$CR-$structures on the Unit Cotangent Bundle and Bochner Type Tensor

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An.St. ale Univ.'Al.I.Cuza' din Iasi, Tomul XLIV, s.I.a, Matematica 1998, f1, 125-136

Introduction

Consider a real submanifold of a complex manifold. The possibility to introduce complex local coordinates on the surrounding manifold, equivalent to a condition of complex involutivity, induces on the real submanifold a certain condition of complex involutivity. However, if the surrounding manifold is only an almost complex manifold, the condition of complex involutivity is not likely to be satisfied on the submanifold.

In [2] it was introduced a torsion free connection adapted to an almost contact structure associated with a pseudoconvex $CR-$structure and, using the curvature tensor of this connection, the authors obtained a Bochner type tensor associated with the $CR-$structure. In this note we show that in the case of the unit cotangent bundle of a Riemannian space of constant sectional curvature, this tensor vanishes if and only if the Riemannian manifold is of constant curvature -1.

The author wishes to express his gratitude to Prof. V.Oproiu for the helpful hints given during the work.

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*Partially supported by the Grant 228 / 1996 - Ministerul Invăţământului; Universitatea "Al.I.Cuza" Iaşi

†Communicated at the Conference "Zilele Universităţii "Al.I.Cuza" Iaşi", October 25-30, 1996

1 The Pseudoconvex $CR-$structure and the Associated Almost Contact Structure

Definition. A $CR-$structure on the differentiable manifold M, is defined by a complex vector subbundle $H(M)$ in the complexification T^cM of the tangent bundle of M so that:

(i) $A(M) \cap H(M) = \{0\}$ where $A(M) = \overline{H(M)}$.

(ii) $H(M)$ is involutive, i.e. for two $H(M)$-valued complex vector fields Z, W, the bracket $[Z, W]$ is $H(M)$-valued too.

The fundamental example of $CR-$structures is supplied by the real submanifolds of the complex manifolds.

Let $\widetilde{M}$ be a complex manifold of complex dimension m and let $M \subset \widetilde{M}$ be a real submanifold. The tangent space T_pM in a fixed point $p \in M$, is a real subspace in the tangent space $T_p\widetilde{M}$, which is a complex vector space. Then we may define:

$$H(T_pM) = \{X_p \in T_pM \mid iX_p \in T_pM\}$$

Remark: Generally, the dimension of $H(T_pM)$ may change when $p \in M$ is changed.

Assuming that dim $(H(T_pM)) = k$, a constant number independent of $p \in M$, then $H(TM) = \bigcup\limits_{p\in M} H(T_pM)$ is a complex vector bundle on M and it is defined, naturally, a $CR-$structure on M. When M is a real hypersurface in a complex manifold, it is not necessary the assumption for the fibre dimension of $H(TM)$ to be constant in order to have an induced $CR-$structure. The involutivity condition of $H(TM)$ is obtained from the property of $\widetilde{M}$ to be a complex manifold. Returning to the case of an arbitrary $CR-$structure and denoting by J the operator on the decomplexification $R(M)$ of $H(M)$, corresponding to the multiplication by i, the condition of complex involutivity can be expressed by:

(i) $[X, Y] - [JX, JY] \in \Gamma(R(M))$; $\forall X, Y \in \Gamma(R(M))$

(ii) $N_J(X, Y) = [JX, JY] - [X, Y] - J\{[JX, Y] + [X, JY]\} = 0$; $\forall X, Y \in \Gamma(R(M))$

where $\Gamma(R(M))$ denotes the set of cross-sections of $R(M)$.

Remark: Since (i) it follows, putting JY in the place of Y and using the property $J^2 = -I_{R(M)}$, that:

$$[JX, Y] + [X, JY] \in \Gamma(R(M)) \ ; \ \forall X, Y \in \Gamma(R(M))$$

thus the operations in the expression of N_J in (ii) are well defined.

Let M be a $(2n+1)-$ dimensional and oriented manifold endowed with the CR-structure of hypersurface type $(M, H(M))$ and let η be a $1-$form having the decomplexification $R(M)$ of $H(M)$ as its null bundle, i.e. $R(M) = \{X \in TM | \eta(X) = 0\}$.

The Levi form of the CR-structure of hypersurface type $(M, H(M))$ is given by the complex valued form, denoted by L and defined by:

$$L(Z, W) = d\eta(X, JY) - d\eta(JX, Y) - i\{d\eta(X, Y) + d\eta(JX, JY)\},$$

where:

$$Z = X - iJX \ , \ W = Y - iJY \ , \ X, Y \in \Gamma(R(M)).$$

Definition: A $CR-$structure of hypersurface type $(M, H(M))$ is pseudoconvex if its Levi form is nondegenerate.

Let $(M, H(M))$ be a pseudoconvex $CR-$structure of hypersurface type. Due to the orientation of M, we can choose the 1-form η defined on the whole M, and the property of pseudoconvexity of $(M, H(M))$ is equivalent with the property of η to be a contact form i.e. $\eta \wedge (d\eta)^n \neq 0$.

Remark. It follows easily that, in the case of a pseudoconvex $CR-$structure of hypersurface type, the restriction of dη to $R(M)$ is nondegenerate.

Let ξ be the Reeb vector field on M, defined by: $\eta(\xi) = 1 \ , \ i_\xi d\eta = 0$. We have $TM \ = \ \text{span } [\xi] \oplus R(M)$. We introduce an almost contact structure associated to the $CR-$structure $(M, H(M))$ as follows:

Proposition ([2]): *The (1,1) type tensor field ϕ defined by:*

(1) $\qquad \phi X = J(X - \eta(X)\xi) \ ; \ \forall X \in \chi(M)$

has the following properties:

$$\eta \circ \phi = 0 \quad , \quad \phi\xi = 0 \quad , \ \phi^2 = -I + \eta \otimes \xi.$$

Hence we obtain the almost contact structure (ϕ, ξ, η).

Next, we give a result which characterizes the complex involutivity condition of $CR-$structure $(M, H(M))$ in terms of the almost contact structure (ϕ, ξ, η):

Theorem ([2]): *The complex involutivity condition of $(M, H(M))$ is equivalent to:*

$$S = 0$$

where S is the tensor field of type (1,2) on M, defined by:

$$S(X,Y) = N_\phi(X,Y) + d\eta(X,Y)\xi + \eta(X)\phi(\mathcal{L}_\xi\phi)Y - \eta(Y)\phi(\mathcal{L}_\xi\phi)X,$$

$\mathcal{L}_\xi\phi$ is Lie derivative of ϕ with respect to ξ and N_ϕ is the Nijenhuis tensor field of ϕ, defined by:

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y].$$

From now on we shall denote $\psi = \frac{1}{2}\mathcal{L}_\xi\phi$.

Let (ϕ, ξ, η) be an almost contact structure associated to the pseudoconvex $CR-$structure $(M, H(M))$. Looking for a torsion free connection ∇ on M related in a natural way to the $1-$form η, the following result has been obtained in [2]:

Theorem : *If (ϕ, ξ, η) is an almost contact structure associated to the pseudoconvex $CR-$structure $(M, H(M))$, then there exists a unique torsion free connection ∇ so that*

$$(2) \qquad \begin{cases} (\nabla_X\eta)(Y) = \frac{1}{2}d\eta(X,Y) \ , \ \nabla_X d\eta = 0 \ , \ \nabla_X\xi = 0 \\ (\nabla_X\phi)Y = 2\eta(X)\psi Y - \frac{1}{2}d\eta(X,\phi Y)\xi \ \ , \ \forall X, Y \in \chi(M). \end{cases}$$

Remarks:

a) The connection found above will be named *the torsion free canonical connection,* adapted to the almost contact structure (ϕ, ξ, η), associated to the pseudoconvex $CR-$structure $(M, H(M))$.

b) The almost contact structure (ϕ, ξ, η) associated to the pseudoconvex CR-structure $(M, H(M))$ is not unique, so the adapted torsion free canonical connection is not unique.

2 Gauge Transformations and the Bochner Tensor of a Pseudoconvex $CR-$structure

Definition: A change $\eta \mapsto \eta' = \varepsilon e^f\eta \ , \ f \in C^\infty(M)$ of the $1-$form η is called a *gauge transformation* $(\varepsilon = \pm 1)$.

Proposition ([2]): *Two almost contact structures* $(\phi, \xi, \eta), (\phi', \xi', \eta')$ *are associated to the same pseudoconvex* $CR-$*structure if and only if there exists a function* $f \in C^\infty(M)$ *so that:*

$$\eta' = \varepsilon e^f \eta \quad , \quad d\eta' = \varepsilon e^f (d\eta + df \wedge \eta)$$

$$\xi' = \varepsilon e^{-f}(\xi + \phi A) \quad , \quad \phi' = \phi + \eta \otimes A$$

where $\varepsilon = \pm 1$ *and* A *is a vector field defined by the conditions:*

$$\eta(A) = 0 \ , \ d\eta(\phi A, X) = df(X - \eta(X)\xi).$$

The complex involutivity is invariant with respect to gauge transformations.

From now on we shall take $\varepsilon = +1$. The case $\varepsilon = -1$ can be discussed in a similar way. Given an almost contact structure (ϕ, ξ, η) associated to the pseudoconvex $CR-$structure $(M, H(M))$ and the adapted torsion free connection ∇ on M, let R be the curvature tensor field of ∇, defined by

(3) $$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ ; \ X, Y, Z \in \chi(M).$$

Introduce the auxiliary (pseudo-)Riemannian metric h on $R(M)$

$$h(X, Y) = d\eta(\phi X, Y) \ ; \ X, Y \in \Gamma(R(M)).$$

The Ricci tensor field $\rho(R)$ of ∇ is defined, in the usual way by:

(4) $$\rho(R)(Y, Z) = \text{trace } (X \mapsto R_{XY}Z) \ ; \ X, Y, Z \in \chi(M).$$

and we denote $\tau(R) = \text{trace } (\rho(R))$ a kind of scalar curvature of the restriction of R to $R(M)$ obtained by using the partial metric h on $R(M)$. We have the following result:

Theorem ([2]): *Let* (ϕ, ξ, η) *be an almost contact structure on* M *associated to the pseudoconvex* $CR-$*structure* $(M, H(M))$*. Then the tensor field*

(5) $$\begin{aligned} B(R)_{XY}Z = {} & R_{XY}Z + L(X,Z)Y - L(Y,Z)X + L(Y,\phi Z)\phi X - \\ & -L(X,\phi Z)\phi Y - \{L(X,\phi Y) - L(Y,\phi X)\}\phi Z - \\ & -d\eta(X,Y)KZ - \tfrac{1}{2}d\eta(X,Z)KY + \tfrac{1}{2}d\eta(Y,Z)KX + \\ & +\tfrac{1}{2}d\eta(X,\phi Z)\phi KY - \tfrac{1}{2}d\eta(Y,\phi Z)\phi KX, \\ & \quad X, Y, Z \in \Gamma(R(M)) \end{aligned}$$

is invariant under the action of gauge transformation, where

$$(6)\qquad \begin{aligned} L(X,Y) = &\tfrac{1}{2(n+2)}\{\rho(R)(X,Y) + 2d\eta(\phi X, \psi Y)\} + \\ &+ \tfrac{1}{8(n+1)(n+2)}\tau(R)d\eta(X, \phi Y)\ , \end{aligned}$$

and $\frac{1}{2}d\eta(KX,Y) = L(X,Y)$.

Remark: Doing the computations in local coordinates, we obtain that trace $(B(R)_{XY}) = 0$ and trace $(X \mapsto B(R)_{XY}Z) = 0$.

Definition: $B(R)$ is called *the Bochner type tensor* associated to the pseudoconvex $CR-$structure $(M, H(M))$.

3 The Unit Cotangent Bundle of a Riemannian Space

Let (M,g) be an $(n+1)$ dimensional Riemannian manifold. Denote by T^*M the cotangent bundle of the manifold M and by $\bar{\pi} : T^*M \longrightarrow M$ the canonical projection. If $(x^1, \ldots, x^{n+1})$ are local coordinates on M, then $(q^1, \ldots, q^{n+1})$ and the vector space coordinates $(p_1, \ldots, p_{n+1})$ with respect to the natural local frame $(dx^1, \ldots, dx^{n+1})$ define together a system of local coordinates on T^*M, where $q^i = x^i \circ \bar{\pi}$. The Levi-Civita connection D of g determines a decomposition of TT^*M in a direct sum, of the vertical distribution VT^*M and the horizontal distribution HT^*M: $TT^*M = VT^*M \oplus HT^*M$. We have that $(\frac{\partial}{\partial p_i})_{i=\overline{1,n+1}}$ is a local frame in VT^*M and $(\frac{\delta}{\delta q^i})_{i=\overline{1,n+1}}$ is a local frame in HT^*M, where $\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^k_{ij}p_k\frac{\partial}{\partial p_j}$ and Γ^k_{ij} are the Christoffel simbols of the connection D. We also put $(dx^i)^V \stackrel{def}{=} \frac{\partial}{\partial p_i}$ and $(dx^i)^H \stackrel{def}{=} g^{ij}\frac{\delta}{\delta q^j}$.

Then the well known almost complex structure on T^*M is defined by:

$$(7)\qquad J\omega^H = \omega^V\ ,\ J\omega^V = -\omega^H\ ,\ \omega \in \Lambda^1(M),$$

where ω^H, ω^V are, respectively, the horizontal and vertical lifts of ω (with respect to D). The Sasaki metric $\bar{g}$ on T^*M is defined by

$$\bar{g}(\omega_1^V, \omega_2^V) = g(\omega_1^\#, \omega_2^\#)\ ,\ \bar{g}(\omega_1^H, \omega_2^H) = g(\omega_1^\#, \omega_2^\#)\ ,\ \bar{g}(\omega_1^V, \omega_2^H) = 0\ ;\ \omega_1, \omega_2 \in \Lambda^1(M),$$

where $\omega^\#$ is defined by $g(X, \omega^\#) = \omega(X)$. Note that $(T^*M, J, \bar{g})$ is only an almost Kaehlerian manifold. Consider the unit cotangent bundle T_1^*M as the bundle of the unit tangent covectors at M. So, if $\omega \in T^*M$, then $\omega \in T_1^*M \iff g(\omega^\#, \omega^\#) = 1$.

If $\omega = p_i dx^i$, we conclude that the unit cotangent bundle $\pi : T_1^*M \longrightarrow M$ is a hypersurface in T^*M, given in the local coordinates by the equation:

$$(8) \qquad g^{ij}(x)p_ip_j - 1 = 0,$$

where g^{ij} are the components of g^{-1}. Looking for a vector field $N \in \chi(T^*M)$, on T_1^*M, unitary and normal to T_1^*M with respect to the Sasaki metric $\bar{g}$, we obtain

$$(9) \qquad N = p_i \frac{\partial}{\partial p_i} \ .$$

Let $\iota : T_1^*M \longrightarrow T^*M$ be the imersion of T_1^*M in T^*M. Denote by G the Riemannian metric $\iota^\star \bar{g}$, induced from $\bar{g}$ on T_1^*M and by $\dot{\nabla}$ its Levi-Civita connection. As a hypersurface of an almost Kählerian structure $(T^*M, J, \bar{g})$, T_1^*M has a metric contact structure (ϕ, ξ, η, G) defined as follows: The vector field ξ is given by

$$(10) \qquad \xi = -JN = p_i g^{ij} \frac{\delta}{\delta q^j}$$

where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial q^i} + \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}$; Γ^j_{ik} being the Christoffel symbols coresponding to the connection D.

The 1-form η is the cannonical form $\eta = p_i dq^i$ of T^*M. Then $\phi \in \mathcal{T}^1_1(T_1^*M)$ is defined by

$$(11) \qquad JX = \phi(X) + \eta(X)N \ ; \ X \in \chi(T_1^*M).$$

Remark that η is a contact 1-form on T_1^*M and, due to (8) we obtain

$$0 = g^{ij} p_i \delta p_i = i_\xi d\eta$$

so that ξ is the Reeb vector field. We have

$$\phi^2 = -I + \eta \otimes \xi \ , \ \phi\xi = 0 \ , \ \eta\phi = 0 \ , \ \eta(\xi) = 1$$

so that we get an almost contact structure. Since the surrounding manifold T^*M is only almost complex, the complex involutivity condition of $H(T_1^*M)$, which is equivalent to the condition $S = 0$ on T_1^*M is not automatically satisfied. A necessary and sufficient condition for the vanishing of the tensor S on T_1^*M, therefore a necessary and sufficient condition for the pair $(T_1^*M, H(T_1^*M))$ to be a $CR-$structure is given by:

Theorem : *The unit cotangent bundle T_1^*M of a Riemannian manifold (M, g) has a Cauchy-Riemann structure obtained from the usual almost Kählerian structure of T^*M, if and only if the base manifold M has constant sectional curvature.*

Proof : The condition for the unit cotangent bundle to have a Cauchy-Riemann structure is equivalent to $S = 0$. If we compute the expression of S in a local frame and if we claim $S = 0$ we get:

$$R_k^{ij0} - g^{i0}R_k^{0j0} + g^{j0}R_k^{0i0} = 0,$$

where $R_k^{ijh} = g^{jl}g^{hm}R^i_{lmk}$, $g^{i0} = g^{ij}p_j$, $R_k^{ij0} = R_k^{ijh}p_h$ and $R_k^{0i0} = R_k^{ijh}p_ip_h$ (R^i_{lmk} being the components of curvature tensor of D). From this relation making ordinary contractions, it follows that M has constant sectional curvature.

4 The Bochner Tensor of the $CR-$structure Induced on the Unit Cotangent Bundle

In this section we shall compute the Bochner tensor in the particular case of T_1^*M as a hypersurface in T^*M, where M is real space form with the constant curvature c. Consider the vector fields

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^k_{ij}p_k\frac{\partial}{\partial p_j}\ ,\ Y^i = \frac{\partial}{\partial p_i} - g^{ij}p_jN\ ;\ i = 1, \ldots, n+1. \tag{12}$$

These vector fields are generators for $\chi(T_1^*M)$. Let us remark that, since $p_iY^i = 0$, the vector fields Y^i ; $i = 1, \ldots, n+1$ are not independent, but in any point z of T_1^*M they span a vector subspace of dimension n in $T_z(T_1^*M)$. Then we have:

$$\begin{cases} \eta = p_idq^i\ ,\ \eta(\frac{\delta}{\delta q^i}) = p_i\ ,\ \eta(Y^i) = 0 \\ \phi(\frac{\delta}{\delta q^i}) = g_{ij}Y^j\ ,\ \phi(Y^i) = -g^{ij}\frac{\delta}{\delta q^j} + g^{i0}\xi\ ;\ i, j = 1, ..., n+1 \end{cases} \tag{13}$$

where $g^{i0} = g^{ij}p_j$.

The Weingarten map of the imersion $\iota : T_1^*M \longrightarrow T^*M$ has the following form:

$$A(U) = -U \quad , \; \forall U \in \chi(T_1M) \; , U - vertical$$
$$A(X) = 0 \quad , \; \forall X \in \chi(T_1M) \; , X - horizontal.$$

Using these formulas we obtain the Levi-Civita connection $\dot{\nabla}$ of G (G being the Riemannian metric induced from $\overline{g}$ on T_1^*M):

$$(14) \qquad \begin{cases} \dot{\nabla}_{Y^i} Y^j = -g^{j0} Y^i \; , \; \dot{\nabla}_{\frac{\delta}{\delta q^i}} Y^j = -\Gamma^j_{ik} Y^k + \frac{1}{2} g^{lk} g^{js} R^0_{ski} \frac{\delta}{\delta q^l} \\ \dot{\nabla}_{Y^i} \frac{\delta}{\delta q^j} = -\frac{1}{2} g^{lk} g^{si} R^0_{sjk} \frac{\delta}{\delta q^l} \; , \; \dot{\nabla}_{\frac{\delta}{\delta q^i}} \frac{\delta}{\delta q^j} = \Gamma^k_{ij} \frac{\delta}{\delta q^k} + \frac{1}{2} R^0_{kij} Y^k \; ; \\ \qquad i, j, k = 1, \ldots, n+1. \end{cases}$$

where $R^0_{ijk} = p_h R^h_{ijk}$ and R^h_{ijk} are the components of the curvature tensor of D.

Since M has constant sectional curvature c, the components of the curvature tensor of D are written as follows

$$(15) \qquad R^h_{kij} = c(g_{jk}\delta^h_i - g_{ik}\delta^h_j) \; ; \; i, j, k, h = 1, \ldots, n+1.$$

Next we obtain the expression of the torsion free canonical connection defined by the almost contact structure (ϕ, ξ, η), determined by the relations (see [2]):

$$(16) \qquad \begin{cases} 2\eta(\nabla_X Y) = 2X(\eta(Y)) - d\eta(X, Y) \\ \\ 2d\eta(\nabla_X Y, Z) = 2\eta(X) d\eta(\phi Y, \psi Z) + 2\eta(Y) d\eta(\phi X, \psi Z) + \\ \quad + \phi Z(d\eta(X, \phi Y)) + d\eta([X, \phi Z], \phi Y) + d\eta([Y, \phi Z], \phi X) + \\ \quad + X(d\eta(Y, Z)) + Y(d\eta(X, Z)) + d\eta([X, Y], Z). \end{cases}$$

The local coordinate expression of the adapted connection ∇ in the local frame $(\frac{\delta}{\delta q^i} \; , \; Y^i)$ is given by:

$$(17) \qquad \begin{cases} \nabla_{Y^i} Y^j = -g^{j0} Y^i \; , \; \nabla_{Y^i} \frac{\delta}{\delta q^j} = \frac{1}{2} g_{js}(g^{is} + g^{i0} g^{s0})\xi - g^{is} p_j \frac{\delta}{\delta q^s} \\ \nabla_{\frac{\delta}{\delta q^i}} Y^j = -\Gamma^j_{ik} Y^k + \frac{1}{2} g_{is}(g^{js} + g^{j0} g^{s0})\xi - g^{js} p_i \frac{\delta}{\delta q^s} \\ \nabla_{\frac{\delta}{\delta q^i}} \frac{\delta}{\delta q^j} = -\Gamma^k_{ij} \frac{\delta}{\delta q^k} + c p_i g_{jl} Y^l \; , \; i, j, k = 1, \ldots, n+1. \end{cases}$$

We find now, the expression of the curvature tensor field R of the connection ∇ determined above. First, we make some notations. From now on we shall deal only with sections in $R(T_1^*M)$. So, we will project the vector fields Y^i, $\frac{\delta}{\delta q^i}$ on $R(T_1^*M)$, i.e. we consider:

$$Y^i - \eta(Y^i)\xi = Y^i$$

$$\frac{\delta}{\delta q^i} - \eta(\frac{\delta}{\delta q^i})\xi = \frac{\delta}{\delta q^i} - p_i\xi \overset{\text{not}}{=} X_i \ , \ i = 1, \ldots, n+1.$$

We denote also

$$h^{ij} = g^{ij} - g^{i0}g^{j0}. \tag{18}$$

Then the expression of the curvature tensor is obtained by using the sections from $\Gamma(R(T_1^*M))$. From the definition (3) we obtain:

$$\begin{cases} R_{Y^iY^j}Y^k = h^{jk}Y^i - h^{ik}Y^j \ , \ R_{Y^iY^j}X_k = g_{kl}(g^{is}h^{jl} - g^{js}h^{il})X_s \\ R_{X_iX_j}Y^k = c(g_{mi}g_{js} - g_{mj}g_{is})h^{ks}Y_m \ , \ R_{Y^iX_j}Y^k = -g_{jl}g^{ks}h_{il}X_s \\ R_{X_iX_j}X_k = cg_{kl}h^{ls}(g_{js}X_i - g_{is}X_j) \\ R_{Y^iX_j}X_k = cg_{mk}g_{js}h^{is}Y^m \ , \ i, j, k, l, m, s = 1, \ldots, n+1. \end{cases} \tag{19}$$

To obtain the Ricci tensor $\rho(R)$ we make the usual contraction and we have:

$$\begin{cases} \rho(R)(Y^b, Y^c) = n\ h^{bc} \quad , \quad \rho(R)(Y^b, X_c) = 0 \\ \rho(R)(X_b, Y^c) = 0 \quad , \quad \rho(R)(X_b, X_c) = cn\ g_{bd}g_{ce}h^{de} \\ \qquad b, c, d, e = 1, \ldots, n. \end{cases} \tag{20}$$

It is easy to prove that the matrix $H = (h^{ij})_{i,j=1,\ldots,n}$ is nonsingular. The matrix H is the matrix of auxiliary metric h on $R(M)$. Hence we get: $\tau(R) = n^2(c+1)$.

Now we can compute the Bochner tensor:

First, we have the following expressions for the tensor field L:

$$\begin{cases} L(Y^a, Y^b) = \frac{k_1}{2}h^{ab} \quad , \quad L(Y_a, X_b) = 0 \\ L(X_a, Y_b) = 0 \quad , \quad L(X_a, X_b) = \frac{k_2}{2}g_{ac}g_{bd}h^{cd} \end{cases} \tag{21}$$

where:

$$k_1=\frac{1}{n+2}(n+1-c-\frac{\tau(R)}{4(n+1)})\ ,\ k_2=\frac{1}{n+2}(nc-1+c-\frac{\tau(R)}{4(n+1)}). \tag{22}$$

From the definition of K it follows:

$$KY^a=-k_1h^{ab}X_b\ ,\ KX_a=k_2g_{ab}Y^b\ ,\ a,b=1,\ldots,n. \tag{23}$$

Finally, making all the computations, we have:

$$\begin{cases} B_{Y^iY^j}Y^k=\frac{(c+1)(n+2)}{4(n+1)}(h^{jk}Y^i-h^{ik}Y^j)\\ B_{Y^iY^j}X_k=\frac{(c+1)(n+2)}{4(n+1)}g_{kl}(g^{is}h^{jl}-g^{js}h^{il})X_s\\ B_{Y^iX_j}Y^k=-\frac{(c+1)(n+2)}{4(n+1)}g_{jl}h^{il}g^{ks}X_s+\\ \qquad +\frac{n(c+1)}{4(n+1)}(h^{ik}X_j+g_{jl}g^{is}h_{kl}X_s+g_{jl}g^{ks}h^{il}X_s)\\ B_{Y^iX_j}X_k=\frac{(c+1)(n+2)}{4(n+1)}g_{jl}g_{ks}h^{il}Y^s-\\ \qquad -\frac{n(c+1)}{4(n+1)}(g_{jl}g_{ks}h^{ls}Y^i+g_{kl}g_{js}h^{il}Y^s+g_{jl}g_{ks}h^{il}Y^s)\\ B_{X_iX_j}Y^k=\frac{(c+1)(n+2)}{4(n+1)}g_{im}g_{jl}(h^{lk}Y^m-h^{mk}Y^l)\\ B_{X_iX_j}X_k=\frac{(c+1)(n+2)}{4(n+1)}g_{ks}h^{ls}(g_{jl}X_i-g_{il}X_j). \end{cases} \tag{24}$$

Now we can state the main result of this paper:

Theorem *Let (M,g) be a Riemannian manifold of dimension $n>2$ and T_1^*M be the unit cotangent bundle with the standard contact Riemannian structure. Then the gauge invariant B of $(1,3)$-type (the Bochner type tensor associated to the CR-structure induced on T_1^*M) vanishes, if and only if (M,g) has constant curvature -1.*

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