An example of planar PH curve of degree 5

In order to have a PH curve of degree 5 we must have: $\lambda + 2\mu = 4$, where $\mu > 0$.

We can have two situations:

(1) $\lambda = 2$ and $\mu = 1$
(2) $\lambda = 0$ and $\mu = 2$

Where μ it is the maximum degree of the polynomials u and v, and λ it is the degree of polynomial w.

Consider: $r : I \to \mathbb{R}$

$t \to (x(t), y(t))$

The necessary and sufficient condition for a polynomial curve to be PH is:

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2 \Longleftrightarrow \exists u, v, w \text{ such that } \begin{cases} x' = w(u^2 - v^2) \\ y' = 2uvw \\ \sigma = w(u^2 + v^2) \end{cases}$$

We give an example in each of the two situations.

Case 1: $\lambda = 2$ and $\mu = 1$

Let us consider $\begin{cases} u = 3t \\ v = t + 2 \\ w = 2t^2 + 1. \end{cases}$

Consequently, we find:

$x'(t) = (2t^2 + 1)(9t^2 - (t + 2)^2) = (2t^2 + 1)(9t^2 - t^2 - 4t - 4) = (2t^2 + 1)(8t^2 - 4t - 4) =$
$= 4(4t^4 - 2t^3 - t - 1),$

$y'(t) = 6t(t + 2)(2t^2 + 1) = (6t^2 + 12t)(2t^2 + 1) = 12t^4 + 24t^3 + 6t^2 + 12t.$

To find the PH curve, we integrate and we obtain:

$$x(t) = \int x'(t)dt = \int (16t^4 - 8t^3 - 4t - 4)dt = \frac{16}{5}t^5 - 2t^4 - 2t^2 - 4t,$$

$$y(t) = \int y'(t)dt = \int (12t^4 + 24t^3 + 6t^2 + 12t)dt = 12\frac{t^5}{5} + 24\frac{t^4}{4} + 6\frac{t^3}{3} + 12\frac{t^2}{2} =$$
$$= \frac{12}{5}t^5 + 6t^4 + 2t^3 + 6t^2.$$

We conclude with:

$$r(t) = \left(\frac{16}{5}t^5 - 2t^4 - 2t^2 - 4t, \frac{12}{5}t^5 + 6t^4 + 2t^3 + 6t^2\right).$$

Check that $\sigma = 2(2 + 2t + 9t^2 + 4t^3 + 10t^4)$.

Case 2: $\lambda = 0$ and $\mu = 2$

Let's take: $\begin{cases} u = 2t^2 \\ v = 3t^2 + 3 \\ w = 1. \end{cases}$

Consequently, we find:

$$x'(t) = 4t^4 - (3t^2 + 3)^2 = t^4 - 9t^4 - 18t^2 - 9 = -8t^4 - 18t^2 - 9$$
$$y'(t) = 2t^2(3t^2 + 3) = 6t^4 + 6t^2.$$

We integrate and we obtain:

$$x(t) = \int x'(t)dt = \int (-8t^4 - 18t^2 - 9)dt = -\frac{8}{5}t^5 - 6t^3 - 9t$$
$$y(t) = \int y'(t)dt = \int (6t^4 + 6t^2)dt = \frac{6}{5}t^5 + 2t^3$$

We conclude with:

$$r(t) = \left(-\frac{8}{5}t^5 - 6t^3 - 9t, \frac{6}{5}t^5 + 2t^3\right)$$

Check that $\sigma = 9 + 18t^2 + 10t^4$.