where a_1 and a_2 are smooth functions of x and $a_1(0) = 0$.

Recall that $d(0, x)$ is, after the notation of the previous section, the distance between $0 \in M$ and $x \in M$. In dimension $m = 1$ we get from:

$$g(x) = d_x^2(0, x) \tag{5.17}$$

the following expression of these functions:

$$a_1(x) = \frac{d(0,x)d_{xx}(0,x)}{2g(x)}, \quad a_2 = 0 \tag{5.18}$$

which yields the vanishing of a_1 in the origin. By using the expression of $d(0, x)$ from section 3 we have:

Example 5.5 (i) Euclidean = Hermite: $a_1 = 0$. The same result $a_1 = a_2 = 0$ holds in the general Euclidean $\mathbb{R}^n$, $n \geq 1$ which means that $f(x, t)$ given by (5.15) is solution of the heat equation. Therefore, the function a_1 is a "measure of how far away" is the metric g from being Euclidean from the heat equation point of view.

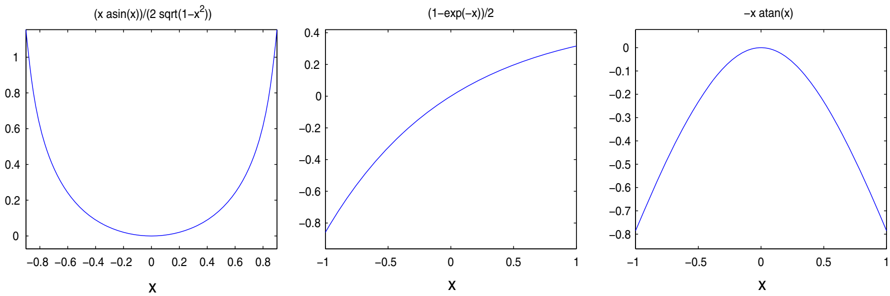
(ii) Jacobi: $a_1^{Jacobi}(x) = \frac{x \arcsin x}{2\sqrt{1-x^2}}$. It is an even function with vertical asymptotics at $x = \pm 1$.

(iii) Nomizu–Sasaki 1: $a_1^{NS1}(x) = \frac{1-e^{-x}}{2}$. Let us remark that the function a_1^{NS1} appears already in literature. More precisely, in order to reparametrize curves in a Finsler (particular Riemannian) manifold, in [13, p. 1646] is introduced the function:

$$\tau_\sigma(x) = \begin{cases} \frac{2}{\sigma}(1 - e^{-\sigma x/2}), & \sigma \neq 0 \\ x, & \sigma = 0 \end{cases} \tag{5.19}$$

Hence $a_1^{NS1} = \frac{1}{2}\tau_2$.

(iv) Nomizu–Sasaki 2: $a_1^{NS2}(x) = -x \arctan x$. Again is an even function.



The function a_1^{Jacobi} (left); the function a_1^{NS1} (center); The function a_1^{NS2} (right).

For a general dimension $m \geq 2$ we determine the functions a_1, a_2 for the Poincaré ball model of the hyperbolic geometry. Recall that the m-dimensional hyperbolic space $H^m = B_1^m = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m; r(x) = \sqrt{(x^1)^2 + \cdots + (x^m)^2} < 1\}$ has the metric $g = (g_{ij})$:

$$g_{ij}(x) = \frac{4\delta_{ij}}{(1 - r^2)^2} \tag{5.20}$$

and the geodesic from the origin O to x is the line: $\gamma_x : [0,1] \to H^m$, $\gamma_x(t) = tx$. It follows:

$$d(O, x) = \int_0^1 \|\gamma_x'(t)\|_g dt = \ln\left(\frac{1+r}{1-r}\right). \tag{5.21}$$

Hence the function (4.14) is:

$$f(t, x) = \frac{1}{(4\pi t)^{\frac{m}{2}}} \exp\left(\frac{-1}{4t} \ln^2\left(\frac{1+r}{1-r}\right)\right) \tag{5.22}$$

and a long computation yields:

$$a_2 = 0, \quad a_1^{hyperbolic}(r) = \frac{1-m}{2} + \frac{(m-1) + (3-m)r^2}{4r} \ln\left(\frac{1+r}{1-r}\right). \tag{5.23}$$

Since:

$$\lim_{r\to 0} \frac{1}{r} \ln\left(\frac{1+r}{1-r}\right) = 2. \tag{5.24}$$

we have that $\lim_{r\to 0} a_1^{hyperbolic}(r) = 0$ as is required.

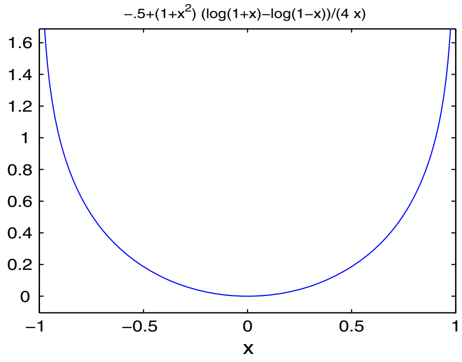
Example 5.6 (The hyperbolic plane ball) For $m = 2$ we have in usual notation $(x^1, x^2) = (x, y)$:

$$a_1^{hyperbolic}(x, y) = -\frac{1}{2} + \frac{1 + x^2 + y^2}{4\sqrt{x^2+y^2}} \ln\left(\frac{1+\sqrt{x^2+y^2}}{1-\sqrt{x^2+y^2}}\right) \tag{5.25}$$

while as function of r we have:

$$a_1^{hyperbolic}(r) = -\frac{1}{2} + \frac{1+r^2}{4r} \ln\left(\frac{1+r}{1-r}\right) \tag{5.26}$$

with the following picture which shows the even property $a_1^{hyperbolic}(-r) = a_1^{hyperbolic}(r)$:



Returning to the general case 5.22 we have with the Taylor expansion of $\ln(1 \pm r)$ that:

$$a_1^{hyperbolic}(r) = \frac{r^2}{2}\left(\frac{m-1}{3} + 3 - m + o(r^2)\right) \tag{5.27}$$

and hence:

$$\lim_{r \to 0} \frac{a_1^{hyperbolic}(r)}{r^2} = \frac{4-m}{3} \tag{5.28}$$

which is: $2/3$ for $m = 2$, $1/3$ for $m = 3$, 0 for $m = 4$ and strictly negative for $m \geq 5$.

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