New structures on the tangent bundles and tangent sphere bundles

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Abstract

In this paper we study a Riemanian metric on the tangent bundle $T(M)$ of a Riemannian manifold M which generalizes Sasaki metric and Cheeger Gromoll metric and a compatible almost complex structure which together with the metric confers to $T(M)$ a structure of locally conformal almost Kählerian manifold. This is the natural generalization of the well known almost Kählerian structure on $T(M)$. We found conditions under which $T(M)$ is almost Kählerian, locally conformal Kählerian or Kählerian or when $T(M)$ has constant sectional curvature or constant scalar curvature. Then we will restrict to the unit tangent bundle and we find an isometry with the tangent sphere bundle (not necessary unitary) endowed with the restriction of the Sasaki metric from $T(M)$. Moreover, we found that this map preserves also the natural almost contact structures obtained from the almost Hermitian ambient structures on the unit tangent bundle and the tangent sphere bundle, respectively.

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1 Introduction

A Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle $T(M)$ of M. Maybe the best known example is the Sasaki metric g_S introduced in [21]. Although the Sasaki metric is *naturally* defined, it is very *rigid* in the following sense. For example, O.Kowalski [13] has shown that the tangent bundle $T(M)$ with the Sasaki metric is never locally symmetric unless the metric g on the base manifold is flat. Then, E.Musso & F.Tricerri [16] have shown a more general result, namely, the Sasaki metric has constant scalar curvature if and only if (M, g) is locally Euclidian. In the same paper, they have given an explicit expression of a positive definite Riemannian metric introduced by J.Cheeger and D.Gromoll in [11] and called this metric *the Cheeger-Gromoll metric*. In [22] M.Sekizawa computed the Levi Civita connection, the curvature

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tensor, the sectional curvatures and the scalar curvature of this metric. These results are completed in 2002 by S.Gudmundson and E.Kappos in [12]. They have also shown that the scalar curvature of the Cheeger Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, M.T.K.Abbassi & M.Sarih have proved that $T(M)$ with the Cheeger Gromoll metric is never a space of constant sectional curvature (cf. [3]). A more general metric is given by M.Anastasiei in [7] which generalizes both of the two metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasaki and the Cheeger Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and hence $T(M)$ becomes an locally conformal almost Käherian manifold.

V.Oproiu and his collaborators constructed a family of Riemannian metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties (cf. [17, 18, 19, 20]). In particular, the scalar curvature of $T(M)$ can be constant also for a non-flat base manifold with constant sectional curvature. Then M.T.K.Abbassi & M.Sarih proved in [4] that the considered metrics by Oproiu form a particular subclass of the so-called *g-natural metrics* on the tangent bundle (see also [1, 2, 4, 5, 6, 14]).

By thinking $T(M)$ as a vector bundle associated with $O(M)$ (the space of orthonormal frames on M), namely $T(M) \equiv O(M) \times \mathbf{R}^n / O(n)$ (where the orthogonal group $O(n)$ acts on the right on $O(M)$), Musso & Tricerri construct natural metrics on $T(M)$ (see §4 in [16]). The idea is to consider Q a symmetric, semi-positive definite tensor field of type $(2,0)$ and rank $2n$ on $O(M) \times \mathbf{R}^n$. Assuming that Q is *basic* for $\psi : O(M) \times \mathbf{R}^n \longrightarrow T(M)$, $(\mathbf{u}, \zeta) \mapsto (p, \zeta^i \mathrm{u}_i)$, where $\mathbf{u} = (p, \mathrm{u}_1, \ldots, \mathrm{u}_n)$ and $\zeta = (\zeta^1, \ldots, \zeta^n)$ (i.e. Q is $O(n)$-invariant and $Q(X,Y) = 0$ for all X tangent to a fiber of ψ) there is a unique Roiemannian metric g_Q on $T(M)$ such that $\psi^* g_Q = Q$. In this paper we will show that the metric introduced in [7] can be construct by using the method of Musso and Tricerri and we study it. Then we will give the conditions under which $T(M)$ is locally conformal Kählerian and respectively Kählerian (Theorems 2.6 and 2.8). These results extend the known result saying that $T(M)$ endowed with the Sasaki metric and the canonical almost complex structure is Kählerian if and only if the base manifold is locally Euclidean.

Next we want to have constant sectional curvature and constant scalar curvature, respectively on $T(M)$. With this end in view, we compute the Levi Civita connection, the curvature tensor, the sectional curvature and the scalar curvature of this metric. We found relations between the sectional curvature (resp. scalar curvature) on $T(M)$ and the corresponding curvature on the base M. We give an example of metric on $T(M)$ of Cheeger Gromoll type which is flat. (Recall the fact that Cheeger Gromoll metric can not have constant sectional curvature.) See Proposition 2.17. We also obtain a locally conformal Kähler structure (cf. Example $2.8/_2$) and a Kähler structure (cf. Remark $2.10/_3$) on $T(M)$. We give some examples of metrics on $T(M)$ (when M is a space form) having constant scalar curvature. See Examples 2.22 and 2.23.

In section 3 we restrict the structure on the unit tangent bundle, obtaining an almost contact metric. We will show that the unit tangent bundle is isometric with a tangent sphere bundle $T_r(M)$ (we find the radius r) endowed with the restriction of Sasaki metric from $T(M)$ (see also [9], Remark 4, p.88). Moreover, this map preserves the almost contact structures. M.Sekizawa & O.Kowalski have studied the geometry of the tangent sphere bundles with arbitrary radii endowed with the induced Sasaki metric (see [15]). They have also noticed that the unit tangent bundle equipped with the induced Cheeger Gromoll metric is isometric to the tangent sphere bundle $T_{\frac{1}{\sqrt{2}}}(M)$, of radius $\frac{1}{\sqrt{2}}$ endowed with the metric induced by the Sasaki metric. Some other generalizations concerning this fact are given in [2]. In the end of the section we obtained some properties for $T_1(M)$ as contact manifold. Among the results we state the following: *The contact metric structure on $T_1(M)$ is K-contact if and only if the base manifold has positive constant sectional curvature. In this case $T_1(M)$ becomes a Sasakian manifold.*

2 The tangent bundle $T(M)$

Let (M,g) be a Riemannian manifold and let ∇ be its Levi Civita connection. Let $\tau : T(M) \longrightarrow M$ be the tangent bundle. If $\mathtt{u} \in T(M)$ it is well known the following decomposition of the tangent space $T_{\mathtt{u}}T(M)$ (in $\mathtt{u}$ at $T(M)$)

$$T_{\mathtt{u}}T(M) = V_{\mathtt{u}}T(M) \oplus H_{\mathtt{u}}T(M)$$

where $V_{\mathtt{u}}T(M) = \ker \tau_{*,\mathtt{u}}$ is the vertical space and $H_{\mathtt{u}}T(M)$ is the horizontal space in $\mathtt{u}$ obtained by using ∇. (A curve $\widetilde{\gamma} : I \longrightarrow T(M)$, $t \mapsto (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \widetilde{\gamma} \circ \tau$. A vector on $T(M)$ is *horizontal* if it is tangent to an horizontal curve and *vertical* if it is tangent to a fiber. Locally, if (U, x^i), $i = 1, \ldots, m$, $m = \dim M$, is a local chart in $p \in M$, consider $(\tau^{-1}(U), x^i, y^i)$ a local chart on $T(M)$. If $\Gamma^k_{ij}(x)$ are the Christoffel symbols, then $\delta_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x) y^j \, \frac{\partial}{\partial y^k}$ in $\mathtt{u}$, $i = 1, \ldots, m$ span the space $H_{\mathtt{u}}T(M)$, while $\frac{\partial}{\partial y^i}$, $i = 1, \ldots, m$ span the vertical space $V_{\mathtt{u}}T(M)$.) We have obtained the horizontal (vertical) distribution HTM (VTM) and a direct sum decomposition

$$TTM = HTM \oplus VTM$$

of the tangent bundle of $T(M)$. If $X \in \chi(M)$, denote by X^H (and X^V, respectively) the horizontal lift (and the vertical lift, respectively) of X to $T(M)$.
If $\mathtt{u} \in T(M)$ then we consider the energy density in $\mathtt{u}$ on $T(M)$, namely $t = \frac{1}{2}\, g_{\tau(\mathtt{u})}(\mathtt{u}, \mathtt{u})$.
The Sasaki metric is defined uniquely by the following relations

$$\text{(1)} \qquad \begin{cases} g_S(X^H, Y^H) = g_S(X^V, Y^V) = g(X, Y) \circ \tau \\ g_S(X^H, Y^V) = 0, \end{cases}$$

for each $X, Y \in \chi(M)$.
On $T(M)$ we an also define an almost complex structure J_S by

$$\text{(2)} \qquad J_S X^H = X^V, \quad J_S X^V = -X^H, \forall X \in \chi(M).$$

It is known that $(T(M), J_S, g_S)$ is an almost Kählerian manifold. Moreover, the integrability of the almost complex structure J_S implies that (M, g) is locally flat (see e.g. [8]).

The Cheeger-Gromoll metric on $T(M)$ is given by

$$(3) \qquad \begin{cases} g_{CG(p,\mathtt{u})}(X^H, Y^H) = g_p(X,Y), \ g_{CG(p,\mathtt{u})}(X^H, Y^V) = 0 \\ g_{CG(p,\mathtt{u})}(X^V, Y^V) = \dfrac{1}{1+2t} \ (g_p(X,Y) + g_p(X,\mathtt{u})g_p(Y,\mathtt{u})) \end{cases}$$

for any vectors X and Y tangent to M.
Since the almost complex structure J_S is no longer compatible with the metric g_{CG}, one defines on $T(M)$ another almost complex structure J_{CG}, compatible with the Chegeer-Gromoll metric, by the formulas

$$(4) \qquad \begin{cases} J_{CG}X^H_{(p,\mathtt{u})} = \mathfrak{r}X^V - \dfrac{1}{1+\mathfrak{r}} \ g_p(X,\mathtt{u})\mathtt{u}^V \\ \\ J_{CG}X^V_{(p,\mathtt{u})} = -\dfrac{1}{\mathfrak{r}} \ X^H - \dfrac{1}{\mathfrak{r}(1+\mathfrak{r})} \ g_p(X,\mathtt{u})\mathtt{u}^H \end{cases}$$

where $\mathfrak{r} = \sqrt{1+2t}$ and $X \in T_p(M)$. Remark that $J_{CG}\mathtt{u}^H = \mathtt{u}^V$ and $J_{CG}\mathtt{u}^V = -\mathtt{u}^H$. We get an almost Hermitian manifold $(T(M), J_{CG}, g_{CG})$. Moreover, if we denote by Ω_{CG} the Kaehler 2-form (namely $\Omega_{CG}(U,V) = g_{CG}(U, J_{CG}V), \forall U,V \in \chi(T(M))$) it is quite easy to prove the following

Proposition 2.1 *We have*

$$(5) \qquad d\Omega_{CG} = \omega \wedge \Omega_{CG},$$

where $\omega \in \Lambda^1(T(M))$ *is defined by* $\omega_{(p,\mathtt{u})}(X^H) = 0$ *and* $\omega_{(p,\mathtt{u})}(X^V) = -\left(\frac{1}{\mathfrak{r}^2} + \frac{1}{1+\mathfrak{r}}\right) g_p(X,\mathtt{u})$, $X \in T_p(M)$.

Proof. A simple computation gives
$\Omega_{CG}(X^H, Y^H) = \Omega(X^V, Y^V) = 0$
$\Omega_{CG}(X^H, Y^V) = -\frac{1}{\mathfrak{r}}\left(g(X,Y) + \frac{1}{1+\mathfrak{r}}\ g(X,\mathtt{u})g(Y,\mathtt{u})\right)$.
(From now on we will omit the point $(p,\mathtt{u})$.)
The differential of Ω_{CG} is given by
$d\Omega_{CG}(X^H, Y^H, Z^H) = d\Omega_{CG}(X^H, Y^H, Z^V) = d\Omega_{CG}(X^V, Y^V, Z^V) = 0$
$d\Omega_{CG}(X^H, Y^V, Z^V) = \frac{1}{\mathfrak{r}}\left(\frac{1}{\mathfrak{r}^2} + \frac{1}{1+\mathfrak{r}}\right)[g(X,Y)g(Z,\mathtt{u}) - g(X,Z)g(Y,\mathtt{u})]$
for any $X, Y, Z \in \chi(M)$.
Hence the statement. ∎

Remark 2.2 The almost Hermitian manifold $(T(M), J_{CG}, g_{CG})$ is never almost Kaehlerian (i.e. $d\Omega_{CG} \neq 0$).

Finally, we obtain a necessary condition for the integrability of J_{CG} namely, the base manifold (M, g) should be locally Euclidian.

A general metric, let's call it g_A, is in fact a family of Riemannian metrics (depending on two parameters) and the Sasaki metric and the Cheeger-Gromoll metric are obtained by taking particular values for the two parameters. It is defined (cf. [7]) by the following formulas

$$
\left\{
\begin{array}{l}
g_{A(p,\mathtt{u})}(X^H, Y^H) = g_p(X, Y) \\
g_{A(p,\mathtt{u})}(X^H, Y^V) = 0 \\
g_{A(p,\mathtt{u})}(X^V, Y^V) = a(t)g_p(X, Y) + b(t)g_p(X, \mathtt{u})g_p(Y, \mathtt{u}),
\end{array}
\right. \tag{6}
$$

for all $X, Y \in \chi(M)$, where $a, b : [0, +\infty) \longrightarrow [0, +\infty)$ and $a > 0$. For $a = 1$ and $b = 0$ one obtains the Sasaki metric and for $a = b = \frac{1}{1+2t}$ one gets the Cheeger-Gromoll metric.

Proposition 2.3 *The metric defined above can be construct by using the method described by Musso and Tricerri in* [16].

PROOF. If we denote by $\theta = (\theta^1, \ldots, \theta^n)$ the canonical 1-form on $O(M)$ (namely, if $\mathbf{p} : O(M) \longrightarrow M$, θ is defined by $d\mathbf{p}_{\mathbf{u}}(X) = \theta^i(X)\mathtt{u}_i$, for $\mathbf{u} = (p, \mathtt{u}_1, \ldots, \mathtt{u}_n)$ and $X \in T_p(M)$) we have $R^*_{\mathbf{u}}(\theta^i) = (a^{-1})^i_h\theta^h$ for each $a \in O(n)$. The vertical distribution of ψ is defined by

$$\theta^i = 0, \quad D\zeta^i := d\zeta^i + \zeta^j\omega^i_j$$

where $\omega = (\omega^j_i)_{i,j}$ denotes the $so(n)$-valued connection 1-form defined by the Levi Civita connection of g. Since $R^*_a(\omega^i_j) = (a^{-1})^i_h\omega^h_k a^k_j$ we can also write $R^*_a(D\zeta^i) = (a^{-1})^i_h D\zeta^h$, for all $a \in O(n)$.
Consider now the following bilinear form on $O(M)$

$$
Q_A = \sum_{i=1}^{n}(\theta^i)^2 + a(\tfrac{1}{2}||\zeta||^2)\sum_{i=1}^{n}(D\zeta^i)^2 + b(\tfrac{1}{2}||\zeta||^2)\left(\sum_{i=1}^{n}\zeta^i D\zeta^i\right)^2. \tag{7}
$$

It is symmetric, semi-positive definite and basic. Moreover, since the following diagram

$$
\begin{array}{ccc}
O(M) \times \mathbf{R}^n & \xrightarrow{\ \psi\ } & T(M) \\
{\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle \tau} \\
O(M) & \xrightarrow{\ \mathbf{p}\ } & M
\end{array}
$$

commutes, we have $\psi^* g_A = Q_A$. (See for details §4 in [16].)

■

Again, we have to find an almost complex structure on $T(M)$, call it J_A, which is compatible with the metric g_A. Inspired from the previous cases we look for the almost complex structure J_A in the following way

$$
\left\{
\begin{array}{l}
J_A X^H_{(p,\mathtt{u})} = \alpha X^V + \beta g_p(X, \mathtt{u})\mathtt{u}^V \\
\\
J_A X^V_{(p,\mathtt{u})} = \gamma X^H + \rho g_p(X, \mathtt{u})\mathtt{u}^H
\end{array}
\right. \tag{8}
$$

where $X \in \chi(M)$ and α, β, γ and ρ are smooth functions on $T(M)$ which will be determined from $J_A^2 = -I$ and from the compatibility conditions with the metric g_A. Following the computations made in [7] we get first $\alpha = \pm\frac{1}{\sqrt{a}}$ and $\gamma = \mp\sqrt{a}$. Without lost of the generality we can take

$$\alpha = \frac{1}{\sqrt{a}} \quad \text{and} \quad \gamma = -\sqrt{a}.$$

Then one obtains

$$\beta = -\frac{1}{2t}\left(\frac{1}{\sqrt{a}} + \epsilon\frac{1}{\sqrt{a+2bt}}\right) \quad \text{and} \quad \rho = \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+bt}\right)$$

where $\epsilon = \pm 1$.

We have the almost complex structure J_A

$$(9) \qquad \begin{cases} J_A X^H = \dfrac{1}{\sqrt{a}} X^V - \dfrac{1}{2t}\left(\dfrac{1}{\sqrt{a}} + \epsilon\dfrac{1}{\sqrt{a+2bt}}\right) g(X,\mathtt{u})\mathtt{u}^V \\ J_A X^V = -\sqrt{a} X^H + \dfrac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right) g(X,\mathtt{u})\mathtt{u}^H \end{cases}$$

and the almost Hermitian manifold $(T(M), g_A, J_A)$.

Remark 2.4 In this general case J_A is defined on $T(M) \setminus 0$ (the bundle of non zero tangent vectors), but if we consider $\epsilon = -1$ the previous relations define J_A on all $T(M)$.

Remark 2.5 If we take $\epsilon = -1$, $a = 1$ and $b = 0$ we get the manifold $(T(M), g_S, J_S)$ and for $\epsilon = -1$, $a = b = \frac{1}{1+2t}$ we obtain the manifold $(T(M), g_{CG}, J_{CG})$.

If we denote by Ω_A the Kähler 2-form (i.e. $\Omega_A(U,V) = g_A(U, J_A V), \forall U, V \in \chi(T(M))$) one obtains

Proposition 2.6 (see [7]) *The almost Hermitian manifold $(T(M), g_A, J_A)$ is locally conformal almost Kählerian, that is*

$$(10) \qquad d\Omega_A = \omega \wedge \Omega_A$$

where ω is a closed and globally defined $1-$form on $T(M)$ given by

$$\omega(X^H) = 0 \quad \textit{and} \quad \omega(X^V) = \frac{1}{\sqrt{a}}\left(\frac{a'}{\sqrt{a}} + \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right)\right) g(X,\mathtt{u}).$$

As consequence one can state the following

Theorem 2.7 *The almost Hermitian manifold $(T(M), g_A, J_A)$ is almost Kählerian if and only if*

$$b(t) = \frac{2a'(t)\,(ta'(t) + a(t))}{a(t)}$$

and for $\epsilon = -1$, $a(t)$ is an increasing function, while for $\epsilon = +1$, $ta(t)$ is a decreasing function.

PROOF. The condition $\omega = 0$ is equivalent to

$$2ta'(t) + a(t) = -\epsilon\sqrt{a(t)} \cdot \sqrt{a(t) + 2tb(t)}\ .$$

From here, we get $b(t)$. Moreover it follows $(a(t)\sqrt{t}$ is a monotone function, namely it is increasing if $\epsilon = -1$ and decreasing for $\epsilon = +1$. Since $b(t)$ is positive we conclude

- if $\epsilon = -1$: $2a't + a > 0 \longleftrightarrow 2(a't + a) > a \longrightarrow a't + a > 0$
  $\longrightarrow a' > 0 \longrightarrow a$ increases (this implies $a\sqrt{t}$, at are also increasing functions);
- if $\epsilon = +1$: $2a't + a < 0 \longleftrightarrow a't + a < -a't \longrightarrow a't + a < 0$
  $\longrightarrow at$ decreases (this implies $a\sqrt{t}$, a are also decreasing functions).

■

The integrability of J_A.

In order to have an integrable structure J_A on $T(M)$ we have to compute the Nijenhuis tensor N_{J_A} of J_A and to ask that it vanishes identically.

For the integrability tensor N_{J_A} we have the following relations

$$(11)\quad \begin{cases} N_{J_A}(X^H, Y^H) = \left(-\frac{a'}{2a^2} + \frac{a+ta'}{a\sqrt{a}}\ A(t)\right)\left(g(X,\mathtt{u})Y - g(Y,\mathtt{u})X\right)^V + (R_{XY}\mathtt{u})^V \\ N_{J_A}(X^V, Y^V) = \left(-aR_{XY}\mathtt{u} + \sqrt{a}\ B(t)g(Y,\mathtt{u})R_{X\mathtt{u}}\mathtt{u} - \sqrt{a}\ B(t)R_{Y\mathtt{u}}\mathtt{u}\right)^V - \\ \qquad -\frac{1}{\sqrt{a}}\left(\frac{a'}{2\sqrt{a}} + B(t)\right)\left(g(Y,\mathtt{u})X - g(X,\mathtt{u})Y\right)^V \end{cases}$$

where $A(t) = \frac{1}{2t}\left(\frac{1}{\sqrt{a}} + \epsilon\frac{1}{\sqrt{a+2bt}}\right)$ and $B(t) = \frac{1}{2t}\left(\sqrt{a} + \epsilon\sqrt{a+2bt}\right)$. (The expression for $N_{J_A}(X^H, Y^V)$ is very complicated.)

Thus if J_A is integrable then

$$R_{XY}\mathtt{u} = \left(-\frac{a'}{2a^2} + \frac{a+ta'}{a\sqrt{a}}\ A(t)\right)\left(g(Y,\mathtt{u})X - g(X,\mathtt{u})Y\right)$$

for every $X, Y \in \chi(M)$ and for every point $\mathtt{u} \in T(M)$. It follows that M is a space form $M(c)$ (c is the constant sectional curvature of M). Consequently,

$$(12)\qquad -\frac{a'}{2a^2} + \frac{a+ta'}{a\sqrt{a}}\ A(t) = c.$$

So

✓ given $a(t)$ and c we can easily find $b(t)$;

✓ given $b(t)$ and c we have to solve an ODE in order to find $a(t)$;

✓ given $a(t)$ and $b(t)$ we have to check if c in (12) is constant.

Example 2.8

1. In Sasaki case $(a(t) = 1, b(t) = 0, \epsilon = -1)$ it follows $c = 0$ i.e. M is flat.

2. Looking for a locally conformal Kähler structure on $T(M)$ with the metric having $a(t) = b(t)$ we obtain

$$a(t) = b(t) = \frac{e^{2\sqrt{1+2t}}}{2\left(ce^{2\sqrt{1+2t}}t + (1 + t + \sqrt{1+2t})k\right)}$$

with k a positive real constant and c must be nonnegative.

Replacing the expression of the curvature R in $(11)_2$ we obtain again (12).

Question: *Can $(T(M), g_A, J_A)$ be a Kaehler manifold?*
If this happens then the base manifold is a space form $M(c)$ and the functions a and b satisfy

$$b = \frac{2a'(ta' + a)}{a} \quad \text{and} \tag{13}$$

$$a' = 2ca(2ta' + a). \tag{14}$$

If $c = 0$ (M is flat) then a is a positive constant and b vanishes.
If $c \neq 0$ the ODE (14) has general solutions

$$a_{1,2}(t) = \frac{1 \pm \sqrt{1 + \kappa t}}{4ct} \tag{15}$$

with κ a real constant. Taking into account that a and b are positive functions, using (13) one gets:
Case 1.

$$a = \frac{1 + \sqrt{1 + \kappa t}}{4ct} \text{ and } b = -\frac{\kappa(1 + \sqrt{1 + \kappa t})}{8ct(1 + \kappa t)}. \tag{16}$$

Here $c > 0$, $t > 0$, $\kappa < 0$, $t < -\frac{1}{\kappa}$ and $\epsilon = +1$.
Case 2.

$$a = -\frac{\kappa}{4c(1 + \sqrt{1 + \kappa t})} \text{ and } b = \frac{\kappa^2}{8c(1 + \kappa t)(1 + \sqrt{1 + \kappa t})}. \tag{17}$$

Here $\kappa c < 0$, $c < 0$ (then $\kappa > 0$), $t < -\frac{1}{\kappa}$ and $\epsilon = -1$.
Consider $B_\kappa = \left\{\mathtt{v} \in T(M) \ : \ g_{\tau(\mathtt{v})}(\mathtt{v}, \mathtt{v}) < -\frac{2}{\kappa}\right\}$ and $\dot{B}_\kappa = B_\kappa \setminus M$.

Theorem 2.9 *The manifolds B_κ in* Case 1 *and $\dot{B}_\kappa$ in* Case 2 *are Kaehler manifolds.*

Remark 2.10 In order to have a positive definite metric g_A, the necessary and sufficient conditions are $a > 0$ and $a + 2bt > 0$ ($b > 0$ is too strong). Hence, the previous theorem can be reformulated as:

1. $(T(M) \setminus M, g_A, J_A)$ where a and b are given by (16), $c > 0$ and $\epsilon = +1$ is a Kaehler manifold.

2. (B_κ, g_A, J_A) where a and b are given by (17), $c > 0$, $k < 0$ and $\epsilon = -1$ is a Kaehler manifold.

3. $(T(M), g_A, J_A)$ where a and b are given by (17), $c < 0$, $k > 0$ and $\epsilon = -1$ is a Kaehler manifold.

Now we give

Proposition 2.11 *Let (M, g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the metric g_A. Then, the corresponding Levi Civita connection $\tilde{\nabla}^A$ satisfies the following relations:*

$$(18) \quad \begin{cases} \tilde{\nabla}^A_{X^H} Y^H = (\nabla_X Y)^H - \frac{1}{2}\,(R_{XY}\mathtt{u})^V \\ \tilde{\nabla}^A_{X^H} Y^V = (\nabla_X Y)^V + \frac{a}{2}\,(R_{\mathtt{u}Y}X)^H \\ \tilde{\nabla}^A_{X^V} Y^H = \frac{a}{2}\,(R_{\mathtt{u}X}Y)^H \\ \tilde{\nabla}^A_{X^V} Y^V = \mathtt{L}\left(g(X,\mathtt{u})Y^V + g(Y,\mathtt{u})X^V\right) + \mathtt{M}g(X,Y)\mathtt{u}^V + \mathtt{N}g(X,\mathtt{u})g(Y,\mathtt{u})\mathtt{u}^V, \end{cases}$$

where $\mathtt{L} = \frac{a'(t)}{2a(t)}$, $\mathtt{M} = \frac{2b(t)-a'(t)}{2(a(t)+2tb(t))}$ *and* $\mathtt{N} = \frac{a(t)b'(t)-2a'(t)b(t)}{2a(t)(a(t)+2tb(t))}$.

Proof. The statement follows from Koszul formula making usual computations. ∎

Having determined Levi Civita connection, we can compute now the Riemannian curvature tensor $\tilde{R}^A$ on $T(M)$. We give

Proposition 2.12 *The curvature tensor is given by*

$$
(19)\quad \begin{cases}
\tilde{R}^A_{X^H Y^H} Z^H = (R_{XY}Z)^H + \frac{a}{4}\ [R_{\mathtt{u}R_{XZ}\mathtt{u}}Y - R_{\mathtt{u}R_{YZ}\mathtt{u}}X + 2R_{\mathtt{u}R_{XY}\mathtt{u}}Z]^H + \\
\qquad + \frac{1}{2}\ [(\nabla_Z R)_{XY}\mathtt{u}]^V \\
\tilde{R}^A_{X^H Y^H} Z^V = \left[R_{XY}Z + \frac{a}{4}\ (R_{YR_{\mathtt{u}Z}X}\mathtt{u} - R_{XR_{\mathtt{u}Z}Y}\mathtt{u})\right]^V + \mathtt{L}g(Z,\mathtt{u})(R_{XY}\mathtt{u})^V + \\
\qquad + \mathtt{M}g(R_{XY}\mathtt{u}, Z)\mathtt{u}^V + \frac{a}{2}\ [(\nabla_X R)_{\mathtt{u}Z}Y - (\nabla_Y R)_{\mathtt{u}Z}X]^H \\
\tilde{R}^A_{X^H Y^V} Z^H = \frac{a}{2}\,[(\nabla_X R)_{\mathtt{u}Y}Z]^H + \\
\qquad + \frac{1}{2}\left[R_{XZ}Y - \frac{a}{2}\ R_{XR_{\mathtt{u}Y}Z}\mathtt{u} + \mathtt{L}g(Y,\mathtt{u})R_{XZ}\mathtt{u} + \mathtt{M}g(R_{XZ}\mathtt{u}, Y)\mathtt{u}\right]^V \\
\tilde{R}^A_{X^H Y^V} Z^V = -\frac{a}{2}\ (R_{YZ}X)^H - \frac{a^2}{4}\ (R_{\mathtt{u}Y}R_{\mathtt{u}Z}X)^H + \\
\qquad + \frac{a'}{4}\left[g(Z,\mathtt{u})(R_{\mathtt{u}Y}X)^H - g(Y,\mathtt{u})(R_{\mathtt{u}Z}X)^H\right] \\
\tilde{R}^A_{X^V Y^V} Z^H = a(R_{XY}Z)^H + \frac{a'}{2}\,[g(X,\mathtt{u})R_{\mathtt{u}Y}Z - g(Y,\mathtt{u})R_{\mathtt{u}X}Z]^H + \\
\qquad + \frac{a^2}{4}\,[R_{\mathtt{u}X}R_{\mathtt{u}Y}Z - R_{\mathtt{u}Y}R_{\mathtt{u}X}Z]^H \\
\tilde{R}^A_{X^V Y^V} Z^V = F_1(t)g(Z,\mathtt{u})\left[g(X,\mathtt{u})Y^V - g(Y,\mathtt{u})X^V\right] + \\
\qquad + F_2(t)\left[g(X,Z)Y^V - g(Y,Z)X^V\right] + \\
\qquad + F_3(t)\,[g(X,Z)g(Y,\mathtt{u}) - g(Y,Z)g(X,\mathtt{u})]\,\mathtt{u}^V,
\end{cases}
$$

where $F_1 = \mathtt{L}' - \mathtt{L}^2 - \mathtt{N}(1+2t\mathtt{L})$, $F_2 = \mathtt{L} - \mathtt{M}(1+2t\mathtt{L})$ *and* $F_3 = \mathtt{N} - (\mathtt{M}' + \mathtt{M}^2 + 2t\mathtt{M}\mathtt{N})$.

Remark 2.13

(a) In the case of Sasaki metric we have:

$$\mathtt{L} = \mathtt{M} = \mathtt{N} = 0\ ,\ F_1 = F_2 = F_3 = 0.$$

(b) In the case of Cheeger Gromoll metric we have (see also [12, 22]):

$$
\begin{array}{l}
\mathtt{L} = -\frac{1}{\mathfrak{r}}\ ,\ \mathtt{M} = \frac{\mathfrak{r}+1}{\mathfrak{r}^2}\ ,\ \mathtt{N} = \frac{1}{\mathfrak{r}^2}\ ,\ \mathtt{L}' = \frac{2}{\mathfrak{r}^2}\ ,\ \mathtt{M}' = -\frac{2(\mathfrak{r}+2)}{\mathfrak{r}^3}\ ,\ 1 + 2t\mathtt{L} = \frac{1}{\mathfrak{r}} \\
F_1 = \frac{\mathfrak{r}-1}{\mathfrak{r}^3}\ ,\ F_2 = -\frac{\mathfrak{r}^2+\mathfrak{r}+1}{\mathfrak{r}^3}\ ,\ F_3 = \frac{\mathfrak{r}+2}{\mathfrak{r}^3}
\end{array}
$$

where $\mathfrak{r} = 1 + 2t$.

In the following let $\tilde{Q}^A(U,V)$ denote the square of the area of the parallelogram with sides U and V for $U, V \in \chi(T(M))$,

$$\tilde{Q}^A(U,V) = g_A(U,U)g_A(V,V) - g_A(U,V)^2.$$

We have

Lemma 2.14 *Let* $X, Y \in T_pM$ *be two orthonormal vectors. Then*

$$
(20)\quad \begin{cases}
\tilde{Q}^A(X^H, Y^H) = 1 \\
\tilde{Q}^A(X^H, Y^V) = a(t) + b(t)g(Y,\mathtt{u})^2 \\
\tilde{Q}^A(X^V, Y^V) = a(t)^2 + a(t)b(t)\left(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})\right).
\end{cases}
$$

We compute now the sectional curvature of the Riemannian manifold $(T(M), g_A)$, namely

$$\tilde{K}^A(U,V) = \frac{g_A(\tilde{R}^A_{UV}V, U)}{\tilde{Q}^A(U,V)}$$

for $U, V \in \chi(T(M))$.

Proposition 2.15 *If $X, Y \in T_pM$ are two orthonormal vectors, then*

$$(21) \quad \begin{cases} \tilde{K}^A(X^H, Y^H) = K(X,Y) - \frac{3a(t)}{4} \ |R_{XY}\mathtt{u}|^2 \\ \tilde{K}^A(X^H, Y^V) = \frac{a(t)^2}{4(a(t)+b(t)g(Y,\mathtt{u})^2} \ |R_{\mathtt{u}Y}X|^2 \\ \tilde{K}^A(X^V, Y^V) = -\frac{F_1(t)a(t)g(Y,\mathtt{u})^2 + F_2(t)\big(a(t)+b(t)g(X,\mathtt{u})^2\big) + F_3(t)(a(t)+2tb(t))g(X,\mathtt{u})^2}{a(t)^2 + a(t)b(t)(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2)} \ . \end{cases}$$

where $K(X,Y)$ is the sectional curvature of the plane spanned by X and Y. Here $|\cdot|$ denotes the norm of the vector with respect to the metric g (in a point).

PROOF. Calculations using (19) show that

$g_A(\tilde{R}^A_{X^HY^H}Y^H, X^H) = g(R_{XY}Y, X) - \frac{3a}{4}\,|R_{XY}\mathtt{u}|^2$

$g_A(\tilde{R}^A_{X^HY^V}Y^V, X^H) = \frac{a^2}{4}\ |R_{\mathtt{u}Y}X|^2$

$g_A(\tilde{R}^A_{X^VY^V}Y^V, X^V) = -aF_1(t)g(Y,\mathtt{u})^2 - F_2(t)(a + bg(X,\mathtt{u})^2) - F_3(t)g(X,\mathtt{u})^2(a+2tb).$

Hence the conclusion. ■

Moreover, if M has constant sectional curvature c, then $|R_{XY}\mathtt{u}|^2 = c^2\big(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2\big)$ and $|R_{\mathtt{u}Y}X|^2 = c^2 g(\mathtt{u}, X)^2$ for any orthonormal $X, Y \in T_p(M)$. Then we have

$\tilde{K}^A(X^H, Y^H) = c - \frac{3a(t)c^2}{4}\ \big(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2\big)$

$\tilde{K}^A(X^H, Y^V) = \frac{a(t)^2c^2g(\mathtt{u},X)^2}{4(a(t)+b(t)g(Y,\mathtt{u})^2}$.

Following an idea from [22] we are interested to study the sign of K^A. We have

(1) If $c < 0$ then $\tilde{K}^A(X^H, Y^H) < 0$, if $c = 0$ then $\tilde{K}^A(X^H, Y^H) = 0$ and if $c > 0$ then

$$\begin{cases} \tilde{K}^A(X^H, Y^H) > 0, \ \text{for } c \in (0, \mathtt{C}_1) \\ \tilde{K}^A(X^H, Y^H) = 0, \ \text{for } c = \mathtt{C}_1 \\ \tilde{K}^A(X^H, Y^H) < 0, \ \text{for } c > \mathtt{C}_1 \end{cases} , \text{ where } \mathtt{C}_1 = \frac{4}{3a(t)(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2)}.$$

Moreover the maximum value for $\tilde{K}^A(X^H, Y^H)$ is $\tilde{K}^A_{\max} = \frac{\mathtt{C}_1}{4}$.

It will be better to have a constant $\mathtt{C} > 0$ (which does not depend on X, Y and t) in the place of $\mathtt{C}_1$ so, we are looking for $\mathtt{C} < \frac{4}{3a(t)(g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2)}$ for all X, Y and for any point $\mathtt{u}$ of $T(M)$. We know that $g(X,\mathtt{u})^2 + g(Y,\mathtt{u})^2 \leq 2t$ for any X, Y orthonormal, so, to have this, it is sufficient for the function a to verify

$$a(t) \leq \frac{2}{3\mathtt{C}t}$$

for any $t > 0$. Remark that in the case of Cheeger Gromoll metric this fact occurs with $\mathtt{C} = \frac{4}{3}$ which is the best constant.

Another remark is that the maximum value for $\tilde{K}^A(X^H, Y^H)$ can be attained: for example take $X = v$ with $|v| = 1$, $Y = \frac{1}{\sqrt{2t - g(u,v)^2}}\ (u - g(u,v)v)$, $a = \frac{2}{3\mathtt{C}t + o}$, with o a positive constant. If we take $\mathtt{u}$ such that $t = \frac{o}{3}$ and $c = \frac{1+\mathtt{C}}{2}$ one gets a maximum value $\frac{1+\mathtt{C}}{4}$.

(2) $\tilde{K}^A(X^H, Y^V)$ is non negative for all $c \neq 0$ and vanishes for $c = 0$.

(3) Unfortunately for $\tilde{K}^A(X^V, Y^V)$ we cannot say anything yet.

Denote by $T_0(M) = T(M) \setminus \mathbf{0}$ the tangent bundle of non-zero vectors tangent to M. For a given point $(p, \mathtt{u}) \in T_0(M)$ consider an orthonormal basis $\{e_i\}_{i=\overline{1,m}}$ for the tangent space $T_p(M)$ of M such that $e_1 = \frac{u}{|u|}$. Consider on $T_{(p,\mathtt{u})}T(M)$ the following vectors

$$\left\{ \begin{array}{l} E_i = e_i^H,\ i = 1, \ldots, m \\ \\ E_{m+1} = \frac{1}{\sqrt{a+2tb}}\ e_1^V \\ \\ E_{m+k} = \frac{1}{\sqrt{a}}\ e_k^V,\ k = 2, \ldots, m. \end{array} \right. \tag{22}$$

It is easy to check that $\{E_1, \ldots, E_{2m}\}$ is an orthonormal basis in $T_{(p,\mathtt{u})}T(M)$ (with respect to the metric g_A). We will rewrite the expressions of the sectional curvature $\tilde{K}^A$ in terms of this basis. We have

$$\left\{ \begin{array}{ll} \tilde{K}^A(E_i, E_j) = K(e_i, e_j) - \frac{3a(t)}{4}\ \left|R_{e_i e_j}\mathtt{u}\right|^2, & i, j = 1, \ldots, m \\ \tilde{K}^A(E_i, E_{m+1}) = 0, & i = 1, \ldots, m \\ \tilde{K}^A(E_i, E_{m+k}) = \frac{1}{4}\left|R_{\mathtt{u} e_k} e_i\right|^2, & i = 1, \ldots, m,\ k = 2, \ldots, m \\ \tilde{K}^A(E_{m+1} E_{m+k}) = -\frac{F_2 + 2tF_3}{a(t)}, & k = 2, \ldots, m \\ \tilde{K}^A(E_{m+k} E_{m+l}) = -\frac{F_2}{a(t)} & k, l = 2, \ldots, m. \end{array} \right. \tag{23}$$

Can we have constant sectional curvature $\tilde{c}$ on $T(M)$?

If this happens, then it must be 0, so $T(M)$ is flat. First, one gets easily that M is locally Euclidean. Then, we should also have $F_2(t) = 0$ and $F_3(t) = 0$ for any t. It follows $\mathtt{M} = \frac{\mathtt{L}}{1+2t\mathtt{L}}$ and $\mathtt{N} = \frac{\mathtt{L}' - \mathtt{L}^2}{1+2t\mathtt{L}}$. (Hence $F_1(t) = 0$.) These equalities yield two ordinary differential equations (involving a and b), namely:

($\diamond$) $t(a')^2 + 2aa' - 2ab = 0$

($\diamond\diamond$) $\dfrac{ab' - 2a'b}{a + 2tb} = \dfrac{2a''a - 3(a')^2}{2(a + ta')}$.

A simple computation shows that ($\diamond\diamond$) is consequence of ($\diamond$). So, we must have

$$b(t) = a'(t)\left(1 + \frac{ta'(t)}{2a(t)}\right)\ . \tag{24}$$

It is interesting to point our attention to two special cases:

Case (i): $b(t) = \mathtt{k}a'(t)$, where $\mathtt{k}$ is a real constant.

If $a' = 0$ then $b = 0$ and a is constant, so, g_A is homothetic to Sasaki metric.

If $a' \neq 0$ then $a(t) = a_0 t^{2(\mathtt{k}-1)}$, ($\mathtt{k} > 1$ or $\mathtt{k} \leq 0$, $a_0 > 0$) and in this case we have to consider $T_0(M)$.

Case (ii): $b(t) = a(t)$. We obtain $\frac{a'}{a} = \frac{-1 \pm \sqrt{1+2t}}{t}$ which gives

$$a(t) = a_0 \frac{e^{2\sqrt{1+2t}}}{(1+\sqrt{1+2t})^2},\ a_0 > 0 \qquad (*)$$

or,

$$a(t) = a_0 \frac{e^{-2\sqrt{1+2t}}}{1+t-\sqrt{1+2t}} \text{ and in this case we have to deal with non zero vectors.}$$

Remark 2.16 The manifold $T(M)$ equipped with the Cheeger Gromoll has non constant sectional curvature.

Putting $a_0 = 1$ in $(*)$, we can state the following

Proposition 2.17 *Consider g_1 on $T(M)$ given by*

$$(25) \qquad \begin{cases} g_1(X^H, Y^H) = g(X,Y),\ g_1(X^H, Y^V) = 0 \\ g_1(X^V, Y^V) = \frac{e^{2\sqrt{1+2t}}}{(1+\sqrt{1+2t})^2}\ \left(g(X,Y) + g(X,\mathtt{u})g(Y,\mathtt{u})\right) \end{cases}$$

The manifold $(T(M), g_1)$ is flat.

Proof. For the metric g_1 given above we have

$$\mathtt{L} = \frac{1}{1+\sqrt{1+2t}}\ ,\mathtt{M} = \frac{1}{1+2t+\sqrt{1+2t}}\ ,\ \mathtt{N} = \frac{1}{(1+2t)(1+\sqrt{1+2t})}$$

$$F_1 = 0,\ F_2 = 0,\ F_3 = 0.$$

Thus ${}^1\tilde{R} = 0$. Here ${}^1\tilde{R}$ is the Riemannian curvature of the metric g_1.

∎

We can now compare the scalar curvatures of (M,g) and $(T(M), g_A)$.

Proposition 2.18 *Let (M,g) be a Riemannian manifold and endow the tangent bundle $T(M)$ with the metric g_A. Let* scal *and* $\widetilde{\mathrm{scal}}^A$ *be the scalar curvatures of g and g_A respectively. The following relation holds:*

$$(26) \qquad \widetilde{\mathrm{scal}}^A = \mathrm{scal} + \frac{2-3a}{2} \sum_{i<j} \left|R_{e_ie_j}\mathtt{u}\right|^2 + \frac{1-m}{a}\ \left(mF_2 + 4tF_3\right),$$

where $\{e_i\}_{i=1,\ldots,m}$ is a local orthonormal frame on $T(M)$.

Proof. Using the fact $\mathrm{scal} = \sum_{i\neq j} K(e_i, e_j)$ and the formula $\sum_{i,j=1}^{m} |R_{e_i\mathtt{u}}e_j|^2 = \sum_{i,j=1}^{m} \left|R_{e_ie_j}\mathtt{u}\right|^2$ we get the conclusion.

∎

Corollary 2.19 (see e.g. [16]) *If (M, g) is a Riemannian manifolds and $T(M)$ is its tangent bundle equipped with the Sasaki metric g_S. Then $(T(M), g_S)$ has constant scalar curvature if and only if the base manifold M is flat. In this case $T(M)$ is also flat.*

Corollary 2.20 *Let $(M(c), g)$ be a space form and equip $T(M)$ with the metric g_A. Then*

$$\widetilde{\mathrm{scal}}^A = (m-1)\left[mc + t(2-3a)c^2 - \frac{mF_2 + 4tF_3}{a}\right] .$$

Corollary 2.21 (see e.g. [22]) *If (M, g) has constant sectional curvature c, then its tangent bundle $T(M)$ endowed with the Chegeer Gromoll metric g_{CG} is not (curvature) homogeneous.*

Could we find functions a and b such that $T(M)$ equipped with the metric g_A has constant scalar curvature?

First of all consider $a = k$ (a positive real constant). After the computations we obtain that $b(t)$ should satisfy the following ODE.

$$(27) \qquad c^2\ (2-3\ k)\ k^3\ t + b(t)\ \big(k\ \big(m + 4\ c^2\ (2-3\ k)\ k\ t^2\big) +$$
$$+2\ t\ \big(-2+m+2\ c^2\ (2-3\ k)\ k\ t^2\big)\ b(t)\big) + 2\ k\ t\ b'(t) = constant.$$

Let us to give some examples:

Example 2.22 If we take $a = \frac{2}{3}$ and $b = 0$ we obtain that $(T(M), g_A)$ has constant scalar curvature $\widetilde{scal}^A = m(m-1)c$.

Example 2.23 For $a = k = \frac{2}{3}$ if the constant in (27) vanishes, then (if $b \neq 0$) we can integrate the ODE obtaining

$$b = e^{-\frac{3}{2}\left[(m-2)t + \frac{m \log t}{3}\right]}.$$

Then, $(T_0(M), g_A)$ has constant scalar curvature $\widetilde{scal}^A = m(m-1)c$.

Example 2.24 If we take $a = k \in \left(0, \frac{2}{3}\right)$ and $b = \frac{c^2k^2(3k-2)t}{2+m+2c^2(2-3k)kt^2}$ then $(T(M), g_A)$ has constant scalar curvature $\widetilde{scal}^A = m(m-1)c$.

Example 2.25 If we take $a = 1$ and $b(t) = \frac{k\ (2+m)+c^2\ m\ t}{m\ (2+m)-2\ k\ (2+m)\ t-2\ c^2\ m\ t^2}$ then $(T_{t_2}(M), g_A)$ (i.e. the bundle of tangent vectors having length greater than the positive solution t_2 of the equation $m\ (2+m) - 2\ k\ (2+m)\ t - 2\ c^2\ m\ t^2 = 0$) has constant scalar curvature $\widetilde{scal}^A = (m-1)(mc+k)$, where k is a real constant.

If we consider $b(t) = a(t)$ (as in the case of Cheeger Gromoll metric) then $(T(M), g_A)$ has constant scalar curvature if and only if a satisfies the following ODE:

$$-\frac{1}{2\ (1+2\ t)^2\ a(t)^3}\Big(-2\ (m+2\ (-2+m)\ t)\ a(t)^2 - 4\ t\ (c+2\ c\ t)^2\ a(t)^3 + \\ +6\ t\ (c+2\ c\ t)^2\ a(t)^4 + (-6+m)\ t\ (1+2\ t)\ a'(t)^2 + \\ +2\ a(t)\ ((m+2\ (-1+m)\ t)\ a'(t) + 2\ t\ (1+2\ t)\ a''(t))\Big) = \text{constant}$$

which seems to be very complicated to solve.

3 The tangent spheres bundle

3.1 T_rM as hypersurface in $(T(M), g_S, J_S)$

Let $T_rM = \{v \in T(M)\ :\ g_{\tau(v)}(v,v) = r^2$, with $r \in (0, +\infty)$ and let $\pi_r : T_rM \longrightarrow M$ be the canonical projection. If we denote by (x^i, v^i) local coordinates on $T(M)$ then, T_rM can be expressed (locally) as

$$g_{\star\star} = r^2, \quad \text{where} \quad g_{i\star} = g_{ij}v^j$$

(g_{ij} are the components of the metric g in the local chart (U, x^i)).
Thus, T_rM is a (real) hypersurface in $T(M)$.
We know a generator system for T_rM, namely $\delta_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}(x)v^j\frac{\partial}{\partial v^k}$ and $Z_i = \frac{\partial}{\partial v^i} - \frac{1}{r^2}g_{i\star}v^k\frac{\partial}{\partial v^k}$, $i = 1, \ldots, m$. (Remark that $\{Z_i\}_{i=1,\ldots,m}$ are not independent (e.g. $v^iZ_i = 0$), but in any point of T_rM they span an $(m-1)$-dimensional subspace of TT_rM.)
Denote by G_r the induced metric from $(T(M), g_S)$; then we can write

$$G_r(\delta_i, \delta_j) = g_{ij},\ G_r(\delta_i, Z_j) = 0,\ G_r(Z_i, Z_j) = g_{ij} - \frac{1}{r^2}\ g_{i\star}g_{j\star}. \tag{28}$$

Since the ambient is almost Hermitian, we can construct an almost contact (metric) structure $(\varphi_r, \xi_r, \eta_r, G_r)$ on T_rM (i.e. $\varphi \in \mathcal{T}^1_1(T(M))$, $\xi_r \in \chi(T(M))$, $\eta_r \in \Lambda^1(T(M))$ verifying $\varphi_r^2 = -I + \eta_r \circ \xi_r$, $\varphi_r\xi_r = 0$, $\eta_r \circ \varphi_r = 0$, $\eta_r(\xi_r) = 1$ and $G_r(\varphi_rU, \varphi_rV) = G_r(U,V) - \eta_r(U)\eta_r(V)$ (the compatibility condition)).
The unit normal of T_rM is $N_r = \frac{1}{r}\ v^i\frac{\partial}{\partial v^i}$. We put

$$\xi_r = -J_SN_r = \frac{1}{r}\ v^i\delta_i \tag{29}$$

(which is unitary and tangent to T_rM).
Next, if U is tangent to the hypersurface T_rM, define φ_r and η_r by

$$\varphi_rU = tan(J_SU) \quad \text{and} \quad \eta_r(U)N_r = nor(J_SU)$$

where $tan : TT(M) \longrightarrow TT_rM$ and $nor : TT(M) \longrightarrow N(T_rM)$ are the usual projection operators. (Here $N(T_rM)$ is the normal bundle.)

Hence

$$(30)\qquad \begin{cases} \varphi_r \delta_i = Z_i, \ \varphi_r Z_i = -\left(\delta_i^h - \frac{1}{r^2}\, g_{i\star} v^h\right) \delta_h \\ \eta_r(\delta_i) = \frac{1}{r}\, g_{i\star}, \ \eta_r(Z_i) = 0. \end{cases}$$

Proposition 3.1 *$(T_r M, \varphi_r, \xi_r, \eta_r, G_r)$ is an almost contact metric manifold.*

We have also

$$d\eta_r'(\delta_i, \delta_j) = 0, \ d\eta_r'(Z_i, Z_j) = 0, \ d\eta_r'(\delta_i, Z_j) = -\frac{1}{2}\left(g_{ij} - \frac{1}{r^2}\, g_{i\star} g_{j\star}\right).$$

(Here we use the formula $d\omega(X,Y) = \frac{1}{2}\ X\omega(Y) - Y\omega(X) - \omega([X,Y])$, for ω a smooth 1-form and for any pair X, Y of vector fields on a manifold.)
In order to have a contact metric structure on $T_r(M)$ (i.e. $d\eta_r(U,V) = G_r(U, \varphi_r V)$, $\forall$ $U, V \in \chi(T_r(M))$), we have to modify the almost contact structure in the following way (see e.g. [8]):

$$\varphi_r^{\text{new}} = \varphi_r^{\text{old}}, \ \xi_r^{\text{new}} = 2r\xi_r^{\text{old}}, \ \eta_r^{\text{new}} = \frac{1}{2r}\ \eta_r^{\text{old}} \text{ and } G_r^{\text{new}} = \frac{1}{4r^2}\ G_r^{\text{old}}.$$

3.2 T_1M as hypersurface in $(T(M), g_A, J_A)$

Let $T_1 M = \{\mathtt{u} \in T(M) \ : \ g_{\tau(\mathtt{u})}(\mathtt{u}, \mathtt{u}) = 1$ and let $\pi : T_1 M \longrightarrow M$ be the canonical projection. If we denote by (x^i, y^i) local coordinates on $T(M)$, then $T_1 M$ can be expressed as

$$g_{00} = 1, \text{ where } g_{i0} = g_{ij} y^j.$$

We know, as above, a generator system for $T_1 M$, namely δ_i and $Y_i = \frac{\partial}{\partial y^i} - g_{i0} y^h \frac{\partial}{\partial y^h}$. Denote by G_A the induced metric from $(T(M), g_A)$ on $T_1 M$; we have

$$(31)\qquad G_A(\delta_i, \delta_j) = g_{ij}, \ G_A(\delta_i, Y_j) = 0, \ G_A(Y_i, Y_j) = a\,(g_{ij} - g_{i0} g_{j0})$$

where a is a real positive constant.
Since the ambient manifold is almost Hermitian, we can construct on $T_1 M$ an almost contact metric structure $(\varphi_A, \xi_A, \eta_A, G_A)$, by using the same method as in previous subsection. We obtain

$$(32)\qquad \begin{cases} \varphi_A \delta_i = \frac{1}{\sqrt{a}}\ Y_i, \ \varphi_A Y_i = -\sqrt{a}\,\left(\delta_i^h - g_{i0} y^h\right) \delta_h \\ \eta_A(\delta_i) = -\epsilon g_{i0}, \ \eta_A(Y_i) = 0, \ \xi_A = -\epsilon y^k \delta_k. \end{cases}$$

We have

Proposition 3.2 *$(T_1 M, \varphi_A, \xi_A, \eta_A, G_A)$ is an almost contact metric manifold.*

We have the following expression for the differential $d\eta_A$:

$$d\eta_A(\delta_i, \delta_j) = 0, \ d\eta_A(Y_i, Y_j) = 0, \ d\eta_A(\delta_i, Y_j) = \frac{\epsilon}{2}\ (g_{ij} - g_{i0} g_{j0}).$$

Similarly to the previous case, in order to have a contact metric structure on $T_1(M)$ we put

$$\varphi_A^{\text{new}} = \varphi_A^{\text{old}}, \ \xi_A^{\text{new}} = -2\epsilon\sqrt{a}\ \xi_A^{\text{old}}, \ \eta_A^{\text{new}} = -\frac{\epsilon}{2\sqrt{a}}\ \eta_A^{\text{old}}, \ G_A^{\text{new}} = \frac{1}{4a}\ G_A^{\text{old}}.$$

3.3 The isometry

Consider the smooth map $\tilde{F} : T(M) \longrightarrow T(M)$ defined by $\tilde{F}(p, \mathtt{u}) = (p, r\mathtt{u})$. We will omit in the following the point p. Remark that if $\mathtt{u}$ is of unit length, then $r\mathtt{u}$ has the length r, so, $\tilde{F}$ restricts to a smooth map $F : (T_1M, \varphi_A, \xi_A, \eta_A, G_A) \longrightarrow (T_rM, \varphi_r, \xi_r, \eta_r, G_r)$.
We have

Theorem 3.3 *The Riemannian manifolds (T_1M, G_A) and (T_rM, G_r) are isometric for $r = \sqrt{a}$.*

Proof. It is an easy computation to prove that $dF(\delta_i) = \delta_i$ and $dF(Y_i) = rZ_i$. Consequently, we have

$$G_r(dF(\delta_i), F(\delta_j)) = G_r(\delta_i, \delta_j) = \frac{1}{4r^2}\ g_{ij} = \frac{a}{r^2}\ G_A(\delta_i, \delta_j)$$

and

$$G_r(dF(Y_i), dF(Y_j)) = G_r(Z_i, Z_j) = \frac{1}{4}\ \left(g_{ij} - \frac{1}{r^2} g_{i\star} g_{j\star}\right) = \frac{1}{4}\ (g_{ij} - g_{i0}g_{j0}) = G_A(Y_i, Y_j).$$

Hence the conclusion.

■

From the contact point of view we can state

Theorem 3.4 *F is a (φ_A, φ_r) map between almost contact manifolds (i.e. $dF \circ \varphi_A = \varphi_r \circ dF$) if and only if $r = \sqrt{a}$.*

Proof. One has

$$dF(\varphi_A \delta_i) = -dF(\frac{1}{\sqrt{a}}\ Y_i) = -\frac{r}{\sqrt{a}}\ Z_i = \frac{r}{\sqrt{a}}\ \varphi_r \delta_i = \frac{r}{\sqrt{a}}\ \varphi_r dF(\delta_i)$$

$$dF(\varphi_A Y_i) = dF(\sqrt{a}\left(\delta_i^h - g_{i0} y^h\right) \delta_h) = \sqrt{a}\left(\delta_i^h - g_{i0} y^h\right) \delta_h = \sqrt{a}\left(\delta_i^h - \frac{1}{r^2}\ g_{i\star} v^h\right) \delta_h =$$
$$= \sqrt{a} \varphi_r Z_i = \frac{\sqrt{a}}{r}\ \varphi_r dF(Y_i)$$

From here, we get the statement. ■

Remark 3.5 The characteristic vector field ξ_A is mapped to the characteristic vector field ξ_r.

3.4 Some properties of $(T_1(M), \varphi_A, \eta_A, \xi_A, G_A)$ as contact manifold

We have already seen that $(T_1(M), \varphi_A, \eta_A, \xi_A, G_A)$ is a contact manifold.
Denote by ∇^A the Levi Civita connection on $T_1(M)$ corresponding to the metric G_A.

Proposition 3.6 *We have*

$$\left\{ \begin{array}{l} \nabla^A_{\delta_i} \delta_j = \Gamma^k_{ij} \delta_k - \frac{1}{2}\ R^k_{0ij} Y_k,\ \ \nabla^A_{Y_i} \delta_j = \frac{a}{2}\ R^k_{j0i} \delta_k \\ \\ \nabla^A_{\delta_i} Y_j = \Gamma^k_{ij} Y_k + \frac{a}{2}\ R^k_{i0j} \delta_k,\ \ \nabla^A_{Y_i} Y_j = -g_{j0} Y_i \end{array} \right. \tag{33}$$

where R^h_{kij} are the local components of the Riemannian curvature on the base manifold M and "0" denotes the contraction with $\mathtt{u}$, e.g. $R^k_{0ij} = R^k_{lij} y^l$.

If we claim that $(T_1(M), \varphi_A, \eta_A, \xi_A, G_A)$ to be a K-contact manifold, i.e. $\nabla^A_U \xi_A = -\varphi_A U$ for all $U \in \chi(T_1(M))$ (see e.g. [8]), we can state

Theorem 3.7 *The contact metric structure $(\varphi_A, \xi_A, \eta_A, G_A)$ on $T_1(M)$ is K-contact if and only if the base manifold (M, g) has positive constant sectional curvature $\frac{1}{a}$. In this case $T_1(M)$ becomes a Sasakian manifold.*

Proof. One can compute

$$\nabla^A_{\delta_i} \xi_A = -\sqrt{a}\ R^k_{0i0} Y_k$$

$$\nabla^A_{Y_i} \xi_A = -\sqrt{a}\left[a R^k_{0i0} - 2\left(\delta^k_i - g_{i0} y^k\right)\right] \delta_k.$$

In order to have a K-contact manifold the following relations must occur

$$(34)\qquad \begin{cases} \left(R^k_{0i0} - \frac{1}{a}\delta^k_i\right) Y_k = 0 \\ \\ \left[a R^k_{0i0} - \left(\delta^k_i - g_{i0} y^k\right)\right] \delta_k = 0. \end{cases}$$

Since $\{\delta_k\}_{k=1,\dots,m}$ are linearly independent, it follows

$$(35)\qquad R^k_{0i0} = \frac{1}{a}\left(\delta^k_i - g_{i0} y^k\right).$$

Remark that (35) implies the first condition in (34).
We obtain by symmetrization

$$(36)\qquad R^k_{lij} + R^k_{jil} = \frac{1}{a}\left(2\delta^k_i g_{jl} - g_{ij}\delta^k_l - g_{il}\delta^k_j\right).$$

Using first Bianchi identity one gets

$$(37)\qquad 2R^k_{lij} + R^k_{ijl} = \frac{1}{a}\left(2\delta^k_i g_{jl} - g_{ij}\delta^k_l - g_{il}\delta^k_j\right).$$

Writting now (36) after a cyclic permutation and substracting obtained formula from (37) one has

$$(38)\qquad R^k_{lij} = \frac{1}{a}\left(g_{jl}\delta^k_i - g_{il}\delta^k_j\right)$$

which shows that the base manifold M is a real space form $M(\frac{1}{a})$.

Conversely, suppose that M has constant sectional curvature $\frac{1}{a}$. Then the curvature can be written as in (38). Let's compute the covariant derivative of φ_A. We have

$$\left(\nabla^A_{\delta_i}\varphi_A\right)\delta_j = \tfrac{1}{2\sqrt{a}}\left(g_{ij} y^h - g_{j0}\delta^h_i\right)\delta_h$$

$$\left(\nabla^A_{\delta_i}\varphi_A\right) Y_j = 0$$

$$\left(\nabla^A_{Y_i}\varphi_A\right)\delta_j = -\tfrac{1}{2\sqrt{a}}\ g_{j0} y^i$$

$$\left(\nabla^A_{Y_i}\varphi_A\right) Y_j = \tfrac{\sqrt{a}}{2}\left(g_{ij} - g_{i0} g_{j0}\right) y^h \delta_h$$

which shows that

$$\left(\nabla^A_U \varphi_A\right) V = G_A(U,V)\xi_A - \eta_A(V)U$$

for all $U,V \in \chi(T_1(M))$. This relation characterizes Sasakian manifolds among the almost contact metric manifolds.

This ends the proof. ■

Remark 3.8 The tensor field φ_A is never parallel. The manifold $(T_1(M), \varphi_A, \xi_A, \eta_A, G_A)$ is never cosymplectic.

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