Minimal submanifolds in $\mathbf{R}^4$ with a g.c.K. structure

Marian-Ioan Munteanu *

Abstract

In this paper we obtain all the invariant, anti invariant and CR submanifolds in $(\mathbf{R}^4, g, J)$ endowed with a globally conformal Kähler structure which are minimal and tangent or normal to the Lee vector field of the g.c.K. structure.

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1 Preliminaries

Although known since 1954 from P. Libermann's paper [Lib54], locally conformal Kähler (l.c.K.) structures have been intensively studied only since 1976 after the impetus given by I. Vaisman in [Vai76]. A great number of research papers has appeared since then studying the main properties of l.c.K. manifolds, generalized Hopf (g.H.) manifolds, the relations with contact metric manifolds, some important classifications of submanifolds in g.H. manifolds. In 1998, the monograph by S.Dragomir & L.Ornea [DO98] brought together all known results in this field at that period. After the book, the geometers continued to study l.c.K. manifolds and many other interesting results have appeared so far.

Let (M, J, g) be a Hermitian manifold of dimension $2n$, where J denotes the complex structure and g the Hermitian metric. Then (M, J, g) is a *locally conformal Kähler* manifold if there is an open cover $\{U_i\}_{i\in I}$ of M and a family $\{f_i\}_{i\in I}$ of smooth functions $f_i : U_i \longrightarrow \mathbf{R}$ such that each local metric $g_i = \exp(-f_i)g|_{U_i}$ is Kählerian. Also (M, J, g) is a *globally conformal Kähler* (g.c.K.) manifold if there is a smooth function $f : M \longrightarrow \mathbf{R}$ such that the metric $\exp(f)g$ is Kählerian. Let Ω be the Kähler 2-form associated with (J, g) (i.e. $\Omega(X, Y) = g(X, JY)$ for $X, Y \in \chi(M)$). Then, the Hermitian manifold (M, J, g) is l.c.K. if and only if there exists a closed 1-form ω, globally defined on M, such that

$$d\Omega = \omega \wedge \Omega$$

(see [DO98] for more details). The closed 1-form ω is called the *Lee form* of the l.c.K. manifold M. Also (M, J, g) is g.c.K. (respectively Kähler) if the Lee form ω is exact (respectively $\omega = 0$). Thus any simply connected l.c.K. manifold is g.c.K.

For a l.c.K. manifold (M, J, g) we define the *Lee vector field* $B = \omega^{\#}$. Here $\#$ denotes the rising of indices with respect to g, namely $g(X, B) = \omega(X)$ for all $X \in \chi(M)$. It is very important that the Levi Civita connections D^i of the local metrics $\{g_i\}_{i\in I}$ glue up to a globally defined torsion free linear connection D on M, called the *Weyl connection* of the l.c.K. manifold M and given by

$$D_X Y = \nabla_X Y - \frac{1}{2}\left(\omega(X)Y + \omega(Y)X - g(X, Y)B\right) \tag{1.1}$$

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for any $X, Y \in \chi(M)$, where ∇ is the Levi Civita connection of (M, g). Moreover, D satisfies $Dg = \omega \otimes g$ and $DJ = 0$. As a consequence, the Hermitian manifold (M, J, g) is l.c.K. if and only if

$$\nabla_X JY = J\nabla_X Y + \frac{1}{2}\left(\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\right) \tag{1.2}$$

for any $X, Y \in \chi(M)$. Here $\theta = \omega \circ J$ and $A = -JB$ are the *anti-Lee form* and the *anti-Lee vector field*, respectively (see [DO98] for details).

2 4-dimensional l.c.K. manifolds

In [IM04] a Hermitian structure on $\mathbf{R}^4$ is defined whose scalar curvature is constant and negative, but $\mathbf{R}^4$ with this Riemannian metric is not of constant curvature.

Let us consider on $\mathbf{R}^4$ global coordinates x, y, v, w and the Riemannian metric g whose matrix with respect to these coordinates is

$$g = \begin{pmatrix} 1 & 0 & -kx & 0 \\ 0 & \alpha & ky\alpha & kx \\ -kx & ky\alpha & \alpha\beta & k^2xy \\ 0 & kx & k^2xy & 1 \end{pmatrix}$$

where $\alpha = 1 + k^2x^2$ and $\beta = 1 + k^2y^2$. Remark that for $k = 0$ one gets the 4-Euclidean space. If $k \neq 0$, $\mathbf{R}^4(k)$ is of negative constant scalar curvature $-\frac{5}{2}k^2$.

Consider on $\mathbf{R}^4$ the $(1,1)$ tensor field J putting

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial w}, \quad J\frac{\partial}{\partial y} = -ky\frac{\partial}{\partial y} + \frac{\partial}{\partial v}, \quad J\frac{\partial}{\partial v} = -\beta\frac{\partial}{\partial y} + ky\frac{\partial}{\partial v}, \quad J\frac{\partial}{\partial w} = -\frac{\partial}{\partial x}.$$

It is easy to prove that J is integrable and $g(JX, JY) = g(X, Y)$. So, $(\mathbf{R}^4, g, J)$ is a Hermitian manifold.

In the following consider the $1-$form $\omega = kdv$ and the Kähler $2-$form Ω. One easily check that the relation $d\Omega = \omega \wedge \Omega$ holds. Hence, $(\mathbf{R}^4, g, J)$ is a locally conformal Kähler manifold. Moreover, since $\mathbf{R}^4$ is simply connected, we get a globally conformal Kähler manifold with the Kähler metric $\widetilde{g} = ce^{-kv}g$, where c is a positive real number.

The Lee vector field is given explicitly as $B = k^2x\frac{\partial}{\partial x} - k^2y\frac{\partial}{\partial y} + k\frac{\partial}{\partial v}$. We immediately have the anti Lee form $\theta = k\ dy + k^2y\ dv$ and the anti Lee vector field $A = k\frac{\partial}{\partial y} - k^2x\frac{\partial}{\partial w}$.

Let us remark that B is parallel if and only if M is the Euclidean space ($k = 0$). Yet, $\nabla_B B = 0$ and $\nabla_B A = 0$. Now, using (1.1), we obtain the Weyl connection D on $\mathbf{R}^4(k)$ given by:

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -\frac{k^2x}{2}\frac{\partial}{\partial x} + \frac{k^2y}{2}\frac{\partial}{\partial y} - \frac{k}{2}\frac{\partial}{\partial v},$$

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{k^2x}{2}\frac{\partial}{\partial y} + \frac{k\gamma}{2}\frac{\partial}{\partial w},$$

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial v} = -\frac{k\gamma}{2} + \frac{k^2x}{2}\frac{\partial}{\partial v} + \frac{k^2y\gamma}{2}\frac{\partial}{\partial w},$$

$$D_{\frac{\partial}{\partial x}}\frac{\partial}{\partial w} = \frac{k}{2}\frac{\partial}{\partial y} - \frac{k^2x}{2}\frac{\partial}{\partial w},$$

$$D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{k^2x\alpha}{2}\frac{\partial}{\partial x} - \frac{k^2y(2+\alpha)}{2}\frac{\partial}{\partial y} + \frac{k(2+\alpha)}{2}\frac{\partial}{\partial v},$$

$$D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial v} = \frac{k^3xy\alpha}{2} - \frac{k(\alpha\beta+2k^2y^2)}{2}\frac{\partial}{\partial y} + \frac{k^2y(2+\alpha)}{2}\frac{\partial}{\partial v} + \frac{k^2x\alpha}{2}\frac{\partial}{\partial w},$$

$$D_{\frac{\partial}{\partial y}}\frac{\partial}{\partial w} = -\frac{k\gamma}{2}\frac{\partial}{\partial x} - \frac{k^3xy}{2}\frac{\partial}{\partial y} + \frac{k^2x}{2}\frac{\partial}{\partial v},$$

$$D_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v}=-\tfrac{k^2x\alpha\rho}{2}\frac{\partial}{\partial x}-\tfrac{k^2y\beta(2+\alpha)}{2}\frac{\partial}{\partial y}+\tfrac{k[(\alpha+2)(\beta-2)+2]}{2}\frac{\partial}{\partial v},$$
$$D_{\frac{\partial}{\partial v}}\frac{\partial}{\partial w}=-\tfrac{k^2y\gamma}{2}\frac{\partial}{\partial x}-\tfrac{k^2x\beta}{2}\frac{\partial}{\partial y}+\tfrac{k^3xy}{2}\frac{\partial}{\partial v}-\tfrac{k\gamma}{2}\frac{\partial}{\partial w},$$
$$D_{\frac{\partial}{\partial w}}\frac{\partial}{\partial w}=\tfrac{k^2x}{2}\frac{\partial}{\partial x}-\tfrac{k^2y}{2}\frac{\partial}{\partial y}+\tfrac{k}{2}\frac{\partial}{\partial v}\ (=\tfrac{1}{2}\ B),$$

where $\gamma=1-k^2x^2$ and $\rho=1-k^2y^2$.

3 Invariant and totally real minimal submanifolds of $(\mathbf{R}^4,g,J)$

A proper *invariant* submanifold M in $(\mathbf{R}^4,g,J)$ is a 2−dimensional submanifold in $\mathbf{R^4}$ such that for all X tangent to M, JX is also tangent to M.

Theorem 1 *A proper invariant submanifold M in $(\mathbf{R}^4,g,J)$ is minimal if and only if it is given by the following implicit equations:*

$$\left[\begin{array}{l} f_1(x,y,v,w)=w+kxy+C_1=0,\\ f_2(x,y,v,w)=x-C_2\mathrm{e}^{kv}=0\end{array}\right. \tag{3.3}$$

where C_1 and C_2 are real constants.

Proof. Let $X,Y=JX$ be unitary tangent vector fields on M. If h denotes the second fundamental form and H is the mean curvature, then $H=\frac{1}{2}\ (h(X,X)+h(JX,JX))$ or, more precisely

$$H=\frac{1}{2}\ (\mathrm{nor}(\nabla_X X)+\mathrm{nor}(\nabla_{JX}JX))$$

where $\mathrm{nor}(V)$ means the normal part of the vector field V. By using (1.2) we get

$$\mathrm{nor}(\nabla_{JX}JX)=\mathrm{nor}(J\nabla_{JX}X)-\frac{1}{2}\ \mathrm{nor}(B)=-\mathrm{nor}(\nabla_X X)-\mathrm{nor}(B).$$

Thus, $H=-\frac{1}{2}\mathrm{nor}(B)$ and consequently M is a minimal invariant submanifold if and only if the Lee and anti Lee vector fields are tangent to M (see also [Dra88]).

Consider M given by $f_j(x,y,v,w)=0$, $j=1,2$, where $f_{1,2}$ are smooth functions on $\mathbf{R}^4$ verifying rank $\left(\frac{D(f_1,f_2)}{D(x,y,v,w)}\right)=2$ in every point of M. Since A and B are tangent to M then they belong to $\ker(df_j)$, $j=1,2$.

Firstly, if $\frac{\partial f_1}{\partial w}$ and $\frac{\partial f_2}{\partial w}$ are both different from 0, then, by using the implicit function theorem, we can consider $f_j=w-F_j(x,y,v)$ with $F_j\in C^\infty(\mathbf{R}^3)$, $j=1,2$. If $F=F_1-F_2$ one obtains that M is given by $f_1=0$ and $F=0$ false (in this case we have supposed that both functions depend on w).

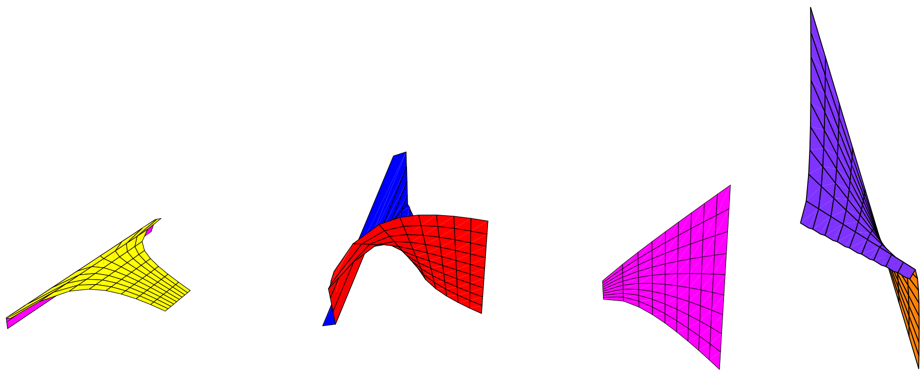
Secondly, if $\frac{\partial f_j}{\partial w}=0$ for $j=1,2$, then $\frac{\partial f_j}{\partial y}=0$ and $kx\ \frac{\partial f_j}{\partial x}-\frac{\partial f_j}{\partial v}=0$, $j=1,2$. It is obvious that $\frac{\partial f_j}{\partial x}\neq 0$ for $j=1,2$ and consequently, by virtue of the implicit function theorem, we can consider that $f_j=x-F_j(v)$, $j=1,2$. This contradicts the fact that the Jacobian has rank 2 on M.

This means that we can suppose, without loss of the generality, that $\frac{\partial f_1}{\partial w}\neq 0$ and $\frac{\partial f_2}{\partial w}=0$. Thus, as above, we can take $f_1=w-F_1(x,y,v)$ and $f_2=x-F_2(v)$, with F_1 and F_2 smooth functions satisfying

$$\frac{\partial F_1}{\partial y}+kF_2=0\quad\text{and}\quad F_2'-kF_2=0\quad\text{along the manifold.}$$

From here, we get the conclusion. ■

Remark 2 We want to know what this submanifold looks like. To do this we project it on the four coordinate planes. One obtains (when the two constants are equal to 1):
on $(x, y, v, 0)$: $y = -\frac{t}{x}$, $v = \frac{\ln x}{k}$ $(x > 0,\ t \in \mathbf{R})$
on $(x, y, 0, w)$: $x = t$, $y = -\frac{w}{kt}$ $(t > 0,\ w \in \mathbf{R})$
on $(x, 0, v, w)$: $v = \frac{\ln x}{k}$, $w = tx$ $(x > 0,\ t \in \mathbf{R})$
on $(0, y, v, w)$: $v = \frac{\ln t}{k}$, $w = -kty$, $(t > 0,\ y \in \mathbf{R})$



A *totally real* submanifold M in $(\mathbf{R}^4, g, J)$ is a (q-dimensional) submanifold such that for all X tangent to M, JX is normal to M. It follows that q must be 1 or 2. It is obvious that any curve is a totally real submanifold, so we are interested in 2-dimensional totally real submanifolds in $\mathbf{R}^4$.
We will study only two cases, namely

a) B is normal to M and thus A is tangent;

b) B is tangent to M and then A is a normal vector field.

Case a) Let $E_1 = \partial_y - kx\partial_w$ be the normalized vector of A. We are looking now for E_2 unitary, orthogonal to E_1 and tangent to M. From the relations $g(E_1, E_2) = 0$, $g(E_2, B) = 0$ and $g(E_2, E_2) = 1$ it follows that E_2 has the form $E_2 = \cos\psi\partial_x + \sin\psi\partial_w$, where ψ is a differentiable function on M. In order for a submanifold M tangent to E_1 and E_2 to exist, it is necessary to have the involutivity condition $[E_1, E_2] \in \text{span}\,\{E_1, E_2\}$. We have

$$[E_1, E_2] = -\sin\psi\ \ (\psi_y - kx\psi_w)\,\partial_x + \cos\psi\ \ (\psi_y - kx\psi_w + k)\,\partial_w$$

(along M). Obviously it is possible to have only $[E_1, E_2] \| E_2$, which yields to the following PDE:

$$\psi_y - kx\psi_w = -k\cos^2\psi \qquad \text{(on } M\text{)}. \tag{3.4}$$

Remark that $\psi = \text{constant}$ implies $\cos\psi = 0$.
Consider now that M is given by

$$M : \begin{cases} f_1(x, y, v, w) = 0 \\ f_2(x, y, v, w) = 0 \end{cases}$$

with $f_{1,2}$ smooth functions on $\mathbf{R}^4$ satisfying rank $\frac{D(f_1,f_2)}{D(x,y,v,w)} = 2$ in any point of M. Since $E_{1,2}$ are tangent to M we have

$$\begin{cases} \frac{\partial f_i}{\partial y} - kx\ \frac{\partial f_i}{\partial w} = 0 \\ \cos\psi\ \frac{\partial f_i}{\partial x} + \sin\psi\ \frac{\partial f_i}{\partial w} = 0\ , \quad i = 1,2 \quad (\text{on } M). \end{cases} \tag{3.5}$$

SUBCASE I. If $\frac{\partial f_2}{\partial w} = 0$, then $\frac{\partial f_2}{\partial y} = 0$ and $\cos\psi\ \frac{\partial f_2}{\partial x} = 0$.

I.1 If $\cos\psi = 0$, we get $\frac{\partial f_1}{\partial w} = 0$. It follows $\frac{\partial f_1}{\partial y} = 0$. Thus $f_1 = f_1(x,v)$ and $f_2 = f_2(x,v)$ with $Jac(f_1,f_2) \neq 0$. Consequently, by virtue of the implicit function theorem, M is given by $x = constant$ and $v = constant$, that is, M is a portion of a 2-plane.

I.2 If $\cos\psi \neq 0$, it follows $\frac{\partial f_2}{\partial x} = 0$ and hence $f_2 = f_2(v)$. Thus, the second equation can be replaced (by using the same argument) with $v = constant$. The Jacobian has rank 2, hence $\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 + \left(\frac{\partial f_1}{\partial w}\right)^2 \neq 0$. If $\frac{\partial f_1}{\partial w} = 0$, we get $\frac{\partial f_1}{\partial y} = 0$ and $\frac{\partial f_1}{\partial x} = 0$, which is impossible. Consequently, $\frac{\partial f_1}{\partial w} \neq 0$. From the implicit function theorem we can get $w = F(x,y)$ since v is constant. The PD equation(3.5) becomes

$$\begin{cases} -F_y - kx = 0 \\ -\cos\psi F_x + \sin\psi = 0 \quad \text{along } M. \end{cases} \tag{3.6}$$

We obtain $F(x,y) = -kxy + \mathbf{q}(x)$, $\mathbf{q} \in C^\infty(I)$, $I \subset \mathbf{R}$ and $\psi = \psi(x,y) = arctan(-ky + \mathbf{q}'(x))$; obviously ψ satisfies (3.4).
Finally, the submanifold M is given by $v = constant$ and $w = -kxy + \mathbf{q}(x)$.

SUBCASE II. Suppose $\frac{\partial f_i}{\partial w} \neq 0$ for $i = 1,2$. We can express M by $\begin{cases} w = F_1(x,y,v) \\ w = F_2(x,y,v) \end{cases}$. After some computations, this subcase yields the previous one.

We can state the following

Theorem 3 *Let M be a 2-dimensional totally real submanifold in $(\mathbf{R}^4, g, J)$ normal to the Lee vector field B. Then, either* **A:** *M is a portion of a 2-plane (namely $x = \text{constant}$, $v = \text{constant}$) or* **B:** *M is given by*

$$M : \begin{cases} v = constant, \\ w = -kxy + \mathbf{q}(x)\ , \quad \mathbf{q} \in C^\infty(I). \end{cases}$$

Remark 4 In the case **A.** $E_2 = \partial_w$ while in the case **B.**

$$E_2 = \frac{1}{\sqrt{1 + (\mathbf{q}'(x) - ky)^2}}\ \left(\partial_x + (\mathbf{q}'(x) - ky)^2 \partial_w\right)\ .$$

Remark 5 The following vector fields

$$\begin{aligned} V_1 &= kx\partial_x - ky\partial_y + \partial_v = \tfrac{1}{k}\ B, \\ V_2 &= -\sin\psi\ \partial_x + \cos\psi\ \partial_w \end{aligned}$$

are unitary, orthogonal and generate the normal bundle of M.

Theorem 6 *There is no minimal, totally real 2-dimensional submanifold M in $(\mathbf{R}^4, g, J)$ normal to the Lee vector field.*

PROOF. After an easy computation one gets $\nabla_{E_1}E_1 = kV_1$, $\nabla_{E_2}E_2 = 0$ (in case **A.**) and

$\nabla_{E_2}E_2 = \frac{k}{2}\sin 2\psi\ E_1 - k\cos^2\psi\ V_1 + \frac{\mathbf{q}''(x)}{\cos\psi}\ V_2$ (in case **B.**).

It follows that the mean curvature is $H = \frac{1}{2}\ B$ (in case **A.**) and

$H = \frac{1}{2}\ \left\{k\sin^2\psi\ V_1 + \frac{\mathbf{q}''(x)}{\cos\psi}\ V_2\right\}$ (in case **B.**).

Hence the conclusion. ∎

Case b) Let B be tangent to M (and A normal to M). Let $E_1 = kx\partial_x - ky\partial_y + \partial_v$ be the normalized vector of B. We are looking for E_2, unitary, orthogonal to E_1 and tangent to M. As in case a) we obtain $E_2 = \cos\psi\partial_x + \sin\psi\partial_w$ with ψ a smooth function on M. Since

$$\begin{aligned}[E_1, E_2] = \ & -(kx\sin\psi\ \psi_x - ky\sin\psi\ \psi_y + \sin\psi\ \psi_v + k\cos\psi)\ \partial_x + \\ & +(kx\psi_x - ky\psi_y + \psi_v)\cos\psi\ \partial_w,\end{aligned}$$

the involutivity condition yields the PDE

$$kx\psi_x - ky\psi_y + \psi_v = -\frac{k}{2}\ \sin 2\psi \quad \text{(along } M). \tag{3.7}$$

Remark that $\psi =$ constant implies $\sin 2\psi = 0$.

Consider M given by

$$M\colon \begin{cases} f_1(x,y,v,w) = 0 \\ f_2(x,y,v,w) = 0 \end{cases}$$

where $f_{1,2}$ are C^∞ functions on $\mathbf{R}^4$ verifying rank $\frac{D(f_1,f_2)}{D(x,y,v,w)} = 2$ in any point of M. Since $E_{1,2}$ are tangent to M, it follows

$$\begin{cases} kx\ \frac{\partial f_i}{\partial x} - ky\ \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial v} = 0, \\ \cos\psi\ \frac{\partial f_i}{\partial x} + \sin\psi\ \frac{\partial f_i}{\partial w} = 0, \quad i = 1,2 \quad \text{(on } M). \end{cases} \tag{3.8}$$

SUBCASE I. If $\frac{\partial f_2}{\partial w} = 0$, then $\cos\psi\ \frac{\partial f_2}{\partial x} = 0$.

I.1. If $\cos\psi \neq 0$, it follows $\frac{\partial f_2}{\partial x} = 0$ and $\frac{\partial f_2}{\partial v} = ky\ \frac{\partial f_2}{\partial y}$. Obviously $\frac{\partial f_2}{\partial y} \neq 0$. If we consider $f_2 = f_2(y,v) = y - G(v)$ one gets $-G' = kG$ and hence the second equation defining M is $ye^{kv} = constant$.

I.2. Let $\cos\psi = 0$. If $\frac{\partial f_2}{\partial x} = 0$ one comes back to the previous case and if $\frac{\partial f_2}{\partial x} \neq 0$ one considers $f_2 = f_2(x,y,v) = x - G(y,v)$. We obtain the PDE

$$G_v - kyG_y - kG = 0 \quad \text{(along } M)$$

having the solution

$$G(y,v) = e^{kv}\mathbf{q}(ye^{kv})$$

where $\mathbf{q}$ is an arbitrary smooth function depending on one variable. Hence the second equation defining M is

$$x = e^{kv}\mathbf{q}(ye^{kv}). \tag{3.9}$$

Let's analyze the first equation.

• If $\frac{\partial f_1}{\partial w} = 0$, we get $\cos\psi = 0$ and as previously $x = e^{kv}\tilde{\mathbf{q}}(ye^{kv})$. This and (3.9) yield $ye^{kv} = constant$ and consequently M is defined by

$$M\colon \begin{cases} xe^{-kv} = constant \\ ye^{kv} = constant\ . \end{cases}$$

• If $\frac{\partial f_1}{\partial w} \neq 0$, consider $f_1 = w - F(x,y,v)$. The two equations in (3.8) corresponding to $i = 1$ yield

$$\begin{cases} F_v - kyF_y + kxF_x = 0 \\ \cos\psi\ F_x - \sin\psi = 0\ . \end{cases}$$

Obviously, $\cos\psi \neq 0$ and hence

$$\begin{cases} F_x = \tan\psi \\ F_v = -kx\tan\psi + ky\ F_y\ . \end{cases} \tag{3.10}$$

The involutivity condition (3.7) becomes

$$kx\rho_x - ky\rho_y + \rho_v = -k$$

where $\rho = \ln\tan\psi$ and has the solution

$$\rho = \rho(ye^{kv}, xe^{-kv}) - kv$$

with ρ an arbitrary smooth function of two variables. The solution of the PDE's system (3.10) is

$$F(x,y,v) = \mathbf{q}(ye^{kv}, xe^{-kv})$$

with $\mathbf{q}$ a differentiable function. Recall that, since $\cos\psi \neq 0$, the second equation defining M is $ye^{kv} = constant$; hence we will take $F(x,y,v) = \mathbf{q}(xe^{-kv})$, $\mathbf{q} \in C^\infty$. Thus M is given by

$$M\colon \begin{cases} ye^{kv} = constant \\ w = \mathbf{q}(xe^{-kv}) \ \text{ or, equivalently, } w = \mathbf{q}(xy)\ . \end{cases}$$

<u>SUBCASE II.</u> Suppose $\frac{\partial f_i}{\partial w} \neq 0$ for $i = 1, 2$. In this situation we can assume that M is given by: $\begin{cases} w = F(x,y,v) \\ w = G(x,y,v) \end{cases}$ or, equivalently, by: $\begin{cases} w = F(x,y,v) \\ \phi(x,y,v) = 0 \end{cases}$ which yields the first subcase.

We can state:

Theorem 7 *Let M be a 2-dimensional totally real submanifold in $(\mathbf{R}^4, g, J)$ tangent to the Lee vector field B. Then M is given by:*

either **A**. $\begin{cases} xe^{-kv} = constant \\ ye^{kv} = constant \end{cases}$ *i.e., $M = \gamma \times \mathbf{R}$, where $\gamma(t) = (c_1e^{kt}, c_2e^{-kt}, t) \subset \mathbf{R}^3$ with c_1 and c_2 real constants;*

or **B**. $\begin{cases} ye^{kv} = constant \\ w = \mathbf{q}(xy) \end{cases}$ *with $\mathbf{q} \in C^\infty$.*

Moreover, all submanifolds belonging to the case **A**. *are minimal.*

Proof. We have to sketch out the proof only for the second part of the statement. The vector fields

$$E_1 = kx\partial_x - ky\partial_y + \partial_v \text{ and } E_2 = \partial_w$$

are unitary, orthogonal and tangent to M. Then, the normal bundle of M is spanned by

$$V_1 = \partial_y - kx\partial_w \text{ and } V_2 = \partial_x\ .$$

One has $\nabla_{E_1}E_1 = \nabla_{E_2}E_2 = 0$ which imply that the mean curvature vanishes.

■

Let's turn our attention to the case **B**.. We have

Theorem 8 *Let M be a minimal, totally real 2-dimensional submanifold in $(\mathbf{R}^4, g, J)$ belonging to the case* **B.**. *Then M is given by*

$$M : \begin{cases} ye^{kv} = constant \\ w = constant\ . \end{cases}$$

($M = \mathbf{R} \times \gamma$ where $\gamma(t) = (t, ce^{-kt}) \subset \mathbf{R}^2$ with c a real constant; hence $M \subset \mathbf{R}^3 \subset \mathbf{R}^4$.)

Proof. The normal bundle of M is spanned by

$$V_1 = \partial_y - kx\partial_w \text{ and } V_2 = -\sin\psi\partial_x + \cos\psi\partial_w.$$

We have $\nabla_{E_1} E_1 = 0$ and

$$\nabla_{E_2} E_2 = -k\cos^2\psi\ E_1 + k\sin\psi\cos\psi\ V_1 + \cos\psi\ V_2.$$

One obtains that the mean curvature is

$$H = \frac{1}{2}\ (k\sin\psi\cos\psi\ V_1 + \cos\psi\ \psi_x\ V_2)$$

and taking into account the fact that $\cos\psi \neq 0$ it follows $\sin\psi = 0$ and $\psi_x = 0$. Thus $\psi = constant$. (As a remark, the involutivity condition is fulfilled.)

After some computations, one gets $F = F(ye^{kv}) = constant$ since $ye^{kv} = constant$; hence the conclusion. ■

4 CR-submanifolds of $(\mathbf{R}^4, g, J)$

A submanifold M in a Hermitian manifold $(\widetilde{M}, J, \widetilde{g})$ is a *CR-submanifold* if it is endowed with a holomorphic distribution $\mathcal{D}$ (i.e. $J_x\mathcal{D}_x = \mathcal{D}_x$ for all $x \in M$) and such that its orthogonal complement $\mathcal{D}^\perp$ (with respect to $g = \jmath^*\widetilde{g}$) of the distribution $\mathcal{D}$ in $T(M)$ is anti-invariant, namely $J_x\mathcal{D}^\perp_x \subseteq T(M)^\perp_x$, $\forall x \in M$. (Here $T(M)^\perp$ is the normal bundle of the immersion $\jmath : M \hookrightarrow \widetilde{M}$.) Let's consider only proper CR-submanifolds, i.e. $\dim\mathcal{D} = 2s \geq 2$, $\dim\mathcal{D}^\perp = q \geq 1$. Since the ambient manifold is $\mathbf{R}^4$ it follows $s = 1$, $q = 1$ and hence M is a generic (i.e. $J_x\mathcal{D}^\perp_x = T(M)^\perp_x$) 3-dimensional CR submanifold in $(\mathbf{R}^4, g, J)$.

We are interested in finding all proper CR submanifolds M (of dimension 3) for which the Lee vector field B is tangent or normal to M. (The case B oblique yields (more) complicated relations.)

Case a) B is tangent to M.

a.1) B belongs to the distribution $\mathcal{D}$; it follows that the anti Lee vector field A lies in $\mathcal{D}$, too. Since $[A, B] = -k^2A$, it results that the holomorphic distribution is integrable. Consider M given by $f(x, y, v, w) \equiv 0$ with *grad* $f \neq 0$ on M. One gets the following PDE's system:

$$\begin{cases} f_y - kxf_w = 0, \\ kxf_x - kyf_y + f_v = 0 \quad \text{(along } M). \end{cases} \tag{4.11}$$

a.1.1) If $f_w = 0$, then $f_y = 0$ and $f_v = -kxf_x$. Thus, $f_x \neq 0$ and by using the same argument as in previous section we can write the function defining the manifold M as $f = x - F(v)$. One obtains the ODE: $kF = F'$ and consequently M is given by $xe^{-kv} = constant$.

a.1.2) If $f_w \neq 0$, then we may assume $f = w - F(x, y, v)$ and thus $f_y = kx$. It follows from $(4.11)_1$ that $F = -kxy + G(x, v)$, with G a smooth function satisfying $G_v + kxG_x = 0$ having the general solution $G(x, v) = \mathbf{q}(xe^{-kv})$, $\mathbf{q} \in C^\infty$. Thus, M is given by $f(x, y, v, w) = w + kxy - \mathbf{q}(xe^{-kv}) \equiv 0$.

We can state now the following

Theorem 9 *Let M be a proper CR submanifold such that the Lee vector field belongs to the holomorphic distribution. Then either* **1.** $M = M^T \times \mathbf{R}$ (M^T is described above) *or* **2.** *M is given by $f = w + kxy - \mathbf{q}(xe^{-kv}) \equiv 0$ where $\mathbf{q} \in C^\infty$. Moreover, every hypersurface in case* **1.** *is minimal.*

Proof. As we have already mentioned, the holomorphic distribution $\mathcal{D}$ is involutive and M^T is the integral submanifold of the distribution $\mathcal{D}$ (generated e.g. by $kx\frac{\partial}{\partial x} + \frac{\partial}{\partial v} - k^2xy\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial y} - kx\frac{\partial}{\partial w}$). Similarly, the anti holomorphic distribution $\mathcal{D}^\perp$ is involutive too. Moreover, the leaves of two distributions are totally geodesic in M.
We have only to study the minimality of the submanifolds in case **a.1.1)**. Consider the orthonormal frame on M: $\{E_1, E_2, E_3\}$ where $E_1 = \frac{1}{k}\ B$, $E_2 = \frac{1}{k}\ A$ and $E_3 = \frac{\partial}{\partial w}$. We have

$$\dot\nabla_{E_1}E_1 = 0 \text{ and } h(E_1, E_1) = 0,$$
$$\dot\nabla_{E_2}E_2 = kE_1 \text{ and } h(E_2, E_2) = 0,$$
$$\dot\nabla_{E_3}E_3 = 0 \text{ and } h(E_3, E_3) = 0$$

and hence, the mean curvature H vanishes. (We have denoted by $\dot\nabla$ and h the Levi Civita connection on M and the second fundamental form, respectively.) ■

Let us study now the minimality of the submanifolds in case **a.1.2)**: Consider the orthonormal frame $\{E_1, E_2, E_3\}$ on M, where $E_1 = \frac{1}{k}\ B$, $E_2 = \frac{1}{k}\ A$ and $E_3 = \frac{1}{\sqrt{1+T^2}}\ \left(\frac{\partial}{\partial x} + T\ \frac{\partial}{\partial w}\right)$. We have denoted $T = e^{-kv}\mathbf{q}'(xe^{-kv}) - ky$.
The unitary normal to M is $V = \frac{1}{\sqrt{1+T^2}}\left(-T\frac{\partial}{\partial x} + \frac{\partial}{\partial w}\right)$. We have

$$\dot\nabla_{E_3}E_3 = -\frac{k}{1+T^2}\ E_1 + \frac{kT}{1+T^2}\ E_2 \text{ and } h(E_3, E_3) = \frac{e^{-2kv}\mathbf{q}''}{(1+T^2)^{\frac{3}{2}}}\ V.$$

Proposition 10 *Let M as in case* **2.** *in previous theorem. Suppose M is minimal in $(\mathbf{R}^4, g)$. Then M is given by*

$$w + kxy - cxe^{-kv} = constant$$

where c is a real constant.

Proof. The statement follows from the relation $H = \frac{e^{-2kv}\mathbf{q}''}{3(1+T^2)^{\frac{3}{2}}}\ V$. ■

If X is a vector field on M we denote be PX the tangent part of JX (see for details [YK83]).

Proposition 11 *Let M be as in case* **1.** *in Theorem* 9*. Then P is parallel* (with respect to the Levi Civita connection $\dot\nabla$ on M).

Remark 12

- The previous result is not surprising (cf. theorem 1, [BD01]).
- Let M be as in case **2.** in theorem 9. It is easy to verify that the leaves of $\mathcal{D}^\perp$ are not totally geodesic in M (for example, the condition (5) in [BD01], page 7, is not satisfied). Moreover, $\dot\nabla P \neq 0$. Consequently, there are no product CR submanifolds in case **2.**

a.2) B belongs to the distribution $\mathcal{D}^\perp$; it follows that the anti Lee vector field A is normal to M. If M is given by $f(x, y, v, w) = 0$, then A and the gradient of f are collinear. From this and the relation $B(f) = 0$ it follows that M has the equation $ye^{kv} = constant$. The holomorphic distribution $\mathcal{D}$ is generated by $E_1 = \frac{\partial}{\partial x}$ and $E_2 = \frac{\partial}{\partial w}$ which means that it is involutive.

Theorem 13 *Let M be a proper CR submanifold in $(\mathbf{R}^4, J, g)$ such that the Lee vector field belongs to the anti holomorphic distribution $\mathcal{D}^\perp$. Then M is the hypersurface in $\mathbf{R}^4$ given by the equation $ye^{kv} = constant$. Moreover, M is minimal.*

Proof. The three vector fields E_1, E_2 and $E_3 = \frac{1}{k}\ B$ form an orthonormal basis in $\chi(M)$. We have

$\dot\nabla_{E_1} E_1 = -kE_3$ and $h(E_1, E_1) = 0$,
$\dot\nabla_{E_2} E_2 = 0$ and $h(E_2, E_2) = 0$,
$\dot\nabla_{E_3} E_3 = 0$ and $h(E_3, E_3) = 0$.

Hence the conclusion. ∎

a.3) B has component both in $\mathcal{D}$ and in $\mathcal{D}^\perp$; let's say $B = B_1 + B_2$, with $B_1 \in \mathcal{D}$ and $B_2 \in \mathcal{D}^\perp$. Then A is oblique and $\tan(A) = -JB_1$ and $\mathrm{nor}(A) = -JB_2$, where $\tan(A)$ and $\mathrm{nor}(A)$ denote the tangent and the normal part of A, respectively. Moreover, the tangent part of A belongs to the holomorphic distribution $\mathcal{D}$. Consider $\tan(A) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + p\frac{\partial}{\partial v} + q\frac{\partial}{\partial w}$, with a, b, p, q smooth functions on $\mathbf{R}^4$. The orthogonality between B and $\mathrm{nor}(A)$ yields $p = 0$. Thus we have

$$\begin{cases} \tan(A) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + q\frac{\partial}{\partial w}, \\ B_1 = -q\frac{\partial}{\partial x} - bky\frac{\partial}{\partial y} + b\frac{\partial}{\partial v} + a\frac{\partial}{\partial w}, \\ B_2 = (q + k^2x)\frac{\partial}{\partial x} - (k-b)ky\frac{\partial}{\partial y} + (k-b)\frac{\partial}{\partial v} - a\frac{\partial}{\partial w} \end{cases}$$

which generate $\chi(M)$ and $\mathrm{nor}(A) = -a\frac{\partial}{\partial x} + (k-b)\frac{\partial}{\partial y} - (k^2x+q)\frac{\partial}{\partial w}$, which is normal to M. Then, since $\tan(A) \perp \mathrm{nor}(A)$, one gets

$$a^2 = b(k-b) - (bkx+q)^2$$

with $a^2+b^2+q^2 \neq 0$ and $a^2+(k-b)^2+(k^2x+q)^2 \neq 0$. It is easy to see that we have $\mathrm{nor}(A) \perp B_{1,2}$ and $B_1 \perp B_2$.

If the submanifold M is given by $f(x, y, v, w) = 0$, one obtains, from the tangency conditions, the following PDE's system (on M):

$$\begin{cases} af_x + bf_y + qf_w = 0, \\ -qf_x - bkyf_y + bf_v + af_w = 0, \\ kxf_x - kyf_y + f_v = 0\ . \end{cases} \tag{4.12}$$

Moreover, grad (f) is parallel to the normal part of A.

From $(4.12)_{2,3}$ we get $af_w = (bkx + q)f_x$.

a.3.1) $a = 0$: It is not difficult to prove that $bkx + q$ cannot be 0 (otherwise $\mathrm{nor}(A) = 0$). It follows that $f_x = 0$ and hence, by replacing in (4.12), we have $f_v = kyf_y$.

a.3.1.i) $f_y = 0$: Then $f_v = 0$ and M is given by $w = constant$. Moreover $q = 0$ and $b = \frac{k}{\alpha}$.

a.3.1.ii) $f_y \neq 0$: Then $f = y - F(v, w)$ and, together with $f_v = kyf_y$, we obtain that $F(v, w) = \mathbf{q}(w)e^{-kv}$, $\mathbf{q} \in C^\infty(I)$, $I \subset \mathbf{R}$. After some computations, one has

$$b = \frac{k(\mathbf{q}'(w)e^{-kv})^2}{1 + (kx + \mathbf{q}'(w)e^{-kv})^2} \quad , \quad q = \frac{k\mathbf{q}'(w)e^{-kv}}{1 + (kx + \mathbf{q}'(w)e^{-kv})^2}$$

and hence $\mathbf{q}$ cannot be constant (otherwise $\tan(A) = 0$). In this case, M is given by $ye^{kv} = \mathbf{q}(w)$.

a.3.2) $a \neq 0$: It follows that $f_w = \frac{q+bkx}{a}\ f_x$. The case $f_x = 0$ yields a contradiction (namely $b = 0$ and $a^2 = -q^2$). Consequently $f_x \neq 0$ and we can consider $f = x - F(y, v, w)$, with F verifying $-F_w = \frac{q+bkF}{a}$. From (4.12) we also obtain a PDE, namely $F_v = kyF_y + kF$ with the general solution $F(y,v,w)=e^{kv}\mathbf{q}(ye^{kv},w)$ ($\mathbf{q}$ is a smooth function depending on two variables). After some computations one has

$$b=\frac{k}{1+(t_1+kx)^2+t_2^2} \quad , \quad q=t_1 b \quad , \quad a=t_2 b$$

where

$$t_1=-\frac{kF+F_yF_w}{1+F_w^2} \text{ and } t_2=\frac{F_y-kFF_w}{1+F_w^2} \; .$$

M is defined by the equation $xe^{-kv}=\mathbf{q}(ye^{kv},w)$.

Theorem 14 *Let M be a proper CR-submanifold in $(\mathbf{R}^4,J,g)$ such that the Lee vector field B is tangent to M and has components both in $\mathcal{D}$ and in $\mathcal{D}^\perp$. Then we have one of the following three situations:*
1. *M is the hyperplane $w=constant$. In this case M is minimal.*
2. *M is given by $ye^{kv}=\mathbf{q}(w)$.*
3. *M is defined by the equation $xe^{-kv}=\mathbf{q}(ye^{kv},w)$.*
If we require M to be minimal, we get:
in case 2: $\mathbf{q}'(w)=c$, a nonzero real constant and hence M is given by

$$cw-ye^{kv}=constant$$

in case 3: $\mathbf{q}'(ye^{kv},w)=cye^{kv}+constant$, where c is a nonzero real number; consequently, M is defined by

$$xe^{-kv}-cye^{kv}=constant \; .$$

Proof. We will sketch the proof only for the second part of the statement.
Let M be given by the equation $ye^{kv}=\mathbf{q}(w)$, with q a nonconstant smooth function. Consider in $\chi(M)$ the following orthonormal frame: $E_1=\frac{\partial}{\partial x}$, $E_2=\frac{1}{k}\,B$, $E_3=\frac{1}{\sqrt{T}}\left(\mathbf{q}'(w)\frac{\partial}{\partial y}+e^{kv}\frac{\partial}{\partial w}\right)$ where $T=\left(\mathbf{q}'(w)kx+e^{kv}\right)^2+\mathbf{q}'(w)^2$. Then $V=\frac{\rho\frac{\partial}{\partial y}+\mu\frac{\partial}{\partial w}}{||\rho\frac{\partial}{\partial y}+\mu\frac{\partial}{\partial w}||}$ is a normal and unitary vector field on M, where $\rho=e^{kv}+kx\mathbf{q}'(w)$ and $\mu=-\left(kxe^{kv}+\alpha\mathbf{q}'(w)\right)$.
The Gauss and Weingarten formulas yield

$\dot\nabla_{E_1}E_1=-kE_2\;,\qquad h(E_1,E_1)=0,$
$\dot\nabla_{E_2}E_2=0\;,\qquad h(E_2,E_2)=0,$
$\dot\nabla_{E_3}E_3=-\frac{k\rho\mathbf{q}'(w)}{T}\,E_1+\frac{k\mathbf{q}'(w)^2}{T}\,E_2+\frac{e^{kv}\mu\mathbf{q}''(w)}{T}\left(\lambda+\frac{2}{\sqrt{T}}\right)E_3\;,\qquad h(E_3,E_3)=\frac{\lambda\mathbf{q}''(w)}{T}\,V$

with λ a certain nonzero function.
Thus M is minimal if and only if $\mathbf{q}''(w)=0$. Hence we get the conclusion.
In this case:
$T=\left(ckx+e^{kv}\right)^2+c^2$, $E_3=\frac{1}{\sqrt{T}}\left(c\frac{\partial}{\partial y}+e^{kv}\frac{\partial}{\partial w}\right)$ and $\dot\nabla_{E_3}E_3=-\frac{ck}{T}\left(ckx+e^{kv}\right)E_1+\frac{c^2k}{T}\,E_2$.

Let now M be given by $xe^{-kv}=\mathbf{q}(ye^{kv},w)$.
First let's make some notations: $P=e^{kv}\mathbf{q}_2'$, $Q=e^{kv}\mathbf{q}_1'-kx\mathbf{q}_2'$ where by $\mathbf{q}_1'$ and $\mathbf{q}_2'$ we have denoted the (first) partial derivatives with respect to the first and to the second coordinate, respectively. Put also $T=\frac{1}{\sqrt{1+P^2}}$ and $S=\frac{1}{\sqrt{1+P^2+e^{2kv}Q^2}}$. Consider now the following orthonormal frame in $\chi(M)$:

$E_1 = \frac{1}{k}\ B,\ E_2 = T\left(P\frac{\partial}{\partial x} + \frac{\partial}{\partial w}\right)$
$E_3 = TS\left(e^{kv}Q\frac{\partial}{\partial x} + (1+P^2)\frac{\partial}{\partial y} - \left(e^{kv}PQ + kx(1+P^2)\right)\frac{\partial}{\partial w}\right).$

Then, a unitary vector field and normal to M is
$V = S\left(\frac{\partial}{\partial x} - e^{kv}Q\frac{\partial}{\partial y} - \left(P - e^{kv}kxQ\right)\frac{\partial}{\partial w}\right).$

The Gauss and Weingarten formulas yield

$\dot{\nabla}_{E_1}E_1 = 0, \quad h(E_1,E_1) = 0,$

$\dot{\nabla}_{E_2}E_2 = -kP^2T^2E_1 + TS\left(kP + e^{2kv}QT^2\mathbf{q}''_{22}\right)E_3, \quad h(E_2,E_2) = e^{kv}T^2S\left(-kPQ + \mathbf{q}''_{22}\right)V,$

$$\begin{aligned}\dot{\nabla}_{E_3}E_3 \ &= \frac{E_3(TS)}{TS}\ E_3 + T^2S^2\Big\{k\left[(1+P^2)^2 - e^{2kv}Q^2\right]E_1 + ke^{kv}P^2(1+P^2)QTE_2\\ &-ke^{2kv}PQ^2\frac{S}{T}\ E_3 + e^{2kv}T\left[e^{kv}P^3\mathbf{q}''_{11} - e^{kv}(1+P^2)Q\mathbf{q}''_{12} - Q\left(e^{kv}PQ + kx(1+P^2)\right)\mathbf{q}''_{22}\right]\\ &+e^{2kv}S\left[e^{2kv}(1+P^2)^2QT\mathbf{q}''_{11} + \frac{2}{T}(1+P^2)\left(P - kxe^{kv}Q\right)\mathbf{q}''_{12}\right.\\ &\left.+\left(e^{2kv}P^2Q^2 - k^2x^2(1+P^2)^2\right)QT\mathbf{q}''_{22}\right]E_3\Big\},\end{aligned}$$

$$\begin{aligned}h(E_3,E_3) &= T^2S^2e^{kv}\Big\{kPQ^3e^{2kv} + e^{2kv}(1+P^2)^2\mathbf{q}''_{11}\\ &-2e^{kv}(1+P^2)\left(e^{kv}PQ + kx(1+P^2)\right)\mathbf{q}''_{12} + \left(e^{kv}PQ + kx(1+P^2)\right)^2\mathbf{q}''_{22}\Big\}V.\end{aligned}$$

Consequently, if the mean curvature of M vanishes, then

$$\begin{aligned}-kPQ(1+P^2) + e^{2kv}(1+P^2)^2\mathbf{q}''_{11} - 2e^{kv}(1+P^2)\left(e^{kv}PQ + kx(1+P^2)\right)\mathbf{q}''_{12}\\ +\left(1 + P^2 + e^{2kv}Q^2\right)\mathbf{q}''_{22} = 0.\end{aligned} \tag{4.13}$$

The expression above is a polynomial of second order in x (since P does not depend on x and Q is affine in x). So, if we look at the coefficient of x^2 one gets $k^2(1+P^2)\mathbf{q}''_{22} = 0$ and hence $\mathbf{q}'_2$ is constant with respect to the second variable. Thus,

$$\mathbf{q}(ye^{kv}, w) = \sigma(ye^{kv})w + \tau(ye^{kv})$$

for some smooth functions σ and τ. We return to (4.13) and find the coefficient of x. This is

$$(1+P^2)\left(ke^{kv}(k\sigma^2 - 2\sigma') + 2ke^{3kv}\sigma^2(1-\sigma')\right)$$

and must vanish on M. It follows $\sigma = 0$. Finally, the remaining term is $e^{2kv}\tau'' = 0$. Thus $\tau(ye^{kv}) = cye^{kv} + constant$ and from this we get the conclusion.

In this case we have

$$P = 0,\ Q = ce^{kv},\ T = 1,\ S = \frac{1}{\sqrt{1 + c^2e^{4kv}}}$$

and

$E_1 = \frac{1}{k}\ B,\ E_2 = \frac{\partial}{\partial w}$
$E_3 = \frac{1}{\sqrt{1+c^2e^{4kv}}}\left(\frac{\partial}{\partial x} - ce^{2kv}\frac{\partial}{\partial y} + ckxe^{2kv}\frac{\partial}{\partial w}\right).$

Moreover, $h(E_1,E_1) = h(E_2,E_2) = h(E_3,E_3) = 0$

■

Case b) B is normal to M. It follows that A belongs to the anti holomorphic distribution $\mathcal{D}^{\perp}$. If M is given by $f(x,y,v,w) = 0$ (with $grad\ f \neq 0$), we have $A(f) = 0$ and $B \| grad\ f$. Consequently, one gets $f_w = 0$, $f_y = 0$ and $f_x = 0$. Hence, $f = f(v) = 0$ which means that M is the hyperplane $v = constant$. We easily obtain that the holomorphic distribution $\mathcal{D}$ is generated by $E_1 = \frac{\partial}{\partial x}$ and $E_2 = \frac{\partial}{\partial w}$. Thus $\mathcal{D}$ is involutive.

Theorem 15 *Let M be a proper CR submanifold in $(\mathbf{R}^4, J, g)$ normal to the Lee vector field B. Then M is a hyperplane in $\mathbf{R}^4$. Moreover, M is minimal but not totally geodesic.*

Proof. The three vector fields E_1, E_2 and $E_3 = \frac{1}{k}\ A$ form an orthonormal basis in $\chi(M)$. We have

$$\dot\nabla_{E_1} E_1 = 0 \text{ and } h(E_1, E_1) = -kV,$$
$$\dot\nabla_{E_2} E_2 = 0 \text{ and } h(E_2, E_2) = 0,$$
$$\dot\nabla_{E_3} E_3 = 0 \text{ and } h(E_3, E_3) = kV$$

where $V = \frac{1}{k}\ B$ is the unitary normal to M. Hence the conclusion. ∎

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Permanent address:
University 'Al.I.Cuza' of Iaşi,
Faculty of Mathematics,
Bd. Carol I, no.11,
700506 Iaşi,
Romania

Temporary address:
Katholieke Universiteit Leuven,
Departement Wiskunde
Celestijnenlaan 200B
B-3001 Leuven (Heverlee)
BELGIUM

e-mail: munteanu@uaic.ro,
munteanu2001@hotmail.com

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