Seminar 14

Exercise 1: Construct a spatial (not planar) PH cubic, such that its projection on xOy is also a PH cubic.

Solution:

Let $r : I \to \mathbb{R}^3$ be the spatial PH cubic, $r(t) = (x(t), y(t), z(t))$.

Its projection: $proj_{xOy}(r) = (x(t), y(t), 0)$.

For the projection: $(\exists)\sigma$ (a polynomial) s. t. $x'(t)^2 + y'(t)^2 = \sigma(t)^2$.

For the curve: $(\exists)\lambda$ (a polynomial) s. t. $x'(t)^2 + y'(t)^2 + z'(t)^2 = \lambda(t)^2$.

$$\begin{cases} x'(t)^2 + y'(t)^2 = \sigma(t)^2 \\ \sigma(t)^2 + z'(t)^2 = \lambda(t)^2 \end{cases}$$

(x', y', σ) and (σ, z', λ) are Pythagorean triples on polynomials.

We know:

I. $(\exists)u, v, w$ polynomials s. t.

$$\begin{cases} x' = w(u^2 - v^2) \\ y' = 2wuv \\ \sigma = w(u^2 + v^2) \end{cases}$$

II. $(\exists)a, b, c$ polynomials s. t.

$$\begin{cases} \sigma = c(a^2 - b^2) \\ z' = 2cab \\ \lambda = c(a^2 + b^2) \end{cases}$$

For cubics:

$$\deg(w) = 0 \Rightarrow w = 1$$
$$\deg(c) = 0 \Rightarrow c = 1$$

So, $\max(\deg(u), \deg(v)) = 1$ and $\max(\deg(a), \deg(b)) = 1$.

$$\begin{cases} x' = u^2 - v^2 \\ y' = 2uv \\ \sigma = u^2 + v^2 \end{cases} \qquad \begin{cases} \sigma = a^2 - b^2 \\ z' = 2ab \\ \lambda = a^2 + b^2 \end{cases}$$

If u and v are defined by $\begin{cases} u = \alpha t + \beta \\ v = pt + q \end{cases}$ it follows that

$u^2 + v^2 = (\alpha^2 + \beta^2)t^2 + 2(\alpha\beta + pq)t + \beta^2 + q^2.$

Its discriminant (as polynomial of degree 2 in t) is

$\Delta = \alpha^2\beta^2 + p^2q^2 + 2\alpha\beta pq - \alpha^2\beta^2 - p^2q^2 - \alpha^2q^2 - p^2\beta^2 = -(\alpha q - p\beta)^2.$

If $\alpha q \neq p\beta$, that is the polynomials u and v are relatively prime, then σ is irreducible.

On the other hand we have $\sigma = (a - b)(a + b)$.

Hence we should have $a - b = 1$ & $a + b = \sigma$ or $a + b = 1$ & $a - b = \sigma$.

This is a contradiction because a and b are of degree at most 1, while σ is of degree 2.

Hence $\alpha q = p\beta$, namely the polynomials u and v are dependent.

Take for example $\begin{cases} u = 2t \\ v = t \end{cases} \quad \Rightarrow \quad \sigma = 5t^2$

For

$$\begin{cases} a - b = t \\ a + b = 5t \end{cases} \quad \Rightarrow \quad a = 3t, b = 2t$$

$$\begin{cases} z' = 12t^2 \\ y' = 4t^2 \\ x' = 3t^2 \end{cases} \quad \Rightarrow \quad \begin{cases} z = 4t^3 + z_0 \\ y = \frac{4}{3}t^3 + y_0 \\ x = t^3 + x_0 \end{cases} ,$$

we obtain a non uniform parametrization for a straight line :

$$\frac{x - x_0}{1} = \frac{y - y_0}{\frac{4}{3}} = \frac{z - z_0}{4}$$

(trivial example).

<u>Exercise 2:</u> Construct a spatial (not planar) PH curve, such that its projection on xOy is also a PH curve.

Solution:

$$\begin{cases} x'(t)^2 + y'(t)^2 = \sigma(t)^2 \\ \sigma(t)^2 + z'(t)^2 = \lambda(t)^2 \end{cases}$$

Do the same steps as in Exercise 1.

Keep the property for the projection to be a cubic and consider: $\begin{cases} u = t+1 \\ v = t. \end{cases}$

Thus, we obtain: $x' = 2t+1$, $y' = 2t(t+1)$, $\sigma = 2t^2 + 2t + 1$.

We still have $\sigma = a^2 - b^2$ and we can take $a = \frac{\sigma+1}{2}$ and $b = \frac{\sigma-1}{2}$.

It follows that

$z' = \frac{1}{2}(2t^2 + 2t + 1 - 1)(2t^2 + 2t + 2) \Leftrightarrow z' = 2t(t+1)(t^2 + t + 1)$

Hence, the PH curve is given by

$$r = \left(t^2 + t, \frac{2}{3}t^3 + t^2, \frac{2}{5}t^5 + t^4 + \frac{4}{3}t^3 + t^2\right) \qquad \text{(r is a quintic).}$$

Matlab check:

```
clear all
syms t
x=t^2+t;
y=(2/3)*t^3+t^2;
z=(2/5)*t^5+t^4+(4/3)*t^3+t^2;
xp=diff(x,t);
yp=diff(y,t);
zp=diff(z,t);
factor(xp^2+yp^2)
factor(xp^2+yp^2+zp^2)
```

Answer:

```
ans =
    (2*t^2 + 2*t + 1)^2
ans =
    (2*t^4 + 4*t^3 + 4*t^2 + 2*t + 1)^2
```