

CONSTANT ANGLE SURFACES IN THE HEISENBERG GROUP

JOHAN FASTENAKELS, MARIAN IOAN MUNTEANU, AND JOERI VAN DER VEKEN

ABSTRACT. In this article we generalize the notion of constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ to general Bianchi-Cartan-Vranceanu spaces. We show that these surfaces have constant Gaussian curvature and we give a complete local classification in the Heisenberg group.

1. INTRODUCTION

Differential geometry of submanifolds started with the study of surfaces in the 3-dimensional Euclidean space. This study has been generalized in two ways. One can generalize the dimension and the codimension of the submanifold, but one can also generalize the ambient space. The most popular ambient spaces are, besides the Euclidean space, the sphere and the hyperbolic space. These spaces are called real space forms and are the easiest Riemannian manifolds from geometric point of view. With the work of, among others, Thurston, 3-dimensional homogeneous spaces have gained interest. A homogeneous space is a Riemannian manifold such that for every two points p and q , there exists an isometry mapping p to q . Roughly speaking, this means that the space looks the same at every point.

There are three classes of 3-dimensional homogeneous spaces depending on the dimension of the isometry group. This dimension can be 3, 4 or 6. In the simply connected case, dimension 6 corresponds to one of the three real space forms and dimension 3 to a general simply connected 3-dimensional Lie group with a left-invariant metric. In this article we restrict ourselves to dimension 4. A 3-dimensional homogeneous spaces with a 4-dimensional isometry group is locally isometric to (an open part of) \mathbb{R}^3 , equipped with a metric depending on two real parameters. Since this two-parameter family of metrics first appeared in the works of Bianchi, Cartan and Vranceanu, these spaces are often referred to as Bianchi-Cartan-Vranceanu spaces, or BCV-spaces for short. More details can be found in the last section. Some well-known examples of BCV-spaces are the Riemannian product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ and the 3-dimensional Heisenberg group.

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There exists a Riemannian submersion from a general BCV-space onto a surface of constant Gaussian curvature. Since this submersion extends the classical Hopf fibration $\pi : \mathbb{S}^3(\kappa/4) \rightarrow \mathbb{S}^2(\kappa)$, it is also called the Hopf fibration. Hence, any BCV-space is foliated by 1-dimensional fibers of the Hopf fibration. Now consider a surface immersed in a BCV-space and consider at every point of the surface the angle between the unit normal and the fiber of the Hopf fibration through this point. The existence and uniqueness theorem for immersions into BCV-spaces, proven in [1], shows that this angle function is one of the fundamental invariants for a surface in a BCV-space. Hence, it is a very natural problem to look for those surfaces for which this angle function is constant.

The constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been classified in [2] and [3], see also [4]. In these cases, the Hopf fibration is just the natural projection $\pi : \mathbb{S}^2 \times \mathbb{R} \rightarrow \mathbb{S}^2$, respectively $\pi : \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2$. For an overview of constant angle surfaces in \mathbb{E}^3 , i.e., surfaces for which the unit normal makes a constant angle with a fixed direction in \mathbb{E}^3 , we refer to [5]. In the present article, we show that all constant angle surfaces in any BCV-space have constant Gaussian curvature and we give a complete local classification of constant angle surfaces in the Heisenberg group by means of an explicit parametrization. We also give some partial results for a classification in general BCV-spaces.

2. PRELIMINARIES

2.1. The Heisenberg group. Let (V, ω) be a symplectic vector space of dimension $2n$. Then the associated Heisenberg group is defined as the set $V \times \mathbb{R}$ equipped with the operation

$$(v_1, t_1) * (v_2, t_2) = \left(v_1 + v_2, t_1 + t_2 + \frac{1}{2}\omega(v_1, v_2) \right).$$

From now on, we restrict ourselves to the 3-dimensional Heisenberg group coming from \mathbb{R}^2 with the canonical symplectic form $\omega((x, y), (\bar{x}, \bar{y})) = x\bar{y} - \bar{x}y$, i.e., we consider \mathbb{R}^3 with the group operation

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = \left(x + \bar{x}, y + \bar{y}, z + \bar{z} + \frac{x\bar{y}}{2} - \frac{\bar{x}y}{2} \right).$$

Remark that the mapping

$$\mathbb{R}^3 \rightarrow \left\{ \left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\} : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $(\mathbb{R}^3, *)$ and a subgroup of $\text{GL}(3, \mathbb{R})$. For every non-zero real number τ the following Riemannian metric on $(\mathbb{R}^3, *)$ is left-invariant:

$$ds^2 = dx^2 + dy^2 + 4\tau^2 \left(dz + \frac{y dx - x dy}{2} \right)^2.$$

After the change of coordinates $(x, y, 2\tau z) \mapsto (x, y, z)$, this metric is expressed as

$$(1) \quad ds^2 = dx^2 + dy^2 + (dz + \tau(y dx - x dy))^2.$$

From now on, we denote by Nil_3 the group $(\mathbb{R}^3, *)$ with the metric (1). By some authors, the notation Nil_3 is only used if $\tau = \frac{1}{2}$.

In the following lemma, we give a left-invariant orthonormal frame on Nil_3 and describe the Levi Civita connection and the Riemann Christoffel curvature tensor in terms of this frame.

Lemma 1. *The following vector fields form a left-invariant orthonormal frame on Nil_3 :*

$$e_1 = \partial_x - \tau y \partial_z, \quad e_2 = \partial_y + \tau x \partial_z, \quad e_3 = \partial_z.$$

The geometry of Nil_3 can be described in terms of this frame as follows.

(i) *These vector fields satisfy the commutation relations*

$$[e_1, e_2] = 2\tau e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

(ii) *The Levi Civita connection $\tilde{\nabla}$ of Nil_3 is given by*

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= 0, & \tilde{\nabla}_{e_1} e_2 &= \tau e_3, & \tilde{\nabla}_{e_1} e_3 &= -\tau e_2, \\ \tilde{\nabla}_{e_2} e_1 &= -\tau e_3, & \tilde{\nabla}_{e_2} e_2 &= 0, & \tilde{\nabla}_{e_2} e_3 &= \tau e_1, \\ \tilde{\nabla}_{e_3} e_1 &= -\tau e_2, & \tilde{\nabla}_{e_3} e_2 &= \tau e_1, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

(iii) *The Riemann Christoffel curvature tensor \tilde{R} of Nil_3 is determined by*

$$\begin{aligned} \tilde{R}(X, Y)Z &= -3\tau^2(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &\quad + 4\tau^2(\langle Y, e_3 \rangle \langle Z, e_3 \rangle X - \langle X, e_3 \rangle \langle Z, e_3 \rangle Y) \\ &\quad + \langle X, e_3 \rangle \langle Y, Z \rangle e_3 - \langle Y, e_3 \rangle \langle X, Z \rangle e_3 \end{aligned}$$

for $p \in \text{Nil}_3$ and $X, Y, Z \in T_p \text{Nil}_3$.

It is clear that the Killing vector field e_3 plays an important role in the geometry of Nil_3 . In fact, the integral curves of e_3 are the fibers of the Hopf fibration:

Definition 1. *The mapping $\text{Nil}_3 \rightarrow \mathbb{E}^2 : (x, y, z) \mapsto (x, y)$ is a Riemannian submersion, called the Hopf fibration. The inverse image of a curve in \mathbb{E}^2 under the Hopf fibration is called a Hopf cylinder.*

2.2. Surfaces in the Heisenberg group. Let $F : M \rightarrow \text{Nil}_3$ be an isometric immersion of an oriented surface in the Heisenberg group. Denote by N a unit normal vector field, by S the associated shape operator, by θ the angle between e_3 and N and by T the projection of e_3 onto the tangent plane to M , i.e. the vector field T on M such that $F_*T + \cos \theta N = e_3$.

If we work locally, we may assume $\theta \in [0, \frac{\pi}{2}]$. Take $p \in M$ and $X, Y, Z \in T_pM$. Then the equations of Gauss and Codazzi are given respectively by

$$(2) \quad \begin{aligned} R(X, Y)Z &= -3\tau^2(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &\quad + 4\tau^2(\langle Y, T \rangle \langle Z, T \rangle X - \langle X, T \rangle \langle Z, T \rangle Y) \\ &\quad + \langle X, T \rangle \langle Y, Z \rangle T - \langle Y, T \rangle \langle X, Z \rangle T \\ &\quad + \langle SY, Z \rangle SX - \langle SX, Z \rangle SY \end{aligned}$$

and

$$(3) \quad \nabla_X SY - \nabla_Y SX - S[X, Y] = -4\tau^2 \cos \theta (\langle Y, T \rangle X - \langle X, T \rangle Y).$$

Remark that (2) is equivalent to the following relation between the Gaussian curvature K of the surface and extrinsic data of the immersion:

$$(4) \quad K = \det S + \tau^2 - 4\tau^2 \cos^2 \theta.$$

Finally, we remark that also the following structure equations hold:

$$(5) \quad \nabla_X T = \cos \theta (SX - \tau JX),$$

$$(6) \quad X[\cos \theta] = -\langle SX - \tau JX, T \rangle,$$

where JX is defined by $JX := N \times X$, i.e., J denotes the rotation over $\frac{\pi}{2}$ in T_pM .

The four equations (3), (4), (5) and (6) are called the compatibility equations since it was proven in [1] that they are necessary and sufficient conditions to have an isometric immersion of a surface into Nil_3 .

3. FIRST RESULTS ON CONSTANT ANGLE SURFACES IN Nil_3

In this section we define constant angle surfaces in Nil_3 and we give some immediate consequences of this assumption.

Definition 2. *We say that a surface in the Heisenberg group Nil_3 is a constant angle surface if the angle θ between the unit normal and the direction e_3 tangent to the fibers of the Hopf fibration is the same at every point.*

Lemma 2. *Let M be a constant angle surface in Nil_3 . Then the following statements hold.*

(i) *With respect to the basis $\{T, JT\}$, the shape operator is given by*

$$S = \begin{pmatrix} 0 & -\tau \\ -\tau & \lambda \end{pmatrix}$$

for some function λ on M .

(ii) *The Levi Civita connection of the surface is determined by*

$$\begin{aligned} \nabla_T T &= -2\tau \cos \theta JT, & \nabla_{JT} T &= \lambda \cos \theta JT, \\ \nabla_T JT &= 2\tau \cos \theta T, & \nabla_{JT} JT &= -\lambda \cos \theta T. \end{aligned}$$

(iii) *The Gaussian curvature of the surface is a negative constant given by*

$$K = -4\tau^2 \cos^2 \theta.$$

(iv) *The function λ satisfies the following PDE:*

$$T[\lambda] + \lambda^2 \cos \theta + 4\tau^2 \cos^3 \theta = 0.$$

Proof. The first statement follows from equation (6) and the symmetry of the shape operator. The second statement follows from equation (5), (i) and the equalities $\langle T, T \rangle = \langle JT, JT \rangle = \sin^2 \theta$ and $\langle T, JT \rangle = 0$. The third statement follows from the equation of Gauss (4) and (i). The last statement follows from (ii) and (iii), or from the equation of Codazzi (3) and (i). \square

4. CLASSIFICATION OF CONSTANT ANGLE SURFACES IN Nil_3

By using Lemma 2, we can prove our main result, namely the complete local classification of all constant angle surfaces in the Heisenberg group.

Theorem 1. *Let M be a constant angle surface in the Heisenberg group Nil_3 . Then M is isometric to an open part of one of the following types of surfaces:*

- (i) *a Hopf-cylinder,*
- (ii) *a surface given by*

$$F(u, v) = \left(\frac{1}{2\tau} \tan \theta \sin u + f_1(v), -\frac{1}{2\tau} \tan \theta \cos u + f_2(v), \right. \\ \left. -\frac{1}{4\tau} \tan^2 \theta u - \frac{1}{2} \tan \theta \cos u f_1(v) - \frac{1}{2} \tan \theta \sin u f_2(v) - \tau f_3(v) \right),$$

with $(f_1')^2 + (f_2')^2 = \sin^2 \theta$ and $f_3'(v) = f_1'(v)f_2(v) - f_1(v)f_2'(v)$. Here, θ denotes the constant angle.

Proof. Let M be a constant angle surface in Nil_3 . If $\theta = \frac{\pi}{2}$, the surface is of the first type mentioned in the theorem. Moreover, $\theta = 0$ gives a contradiction with Lemma 1 (i), since $\tau \neq 0$. From now on, we assume that θ lies strictly between 0 and $\frac{\pi}{2}$. There exists a locally defined real function ϕ on M such that the unit normal on M is given by

$$(7) \quad N = \sin \theta \cos \phi e_1 + \sin \theta \sin \phi e_2 + \cos \theta e_3.$$

Remark that with this notation, we have

$$(8) \quad T = -\sin \theta (\cos \theta \cos \phi e_1 + \cos \theta \sin \phi e_2 - \sin \theta e_3),$$

$$(9) \quad JT = \sin \theta (\sin \phi e_1 - \cos \phi e_2).$$

By a straightforward computation, using Lemma 1 (ii), (7), (8) and (9), we obtain that the shape operator S satisfies

$$ST = -\tilde{\nabla}_T N = (T[\phi] - \tau \sin^2 \theta + \tau \cos^2 \theta)JT,$$

$$SJT = -\tilde{\nabla}_{JT} N = -\tau T + (JT)[\phi]JT.$$

Comparing this to Lemma 2 (i), gives

$$(10) \quad \begin{cases} T[\phi] = -2\tau \cos^2 \theta, \\ (JT)[\phi] = \lambda. \end{cases}$$

The integrability condition for this system of equations is precisely the PDE from Lemma 2 (iv).

Remark that the solutions of this system describe the possible tangent distributions to M .

In order to solve the system (10), let us choose coordinates (u, v) on M such that $\partial_u = T$ and $\partial_v = aT + bJT$, for some real functions a and b which are locally defined on M . The condition $[\partial_u, \partial_v] = 0$ is equivalent to the following system of equations:

$$(11) \quad \begin{cases} \partial_u a = -2\tau b \cos \theta, \\ \partial_u b = \lambda b \cos \theta. \end{cases}$$

The PDE from Lemma 2 (iv) is now equivalent to $\partial_u \lambda + \lambda^2 \cos \theta + 4\tau^2 \cos^3 \theta = 0$, for which the general solution is given by

$$\lambda(u, v) = 2\tau \cos \theta \tan(\varphi(v) - 2\tau \cos^2 \theta u),$$

for some function $\varphi(v)$. We can now solve system (11). Remark that we are interested in only one coordinate system on the surface M and hence we only need one solution for a and b , for example

$$(12) \quad a(u, v) = \frac{1}{\cos \theta} \sin(\varphi(v) - 2\tau \cos^2 \theta u),$$

$$(13) \quad b(u, v) = \cos(\varphi(v) - 2\tau \cos^2 \theta u).$$

System (10) is then equivalent to

$$\begin{cases} \partial_u \phi = -2\tau \cos^2 \theta, \\ \partial_v \phi = 0, \end{cases}$$

for which the general solution is given by

$$(14) \quad \phi(u, v) = \phi(u) = -2\tau \cos^2 \theta u + c,$$

where c is a real constant.

To finish the proof, we have to integrate the distribution spanned by T and JT . Denote the parametrization of M by

$$F : U \subseteq \mathbb{R}^2 \rightarrow M \subset \text{Nil}_3 : (u, v) \mapsto F(u, v) = (F_1(u, v), F_2(u, v), F_3(u, v)).$$

Then we know from (8), (9) and the choice of coordinates (u, v) that

$$(\partial_u F_1, \partial_u F_2, \partial_u F_3) = -\sin \theta (\cos \theta \cos \phi e_1 + \cos \theta \sin \phi e_2 - \sin \theta e_3),$$

$$(\partial_v F_1, \partial_v F_2, \partial_v F_3) = \sin \theta ((-a \cos \theta \cos \phi + b \sin \phi) e_1 - (a \cos \theta \sin \phi + b \cos \phi) e_2 + a \sin \theta e_3).$$

Moreover, at the point (F_1, F_2, F_3) , we have $e_1 = (1, 0, -\tau F_2)$, $e_2 = (0, 1, \tau F_1)$, $e_3 = (0, 0, 1)$ and a , b and ϕ are given by (12), (13) and (14). Direct integration, followed by the reparametrization $-2\tau \cos^2 \theta u + c \mapsto u$ yields the parametrization given in the theorem, where $f_1(v)$ and $f_2(v)$ are primitive functions of $\sin \theta \sin(c - \varphi(v))$ and $\sin \theta \cos(c - \varphi(v))$ respectively. \square

Remark 1. We may assume that $f_1(v)$ and $f_2(v)$ are polynomials of degree at most one, by changing the v -coordinate if necessary. $f_3(v)$ is then a polynomial of degree at most two.

We end this section with a concrete example of a non-trivial constant angle surface in Nil_3 .

Example 1. Take $f_1(v) = f_3(v) = 0$ and $f_2(v) = \frac{1}{\sqrt{2}}v$. Then it follows from Theorem 1 that the surface

$$F(u, v) = \left(\frac{1}{2\tau} \sin u, -\frac{1}{2\tau} \cos u + \frac{1}{\sqrt{2}}v, -\frac{1}{4\tau}u - \frac{1}{2\sqrt{2}}v \sin u \right)$$

is a constant angle surface in Nil_3 , with $\theta = \frac{\pi}{4}$. This surface is a ruled surface based on a helix.

5. A POSSIBLE GENERALIZATION TO BCV-SPACES

5.1. Bianchi-Cartan-Vranceanu spaces. Constant angle surfaces have now been classified in the homogeneous 3-spaces \mathbb{E}^3 , $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and Nil_3 . All of these spaces belong to a larger family, called the BCV-spaces.

Definition 3. Let κ and τ be real numbers, with $\tau \geq 0$. The Bianchi-Cartan-Vranceanu space (BCV-space) $\widetilde{M}^3(\kappa, \tau)$ is defined as the set

$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\}$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2} + \left(dz + \tau \frac{y dx - x dy}{1 + \frac{\kappa}{4}(x^2 + y^2)} \right)^2.$$

It was Cartan who obtained this family of spaces, with $\kappa \neq 4\tau^2$, as the result of the classification of 3-dimensional Riemannian manifolds with 4-dimensional isometry group. They also appear in the work of Bianchi and Vranceanu. The complete classification of BCV-spaces is as follows:

- if $\kappa = \tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{E}^3$,
- if $\kappa = 4\tau^2 \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{S}^3(\kappa/4) \setminus \{\infty\}$,
- if $\kappa > 0$ and $\tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong (\mathbb{S}^2(\kappa) \setminus \{\infty\}) \times \mathbb{R}$,
- if $\kappa < 0$ and $\tau = 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \mathbb{H}^2(\kappa) \times \mathbb{R}$,
- if $\kappa > 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \text{SU}(2) \setminus \{\infty\}$,
- if $\kappa < 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \widetilde{\text{SL}}(2, \mathbb{R})$,
- if $\kappa = 0$ and $\tau \neq 0$, then $\widetilde{M}^3(\kappa, \tau) \cong \text{Nil}_3$.

The Lie groups in the classification are equipped with a standard left-invariant metric. This classification implies that $\widetilde{M}^3(\kappa, \tau)$ has constant sectional curvature if and only if $\kappa = 4\tau^2$. The curvature is then equal to $\frac{\kappa}{4} = \tau^2 \geq 0$. The following lemma generalizes Lemma 1.

Lemma 3. *The following vector fields form an orthonormal frame on $\widetilde{M}^3(\kappa, \tau)$:*

$$e_1 = \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \partial_x - \tau y \partial_z, \quad e_2 = \left(1 + \frac{\kappa}{4}(x^2 + y^2)\right) \partial_y + \tau x \partial_z, \quad e_3 = \partial_z.$$

The geometry of the BCV-space can be described in terms of this frame as follows.

(i) *These vector fields satisfy the commutation relations*

$$[e_1, e_2] = -\frac{\kappa}{2} y e_1 + \frac{\kappa}{2} x e_2 + 2\tau e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

(ii) *The Levi Civita connection $\widetilde{\nabla}$ of $\widetilde{M}^3(\kappa, \tau)$ is given by*

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= \frac{\kappa}{2} y e_2, & \widetilde{\nabla}_{e_1} e_2 &= -\frac{\kappa}{2} y e_1 + \tau e_3, & \widetilde{\nabla}_{e_1} e_3 &= -\tau e_2, \\ \widetilde{\nabla}_{e_2} e_1 &= -\frac{\kappa}{2} x e_2 - \tau e_3, & \widetilde{\nabla}_{e_2} e_2 &= \frac{\kappa}{2} x e_1, & \widetilde{\nabla}_{e_2} e_3 &= \tau e_1, \\ \widetilde{\nabla}_{e_3} e_1 &= -\tau e_2, & \widetilde{\nabla}_{e_3} e_2 &= \tau e_1, & \widetilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

(iii) *The Riemann Christoffel curvature tensor \widetilde{R} of $\widetilde{M}^3(\kappa, \tau)$ is determined by*

$$\begin{aligned} \widetilde{R}(X, Y)Z &= (\kappa - 3\tau^2)(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &\quad - (\kappa - 4\tau^2)(\langle Y, e_3 \rangle \langle Z, e_3 \rangle X - \langle X, e_3 \rangle \langle Z, e_3 \rangle Y) \\ &\quad + \langle X, e_3 \rangle \langle Y, Z \rangle e_3 - \langle Y, e_3 \rangle \langle X, Z \rangle e_3, \end{aligned}$$

for $p \in \widetilde{M}^3(\kappa, \tau)$ and $X, Y, Z \in T_p \widetilde{M}^3(\kappa, \tau)$.

Also the notions of Hopf fibration and Hopf cylinder can be generalized to BCV-spaces.

Definition 4. Let $\widetilde{M}^2(\kappa)$ be the following Riemannian surface of constant Gaussian curvature κ :

$$\widetilde{M}^2(\kappa) = \left(\left\{ (x, y) \in \mathbb{R}^2 \mid 1 + \frac{\kappa}{4}(x^2 + y^2) > 0 \right\}, \frac{dx^2 + dy^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2)\right)^2} \right).$$

Then the mapping $\pi : \widetilde{M}^3(\kappa, \tau) \rightarrow \widetilde{M}^2(\kappa) : (x, y, z) \mapsto (x, y)$ is a Riemannian submersion, called the Hopf-fibration. The inverse image of a curve in $\widetilde{M}^2(\kappa)$ under π is called a Hopf-cylinder and by a leaf of the Hopf-fibration one denotes a surface which is everywhere orthogonal to the fibres of π .

Remark that in the special case that $\kappa = 4\tau^2 \neq 0$, π coincides locally with the classical Hopf-fibration $\pi : \mathbb{S}^3(\kappa/4) \rightarrow \mathbb{S}^2(\kappa)$. From the theorem of Frobenius and Lemma 3 (i), it is clear that leaves of π only exist if $\tau = 0$. They are nothing but open parts of surfaces of type $\mathbb{E}^2 \times \{t_0\}$, $(\mathbb{S}^2(\kappa) \setminus \{\infty\}) \times \{t_0\}$ or $\mathbb{H}^2(\kappa) \times \{t_0\}$.

5.2. Constant angle surfaces in BCV-spaces. Let $F : M \rightarrow \widetilde{M}^3(\kappa, \tau)$ be an isometric immersion of an oriented surface in a BCV-space. We can define N , S , θ and T as in the case of Nil_3 . The four compatibility equations become

$$\begin{aligned} \nabla_X SY - \nabla_Y SX - S[X, Y] &= (\kappa - 4\tau^2) \cos \theta (\langle Y, T \rangle X - \langle X, T \rangle Y), \\ K &= \det S + \tau^2 + (\kappa - 4\tau^2) \cos^2 \theta, \\ \nabla_X T &= \cos \theta (SX - \tau JX), \\ X[\cos \theta] &= -\langle SX - \tau JX, T \rangle. \end{aligned}$$

One can define constant angle surfaces in a general BCV-space in the same way as in the Heisenberg group. In the special case that the BCV-space is $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, this definition coincides with the ones used in [2] and [3]. Also Lemma 2 can be easily adapted.

Lemma 4. Let M be a constant angle surface in a general BCV-space $\widetilde{M}(\kappa, \tau)$. Then the following statements hold.

- (i) The shape operator with respect to $\{T, JT\}$ is the same as in Lemma 2 (i).
- (ii) The Levi Civita connection of the surface is determined by the same formulas as in Lemma 2 (ii).

(iii) *The Gaussian curvature of the surface is a constant given by*

$$K = (\kappa - 4\tau^2) \cos^2 \theta.$$

(iv) *The function λ satisfies the following PDE:*

$$T[\lambda] + \lambda^2 \cos \theta + \kappa \cos \theta \sin^2 \theta + 4\tau^2 \cos^3 \theta = 0.$$

In a further study of constant angle surfaces in BCV-spaces, one may assume that $\kappa \neq 0$ and $\tau \neq 0$, since in all other cases there is now a complete classification. We see that $\theta = 0$ cannot occur, since the distribution spanned by e_1 and e_2 is not integrable, and that $\theta = \frac{\pi}{2}$ gives a Hopf cylinder.

In the final remark, we give some partial results that may lead to a complete classification.

Remark 2. To get an explicit local classification for a constant angle surface with $\theta \in]0, \frac{\pi}{2}[$ in a BCV-space with $\kappa\tau \neq 0$, we solve the PDE for λ , given in Lemma 4 (iv), by choosing local coordinates (u, v) on the surface, for instance such that $T = \partial_u$ and $\partial_v = aT + bJT$. This solution depends on the sign of $\kappa \sin^2 \theta + 4\tau^2 \cos^2 \theta$. If we suppose that we are in the case that this is strictly positive and denote this constant by r^2 , then we find

$$\lambda(u, v) = r \tan(\varphi(v) - r \cos \theta u),$$

$$a(u, v) = \frac{2\tau}{r} \sin(\varphi(v) - r \cos \theta u),$$

$$b(u, v) = \cos(\varphi(v) - r \cos \theta u)$$

for some function $\varphi(v)$. Using the same notations as in the previous section, this means that we have to solve the system

$$(15) \quad \phi_u = -\frac{\kappa}{2} \sin \theta \cos \theta (F_1 \sin \phi - F_2 \cos \phi) - 2\tau \cos^2 \theta,$$

$$(16) \quad \phi_v = a(u, v)\phi_u + b(u, v) \left(\lambda(u, v) - \frac{\kappa}{2} \sin \theta (F_1 \cos \phi + F_2 \sin \phi) \right),$$

$$(17) \quad (F_1)_u = -\sin \theta \cos \theta \cos \phi \left(1 + \frac{\kappa}{4} (F_1^2 + F_2^2) \right),$$

$$(18) \quad (F_1)_v = a(u, v)(F_1)_u + b(u, v) \sin \theta \sin \phi \left(1 + \frac{\kappa}{4} (F_1^2 + F_2^2) \right),$$

$$(19) \quad (F_2)_u = -\sin \theta \cos \theta \sin \phi \left(1 + \frac{\kappa}{4} (F_1^2 + F_2^2) \right),$$

$$(20) \quad (F_2)_v = a(u, v)(F_2)_u - b(u, v) \sin \theta \cos \phi \left(1 + \frac{\kappa}{4} (F_1^2 + F_2^2) \right),$$

$$(21) \quad (F_3)_u = -\sin \theta (-\tau F_2 \cos \theta \cos \phi + \tau F_1 \cos \theta \sin \phi - \sin \theta),$$

$$(22) \quad (F_3)_v = a(u, v)(F_3)_u - b(u, v)\tau \sin \theta (F_2 \sin \phi + F_1 \cos \phi).$$

By solving (15), (17) and (19), we obtain

$$\begin{aligned} F_1 &= \frac{\sin 2\theta}{2D(v)} \sin \phi + L(v) \cos(\rho(v)), \\ F_2 &= -\frac{\sin 2\theta}{2D(v)} \cos \phi + L(v) \sin(\rho(v)), \\ \phi &= \rho(v) + 2 \arctan \left(\frac{-A + \sqrt{B^2 - A^2} \tan \left(-\frac{1}{2} \sqrt{B^2 - A^2} u + C(v) \right)}{B} \right), \end{aligned}$$

where $D(v)$, $L(v)$, $\rho(v)$ and $C(v)$ are integration constants and

$$A(v) = \frac{\kappa}{4} \sin 2\theta L(v), \quad B(v) = D(v) + \frac{\kappa}{4} \left(\frac{\sin^2 2\theta}{4D(v)} + D(v)L^2(v) \right).$$

Remark here that $B^2 - A^2 = r^2 \cos^2 \theta$ is a strictly positive constant. When we substitute these solutions in the remaining equations (16), (18), (20), (21) and (22), calculations get rather complicated. However, we hope that this partial results can inspire other mathematicians to construct examples of constant angle surfaces in a general BCV-space or even to obtain a full classification.

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KATHOLIEKE UNIVERSITEIT LEUVEN, CELESTIJNENLAAN 200 B, B-3001 LEUVEN, BELGIUM

E-mail address, J. Fastenakels: johan.fastenakels@wis.kuleuven.be

E-mail address, J. Van der Veken: joeri.vanderveken@wis.kuleuven.be

“AL.I.CUZA” UNIVERSITY OF IASI, BD. CAROL I, N. 11, 700506 - IASI, ROMANIA

E-mail address, M. I. Munteanu: marian.ioan.munteanu@gmail.com