Invariant distributions and holomorphic vector fields in paracontact geometry

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Abstract: Having as a model the metric contact case of V. Brînzănescu; R. Slobodeanu, we study two similar subjects in the paraccontact (metric) geometry: a) distributions that are invariant with respect to the structure endomorphism \( \varphi \); b) the class of vector fields of holomorphic type. As examples we consider both the 3-dimensional case and the general dimensional case through a Heisenberg-type structure inspired also by contact geometry.

Key words: Paracontact metric manifold, invariant distribution, paracontact-holomorphic vector field

1. Introduction
Paracontact geometry [7, 13] appears as a natural counterpart of the contact geometry in [9]. Compared with the huge literature in (metric) contact geometry, it seems that new studies are necessary in almost paracontact geometry; a very interesting paper connecting these fields is [5]. The present work is another step in this direction, more precisely from the point of view of some subjects of [4].

The first section deals with the distributions \( \mathcal{V} \), which are invariant with respect to the structure endomorphism \( \varphi \), one trivial example being the canonical distribution \( \mathcal{D} \) provided by the annihilator of the paraccontact 1-form \( \eta \). As in the contact case, the characteristic vector field \( \xi \) must belong to \( \mathcal{V} \) or \( \mathcal{V}^\perp \). Two important tools in this study are the second fundamental form and the integrability tensor field, both satisfying important (skew)-commutation formulas in the paracontact metric and para-Sasakian geometries. Let us remark that another important class of paracontact geometries, namely the para-Kenmotsu case, was studied recently in [2] from the same points of view.

The second subject of the present paper is the class of paracontact-holomorphic vector fields that form a Lie subalgebra on a normal almost paracontact manifold; recently this type of vector fields was studied as providing the potential vector field of Ricci solitons in (3-dimensional) almost paracontact geometries in [1]. These vector fields vanish a \( \partial \)-operator expressed in terms of Levi-Civita as well as the canonical paracontact connection from [14]. We also give a relationship between the paracontact-holomorphicity on the manifold \( M \) and the holomorphicity on the cone manifold \( C(M) \). The last result gives a characterization of paracontact-holomorphic vector fields \( X \) in terms of para-Cauchy–Riemann equations for the components of \( X \) in a paracontact-holomorphic frame.

Two types of examples are examined: firstly in dimension 3 and secondly in arbitrary dimension following the Heisenberg-type example of contact metric geometry from [3, p. 60–61]. For the former case we compute the

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fundamental functions $\alpha, \beta$ occurring in the Levi-Civita differential of $\varphi$ while for the latter we use an adapted frame of $D$. Let us remark that our Heisenberg-type example 2.11 is different from the hyperbolic Heisenberg group of [8, p. 85]. For the 3-dimensional example we point out the vanishing of the mixed sectional curvature of the pair $(D, \xi)$ of invariant distributions in a short Appendix.

2. Invariant distributions on almost paracontact metric manifolds

Let $M$ be a $(2n+1)$-dimensional smooth manifold, $\varphi$ a $(1,1)$-tensor field called the structure endomorphism, $\xi$ a vector field called the characteristic vector field, $\eta$ a 1-form called the paracontact form, and $g$ a pseudo-Riemannian metric on $M$ of signature $(n+1, n)$. In this case, we say that $(\varphi, \xi, \eta, g)$ defines an almost paracontact metric structure on $M$ if [14]:

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

(2.1)

From the definition it follows $\varphi(\xi) = 0, \eta \circ \varphi = 0, \eta(X) = g(X, \xi), g(\xi, \xi) = 1$ and the fact that $\varphi$ is $g$-skew-symmetric: $g(\varphi X, Y) = -g(\varphi Y, X)$. The associated 2-form $\omega(X, Y) := g(X, \varphi Y)$ is skew-symmetric and is called the fundamental form of the almost metric paracontact manifold $(M, \varphi, \xi, \eta, g)$.

The $2n$-dimensional distribution $D := \ker \eta$ is called the canonical distribution associated to the almost paracontact metric structure $(\varphi, \xi, \eta, g)$. The vector field $\xi$ is $g$-orthogonal to $D$ and we have the orthogonal splitting of the tangent bundle $TM = D \oplus \text{span}\{\xi\}$; let $v_\xi$ and $h_\xi$ be the corresponding projectors; thus $v_\xi(X) = X - \eta(X)\xi$.

We assume given a distribution $V$ on $M$. The main hypothesis for our framework is the existence of a $g$-orthogonal complementary distribution $V^\perp$. Let $\Gamma(V)$ be the $C^\infty(M)$-module of its sections. We denote with $v$ and $h$ the orthogonal projectors with respect to the decomposition $TM = V \oplus V^\perp$.

Inspired by [4] we introduce:

**Definition 2.1** The distribution $V$ is called invariant if $\varphi(V) \subseteq V$, i.e. $h \circ \varphi \circ v = 0$.

The first result provides an example and a characterization:

**Proposition 2.2** On $(M, \varphi, \xi, \eta, g)$ we have: i) $D$ is an invariant distribution; ii) $V$ is invariant if and only if $V^\perp$ is invariant. Hence the invariance means $\varphi \circ v = v \circ \varphi$ respectively $\varphi \circ h = h \circ \varphi$.

**Proof** i) From $\eta \circ \varphi = 0$. ii) From the skew-symmetry of $\varphi$. \hfill $\Box$

With the same proof as that of Lemma 2.1. from [4, p. 194] we have:

**Proposition 2.3** If $V$ is an invariant distribution then $\xi \in \Gamma(V)$ or $\xi \in \Gamma(V^\perp)$. Moreover, if $\xi \in \Gamma(V)$ then $V^\perp \subseteq D$.

We consider a particular class of almost paracontact metric geometry after [14, p. 39]:

**Proposition 2.4** The almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is a paracontact metric manifold if $\omega = d\eta$ where $d$ is given by:

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])$$

(2.2)

for all vector fields $X, Y$. 2
The same proof as that of Proposition 2.1 from [4, p. 195] yields:

**Proposition 2.5** Suppose that $V$ is an invariant distribution in a paracontact metric manifold satisfying one of the following conditions:

(i) $\dim(V) = 2k + 1$ with $k \leq n$,

(ii) $V$ is integrable.

Then $\xi \in \Gamma(V)$. In particular, an integrable invariant distribution must be odd-dimensional.

Recall now two important tensor fields associated to a given distribution:

**Definition 2.6** If $V$ is a distribution on the Riemannian manifold $(M, g)$ then:

i) its second fundamental form is $B^V : \Gamma(V) \times \Gamma(V) \to \Gamma(V^\perp)$ given by:

$$B^V(X, Y) = \frac{1}{2} h(\nabla_X Y + \nabla_Y X)$$  \hspace{1cm} (2.3)

where $\nabla$ is the Levi-Civita connection of $g$;

ii) its integrability tensor is $I^V : \Gamma(V) \times \Gamma(V) \to \Gamma(V^\perp)$ given by:

$$I^V(X, Y) = h([X, Y]).$$  \hspace{1cm} (2.4)

For the class of paracontact metric structures we determine a relationship between the second fundamental form and the integrability tensor for invariant distributions transversally to the characteristic vector field:

**Proposition 2.7** Let $V$ be an invariant distribution on the paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(V^\perp)$. If $X, Y \in \Gamma(V)$ then

$$2 \left[ B^V(\varphi X, Y) - B^V(X, \varphi Y) \right] = \varphi \circ I^V(\varphi X, \varphi Y) - \varphi \circ I^V(X, Y).$$  \hspace{1cm} (2.5)

In particular, for $V = D$ we have the symmetry

$$B^D(\varphi X, Y) = B^D(X, \varphi Y), \quad B^D(\varphi X, \varphi Y) = B^D(X, Y).$$  \hspace{1cm} (2.6)

**Proof** From Lemma 2.7 of [14, p. 42] we have for all vector fields $X, Y$:

$$(\nabla_{\varphi X} \varphi) Y - (\nabla_X \varphi) \varphi Y = 2g(X, Y)\xi - (X - hX + \eta(X)\xi)\eta(Y)$$  \hspace{1cm} (2.7)

where $h = \frac{1}{2} L_\xi \varphi$. The Proposition 2.3 gives $V \subseteq D$ and then the second term in the right hand-side is zero. Hence

$$\nabla_{\varphi X} Y - \varphi(\nabla_{\varphi X} \varphi Y) - \nabla_X \varphi Y + \varphi(\nabla_X Y) = 2g(X, Y)\xi = \nabla_{\varphi Y} X - \varphi(\nabla_{\varphi Y} \varphi X) - \nabla_{\varphi Y} \varphi X + \varphi(\nabla_Y X)$$

gives

$$(\nabla_{\varphi X} Y + \nabla_Y \varphi X) - (\nabla_{\varphi Y} X + \nabla_X \varphi Y) = \varphi([\varphi X, \varphi Y] - [X, Y])$$

yielding

$$2 \left( B^V(\varphi X, Y) - B^V(X, \varphi Y) \right) = h \circ \varphi([\varphi X, \varphi Y] - [X, Y])$$  \hspace{1cm} (2.8)
which is (2.5). For $V = D$ we take the $g$-inner product of (2.8) with $\xi$ and use the $g$-skew-symmetry of $\varphi$ and $\varphi(\xi) = 0$ to obtain (2.61). With $Y$ replaced by $\varphi Y$ in (2.61) it results (2.62).

Let us study now the complementary case when $\xi \in \Gamma(V)$. We recall that a para-Sasakian manifold is a normal paracontact metric manifold; the normality means the integrability of the almost paracomplex structure $J$ on the cone $C(M) = M \times \mathbb{R}$:

$$J \left( X, f \frac{d}{dt} \right) = \left( \varphi X + f \xi, \eta(X) \frac{d}{dt} \right). \quad (2.9)$$

A characterization of this case is given in [14, p. 42]

$$(\nabla_X \varphi) Y = -g(X,Y) \xi + \eta(Y)X \quad (2.10)$$

for all vector fields $X, Y$. In a para-Sasakian manifold we have

$$\nabla_X \xi = -\varphi X \quad (2.11)$$

which yields the commutation formula

$$\nabla_{\varphi X} \xi = \varphi(\nabla_X \xi) = -\varphi^2 X. \quad (2.12)$$

**Proposition 2.8** Let $V$ be an invariant distribution with $\xi \in \Gamma(V)$ in a para-Sasakian manifold. Then for all $X, Y \in \Gamma(V)$ we have

$$2 \left[ B^V(X, \varphi Y) - \varphi \circ B^V(X, Y) \right] = -\varphi \circ I^V(X, \varphi Y) - \varphi \circ I^V(X, Y). \quad (2.13)$$

In particular,

$$2B^V(X, \xi) = -I^V(X, \xi) \quad (2.14)$$

and if $V$ is integrable then

$$B^V(\varphi X, Y) = \varphi \circ B^V(X, Y) = B^V(X, \varphi Y). \quad (2.15)$$

**Proof** By using the relation (2.10) the left-hand side of (2.13) is

$$h(\nabla_X \varphi Y + \nabla_{\varphi Y} X - \varphi(\nabla_X Y) - \varphi(\nabla_Y X)) = h(\nabla_{\varphi Y} X - \varphi(\nabla_Y X)).$$

Now, using the metric character of $\nabla$, the last term is $h(\nabla_X \varphi Y - [X, \varphi Y] - \varphi(\nabla_X Y) - \varphi([X, Y]))$ and we get the conclusion (2.13). With $Y = \xi$ in (2.13) we obtain (2.14) while (2.15) is a direct consequence of (2.13).

**Corollary 2.9** Let $N$ be an invariant submanifold of the para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ containing $\xi$ and $B$ its second fundamental form. Then for all $X, Y \in \Gamma(N)$ we have:

$$B(X, \xi) = 0, \quad B(\varphi X, Y) = \varphi \circ B(X, Y) = B(X, \varphi Y). \quad (2.16)$$

We finish this section with some examples other than those of [8]:

**Example 2.10** Suppose that $n = 1$. After [11, p. 379] we have
\[(\nabla_X \varphi)Y = g(\varphi(\nabla_X Y), Y) - \eta(Y)\varphi(\nabla_X X) \] (2.17)

and \((M, \varphi, \xi, \eta, g)\) is normal if and only if there exist smooth functions \(\alpha, \beta\) on \(M\) such that

\[ (\nabla_X \varphi)Y = \beta (g(X, Y)\xi - \eta(Y)X) + \alpha (g(\varphi X, Y)\xi - \eta(\varphi Y)\varphi X), \nabla_X \xi = \alpha (X - \eta(X)\xi) + \beta \varphi(X). \] (2.18)

Hence, the para-Sasakian case is provided by \(\alpha = 0\) and \(\beta = -1\). \((M, \varphi, \xi, \eta, g)\) admits locally a frame \(\{\xi, E, \varphi E\}\) with \(g(E, E) = 1 = -g(\varphi E, \varphi E)\), which means that \(\xi\) and \(E\) are space-like vector fields while \(\varphi E\) is a time-like vector field. We have \(I^P(E, \varphi E) = \eta([E, \varphi E])\xi\).

In order to handle a concrete example let \(N\) be an open connected subset of \(\mathbb{R}^2\), \((a, b)\) an open interval in \(\mathbb{R}\), and let us consider the manifold \(M = N \times (a, b)\). Let \((x, y)\) be the coordinates on \(N\) induced from the Cartesian coordinates on \(\mathbb{R}^2\) and let \(z\) be the coordinate on \((a, b)\) induced from the Cartesian coordinate on \(\mathbb{R}\). Thus \((x, y, z)\) are the coordinates on \(M\). Now we choose the functions

\[ \omega_1, \omega_2 : N \to \mathbb{R}, \quad \sigma, f : M \to \mathbb{R}^*_+, \] (2.19)

and following the idea from [10] we define

\[ g = \frac{1}{4} \begin{pmatrix} \omega_1^2 + \sigma^2 & \omega_1 \omega_2 & \omega_1  \\ \omega_1 \omega_2 & \omega_2^2 - \sigma^2 & \omega_2  \\ \omega_1 & \omega_2 & 1 \end{pmatrix} = \frac{1}{4} \sigma e^{2f} (dx^2 - dy^2) + \eta \otimes \eta, \eta = \frac{1}{2} (dz + \omega_1 dx + \omega_2 dy), \] (2.20)

\[ \xi = 2 \frac{\partial}{\partial z}, \quad \varphi = \begin{pmatrix} 0 & 1 & 0  \\ 1 & 0 & 0  \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}. \] (2.21)

It follows an almost paracontact metric manifold with

\[ E = \frac{2e^{-f}}{\sqrt{\sigma}} \left( \frac{\partial}{\partial x} - \omega_1 \frac{\partial}{\partial z} \right), \quad \varphi E = \frac{2e^{-f}}{\sqrt{\sigma}} \left( \frac{\partial}{\partial y} - \omega_2 \frac{\partial}{\partial z} \right). \] (2.22)

From

\[ \left\{ \begin{array}{l}
[E, \xi] = \frac{2f}{\sigma} \xi + \varphi E, \\
[E, \varphi E] = \frac{\sqrt{\sigma}}{e^f} \left[ E(\frac{e^{-f}}{\sqrt{\sigma}}) \varphi E - \varphi E(\frac{e^{-f}}{\sqrt{\sigma}}) E \right] + \frac{2e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right) \xi
\end{array} \right. \] (2.23)

it follows that \(\mathcal{D}\) is integrable if and only if the 1-form \(\omega_1 dx + \omega_2 dy\) is closed; hence \(\eta\) is closed. We have the Levi-Civita connection

\[ \nabla_E E = -\frac{\sqrt{\sigma}}{e^f} E(\frac{e^{-f}}{\sqrt{\sigma}}) \varphi E + \frac{2f}{\sigma} \xi + \frac{2f}{\sigma} \varphi E \] (2.24)

\[ \nabla_E \varphi E = -\frac{1}{\sqrt{\sigma}} e^{-f} \varphi E(\frac{e^{-f}}{\sqrt{\sigma}}) E + \frac{e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right) \xi \]

\[ \nabla_E \xi = \frac{2f}{\sigma} (\varphi E(\frac{e^{-f}}{\sqrt{\sigma}}) E + \frac{e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right) \varphi E \] (2.25)

\[ \nabla_E E = \frac{e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right) \varphi E, \quad \nabla_E \varphi E = \frac{e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right) E, \quad \nabla_E \xi = 0 \] (2.26)
and then

$$\alpha = 2f + \frac{\sigma_2}{\sigma}, \quad \beta = \frac{e^{-2f}}{\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right).$$  (2.27)

Hence, \((M, \varphi, \xi, \eta, g)\) is a para-Sasakian manifold if and only if

$$\sigma e^{2f} = \sigma e^{2f}(x, y), \quad \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} = -\sigma e^{2f}. \quad (2.28)$$

The first relation expresses the normality of the paracontact structure while the second condition means the metrical condition of the Definition 2.4 and yields the nonintegrability of \(D\) since \(I^D(E, \varphi E) = -2\xi\). Some cases when both equations hold are: i) \(\omega_1 = -y, \ \omega_2 = 0 = f, \ \sigma = 1\); ii) \(\omega_1 = -y, \ \omega_2 = x, \ \sigma = 2, \ f = 0\).

Other examples of 3-dimensional (almost) paracontact manifolds appear in [6, 11, 12].

**Example 2.11** On \(M = \mathbb{R}^{2n+1}\) with the splitting \(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) we consider a Heisenberg-type structure inspired by the contact metric example from [3, p. 60-61]:

$$g = \frac{1}{4} \begin{pmatrix} \delta_{ij} + y_i y_j & 0 & -y_i \\ 0 & \delta_{ij} & 0 \\ -y_i & 0 & 0 \end{pmatrix}, \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ \delta_{ij} & 0 & 0 \\ 0 & y_i & 0 \end{pmatrix}, \xi = 2 \frac{\partial}{\partial z}, \eta = \frac{1}{2}(dz - \sum_{i=1}^n y_i dx^i).$$  (2.29)

It follows that \((\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)\) is a paracontact metric manifold with

$$\mathcal{D} = \text{span} \left\{ A_i = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, B_i = \frac{\partial}{\partial y^i}; 1 \leq i \leq n \right\}. \quad (2.30)$$

Two classes of invariant distributions are indexed by \(k \in \{1, \ldots, n-1\} : 

$$V_k^{\text{even}} = \text{span} \left\{ A_\alpha, B_\alpha; 1 \leq \alpha \leq k \right\}, V_k^{\text{odd}} = V_k^{\text{even}} \cup \{ \xi \}. \quad (2.31)$$

Let us remark that for \(n = 1\) we recover the previous Example with: \(\omega_1 = -y, \ \omega_2 = 0 = f, \ \sigma = 1\). It is a para-Sasakian manifold with nonintegrable \(D\): \([E, \xi] = [\varphi E, \xi] = 0, \ [E, \varphi E] = -2\xi\). The sectional curvature of the plane spanned by \(E\) and \(\varphi E\) is

$$pK = K^M(E, \varphi E) = g(R(E, \varphi E) \varphi E, E) = g(\nabla_{\varphi E} \xi + 2\nabla_{\xi} \varphi E, E) = g(-E - 2E, E) = -3$$  (3.2)

similar to the metric contact case.

### 3. Infinitesimal paracontact-holomorphicity

**Definition 3.1** The vector field \(X \in \Gamma(TM)\) is called paracontact-holomorphic if

$$v_\xi \circ \mathcal{L}_X \varphi = 0. \quad (3.1)$$

Let \(\mathfrak{phol}(M)\) be the set of all paracontact-holomorphic vector fields. The distribution \(\mathcal{V}\) is paracontact-holomorphic if its sections are elements of \(\mathfrak{phol}(M)\).

The condition (3.1) says that for all vector fields \(Y\) we have that \((\mathcal{L}_X \varphi) Y\) is collinear with \(\xi\); let us denote \(\alpha_X(Y)\) the collinearity factor. We have

$$\alpha_X(Y) = g([X, \varphi Y] - \varphi([X, Y]), \xi) = \eta([X, \varphi Y]). \quad (3.2)$$

The next result shows the invariance of the above defined holomorphicity and its proof is exactly as in [4]:


Proposition 3.2  Let $X$ be a paracontact-holomorphic vector field on the normal almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$. Then $\varphi X$ is also a paracontact-holomorphic vector field.

Remarks 3.3  i) Fix $X$ a paracontact-holomorphic vector field. Then computing $X(Y)$ with (3.2) we get

$$\alpha_X(\xi) = 0$$  \hspace{1cm} (3.3)

which means that $[X, \xi]$ is collinear with $\xi$, i.e. $v_\xi([X, \xi]) = 0$.

ii) The vanishing of the tensor field $N(3) = \mathcal{L}_\xi \varphi$ means that $\xi$ is a paracontact-holomorphic vector field with $\alpha_\xi = 0$.

iii) The paracontact-holomorphicity of a fixed $X$ implies for every vector field $Y$

$$\mathcal{L}_X Y = \eta(\mathcal{L}_X Y) \xi + \varphi(\mathcal{L}_X \varphi Y), \quad \mathcal{L}_X \varphi Y = \alpha_X(Y) \xi + \varphi([X, Y]).$$  \hspace{1cm} (3.4)

In both relations, the first term in the right-hand side belongs to $\text{span}\xi$ while the second belongs to $\mathcal{D}$.

By using these remarks we get:

Proposition 3.4  If $(M, \varphi, \xi, \eta, g)$ is a normal almost paracontact manifold then $\text{phol}(M)$ is a Lie subalgebra in the Lie algebra of vector fields of $M$.

Proof  Let $X$ and $Y$ be paracontact-holomorphic vector fields and $Z$ an arbitrary vector field. Then

$$([\mathcal{L}_{[X, Y]} \varphi] Z) = [X, ([\mathcal{L}_Y \varphi] Z)] - ([\mathcal{L}_X \varphi] ([X, Z])) - [Y, ([\mathcal{L}_X \varphi] Z)] + ([\mathcal{L}_X \varphi] ([Y, Z])).$$  \hspace{1cm} (3.5)

From the property of $X$, $Y$ we have that the second and fourth terms are collinear with $\xi$. Also

$$[X, ([\mathcal{L}_Y \varphi] Z)] = X(\alpha_Y(Z)) \xi - \alpha_Y(Z)[X, \xi]$$

and the first relation (3.4) gives that this expression is collinear with $\xi$. The same fact holds for the third term of (3.5). \hfill \Box

As in the contact case we can express the paracontact-holomorphicity by the vanishing of some $\bar{\partial}$-operator. More precisely, we define the map $\bar{\partial} : \Gamma(TM) \to \text{End}(TM)$ given by

$$\bar{\partial}(X)(Y) = \varphi(\nabla_Y X - \varphi(\nabla_{\varphi Y} X) + \varphi(\nabla_X \varphi Y)).$$  \hspace{1cm} (3.6)

Thus, $X$ is a paracontact-holomorphic vector field if and only if $\bar{\partial}(X) = 0$. For a general vector field $X$, if $(M, \varphi, \xi, \eta, g)$ is a para-Sasakian manifold then

$$\bar{\partial}(X)(\xi) = \varphi([\xi, X])$$  \hspace{1cm} (3.7)

and for $Y \in \mathcal{D}$ we have

$$\bar{\partial}(X)(Y) = \varphi(\nabla_Y X - \varphi(\nabla_{\varphi Y} X)).$$  \hspace{1cm} (3.8)

If $n = 1$ then the expression (3.6) reduces to

$$\bar{\partial}(X)(Y) = \varphi(\nabla_Y X - \varphi(\nabla_{\varphi Y} X) - \eta(Y)(\alpha X + \beta \varphi X)).$$  \hspace{1cm} (3.9)
For the general $n$ and using the canonical paracontact connection $\tilde{\nabla}$ of [14, p. 49] we have

$$\partial(X)(Y) = \varphi \left( \tilde{\nabla}_Y X - \varphi(\tilde{\nabla}_Y \varphi X) + \varphi(\tilde{\nabla}_X \varphi) Y + 2\eta(X)(\varphi N^{(3)} Y - \varphi^2 N^{(3)} \varphi Y) - \eta(Y)\varphi N^{(3)} X \right).$$  (3.6c\alpha)

Recall now that on the cone $\mathcal{C}(M)$ we have

$$[(X, f \frac{d}{dt}), (Y, g \frac{d}{dt})] = \left( [X, Y], (X(g) - Y(f) + f \frac{dg}{dt} - g \frac{df}{dt}) \frac{d}{dt} \right)$$  (3.10)

which yields:

**Proposition 3.5** Fix $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$. Then $(X, f \frac{d}{dt})$ is a paraholomorphic vector field on the cone $\mathcal{C}(M)$ if and only if the following three conditions hold:

i) $\mathcal{L}_X \varphi Y = -Y(f) \xi$,

ii) $\mathcal{L}_X \eta(Y) = \varphi Y(f) + \eta(Y) \frac{df}{dt}$,

iii) $\mathcal{L}_X \xi = -\frac{df}{dt} \xi$,

where $Y \in \Gamma(TM)$ is arbitrary. Consequently, if $(X, f \frac{d}{dt})$ is a paraholomorphic vector field on $\mathcal{C}(M)$ then $X$ is paracontact-holomorphic vector field on $M$ and $f$ is a first integral if $\xi$.

**Proof** By using (3.10) we get with respect to $J$ of (2.9)

$$(\mathcal{L}_{(X, f \frac{d}{dt})})J)(Y, 0) = \left( \left( \mathcal{L}_X \varphi Y + Y(f) \xi, (X(\eta(Y)) - \varphi Y(g) - \eta(Y) \frac{df}{dt} - \eta([X, Y]) \frac{d}{dt} \right) \right)$$  (3.11)

$$(\mathcal{L}_{(X, f \frac{d}{dt})})J)(0, \frac{d}{dt}) = \left( [X, \xi] + df, -\xi(f) \frac{d}{dt} \right).$$  (3.12)

The paraholomorphicity of $(X, f \frac{d}{dt})$ means the vanishing of the above left-hand sides and this is equivalent with $f$ being first integral of $\xi$ and the relations i)-iii). However, with $Y = \xi$ in i) and using iii) it follows that $\xi(f) = 0$. The equation i) means that $X$ is a paracontact-holomorphic vector field.

**Corollary 3.6** The paracontact-holomorphic vector fields on $M$, which come about by projection of the paraholomorphic fields on $\mathcal{C}(M)$, form a Lie subalgebra of $\text{phol}(M)$, denoted by $\text{phol}_p(M)$. They are paracontact-holomorphic fields $X$ with two additional properties:

a) The 1-form $\alpha_X$ is exact: there exists a smooth function $f$ on $M$ such that $\alpha_X = d(-f)$,

b) $\eta([X, \xi])$ is a (locally) constant, i.e. constant on any connected component of $M$.

**Proof** a) it results by applying $\eta$ to i); more precisely $Y(-f) = \eta([X, \varphi Y])$ for all vector fields $Y$. By applying $\eta$ to iii) we get $\frac{df}{dt} = \eta([\xi, X])$ and then around a point $p_0 \in M$ we have the following expression of $f$:

$$f(p, t) = \eta([\xi, X])(p)t - F(p).$$  (13.13)

Plugging this expression in a) we get: $Y(F) + Y(\eta([X, \xi]))t = \eta([X, \varphi Y])$ and it results in b).
Corollary 3.7 On a normal almost paracontact metric manifold \((M, \varphi, \xi, \eta, g)\) we have:

iv) \(a \xi\) is a contact-holomorphic vector field, for any function \(a \in M\); so \(a \xi \in \mathfrak{phol}(M)\) but it is not necessary the case that \(a \xi \in \mathfrak{phol}_{pr}(M)\).

v) \((\xi, c \frac{d}{dt})\) is a holomorphic vector field on \(C(M)\) if and only if \(c\) is a constant.

Proof The first part is a direct consequence of

\[
(\mathcal{L}a \xi \varphi)Y = a(\mathcal{L} \xi \varphi)Y - \varphi Y(a) \xi. \tag{3.14}
\]

Let us remark that the normality implies that \(a \alpha(Y) = -\varphi Y(a)\). For the second part, from iii) of Proposition 3.5 it results that \(\frac{dc}{dt} = 0\) while i) gives that \(Y(c) = 0\) for all vector fields \(Y\).

Proposition 3.8 Let \((M, \varphi, \xi, \eta, g)\) be a paracontact metric manifold. Then any two of the following conditions imply the third one:

(i) \((\mathcal{L}X g)(Y, Z) = 0\) for all \(Y, Z \in \Gamma(D)\),

(ii) \(i_X d\eta\) is a closed form,

(iii) \(X\) is a paracontact-holomorphic vector field.

Proof It is a direct consequence of the formula

\[
(\mathcal{L}X g)(Y, \varphi Z) = (\mathcal{L}X d\eta)(Y, Z) - g(Y, (\mathcal{L}X \varphi)Z) \tag{3.15}
\]

for all vector fields \(Y, Z\).

Example 3.9 Returning to Example 2.11, let

\[
X = \alpha^i A_i + \beta^i B_i + \gamma \xi = \alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial y^i} + (2\gamma + (\sum_{j=1}^n y^j \alpha^j)) \frac{\partial}{\partial z}. \tag{3.16}
\]

Then \(X \in \mathfrak{phol}(M)\) if and only if the coefficients \(\alpha\) and \(\beta\) satisfy the para-Cauchy–Riemann equations with respect to the variables \((x, y)\) and are constant with respect to \(z\):

\[
\begin{align*}
\frac{\partial \alpha^i}{\partial x^i} &= \frac{\partial \beta^i}{\partial y^i}, \\
\frac{\partial \alpha^i}{\partial y^i} &= \frac{\partial \beta^i}{\partial x^i}, \\
\frac{\partial \beta^i}{\partial z} &= 0.
\end{align*} \tag{3.17}
\]

The following analogy with the contact case shows that these computations have a general nature:

Proposition 3.10 On a normal almost paracontact metric manifold there always exist (local) adapted frames \((E_i, \varphi E_i, \xi)\) consisting of contact-holomorphic vector fields. If the vector field \(X\) has the expression \(X = \alpha^i E_i + \beta^i \varphi E_i + \gamma \xi\) then \(X\) is a paracontact-holomorphic vector field if and only if the coefficients \(\alpha, \beta\) satisfy the generalized para-Cauchy–Riemann equations:

\[
E_j(\alpha^i) = \varphi E_j(\beta^i), \quad \varphi E_j(\alpha^i) = E_j(\beta^i) \tag{3.18}
\]

and are first integrals of \(\xi\).
4. Appendix: The mixed sectional curvature

The main result of [4] is the Bochner-type Theorem 5.1 stated on page 206. The technical ingredient of this result is the mixed sectional curvature:

\[ s_{\text{mix}}(V, V^\perp) = \sum K^M(e_i \wedge f_\alpha) \]  

(a.1)

where \( \{e_i\} \) respectively \( \{f_\alpha\} \) are local orthonormal frames for the given distribution. The cited Bochner-type result deals with an invariant distribution \( V \) of dimension \( 2p + 1 \) in the Sasakian case and concerns the case \( s_{\text{mix}} \geq 2(n - p) \).

The aim of this short Appendix is to compute this quantity for our example 2.10:

\[ s_{\text{mix}}(D, \xi) = K^M(E \wedge \xi) + K^M(\varphi E \wedge \xi) = g(R(E, \xi)\xi, E) + g(R(\varphi E, \xi)\xi, \varphi E) \]  

(a.2)

Since \( E \) is a space-like vector field while \( \varphi E \) is a time-like one, a direct computation yields the vanishing:

\[ s_{\text{mix}}(D, \xi) = 0. \]

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References


