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CR-structures of CR-codimension 2 on hypersurfaces in Sasakian manifolds

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Abstract. In [**7**] P.Matzeu & V.Oproiu have introduced a torsion free linear connection adapted to an almost contact structure associated with a given pseudoconvex CR-manifold. In this paper we consider CR-structures of codimension 2 on hypersurfaces in Sasakian manifolds. We use a natural f-structure with complemented frames in order to obtain a torsion free linear connection. Then, we examine some symmetry properties of the curvature tensor field of this connection. In the end of the paper we present some examples in the case when the ambient is $\mathbf{R}^5$ and S^5 endowed with the canonical Sasakian structures.

1. The adapted torsion free canonical connection

The study of CR-manifolds of higher codimension in general and of codimension two in particular was intensively developed in the last period (see for example [**4**], [**10**]). For CR-structures of higher codimension (e.g. real submanifolds of higher codimension in complex manifolds) the situation is much more complicated (the main problem is that the Levi form takes values in a real vector space of dimension bigger than one). In this case it is quite difficult to find geometric structures similar to Cartan connections. There are, however, few cases in which the situation is more controlable. The simplest of these cases is that of CR dimension two and codimension two treated in [**3**].

In this paper we study CR-structures of codimension two defined in a natural way on hypersurfaces in a Sasakian manifold.

Let $\widetilde{M}$ be a smooth Sasakian manifold of dimension $2n+3$ with the contact metric structure $(\widetilde{\varphi},\ \widetilde{\xi},\ \widetilde{\eta},\ \widetilde{g})$ and let $\widetilde{\nabla}$ be its Levi-Civita connection verifying

$$(\widetilde{\nabla}_X\widetilde{\varphi})Y=-\widetilde{g}(X,Y)\widetilde{\xi}+\widetilde{\eta}(Y)X, \tag{1}$$

for $X,Y\in\chi(\widetilde{M})$ (see for details [**1**], [**2**]).

Let M be an oriented C^∞ hypersurface in $\widetilde{M}$ tangent to the structure vector field $\widetilde{\xi}$ and let $\iota: M\hookrightarrow\widetilde{M}$ the immersion of M in $\widetilde{M}$. Since $\widetilde{M}$ is a contact manifold we have

$$\widetilde{g}(X,Y)=d\widetilde{\eta}(X,\widetilde{\varphi}Y)+\widetilde{\eta}(X)\widetilde{\eta}(Y)\ ,\quad \forall\ X,Y\in\chi(\widetilde{M})\ . \tag{2}$$

On M we set the 1-form $\eta=\iota^*\widetilde{\eta}$ and let ξ be the restriction of $\widetilde{\xi}$ to M, i.e. $\iota_*\xi=\widetilde{\xi}$. Let $N\in\chi(\widetilde{M})$ be the unit vector field normal to M and define $U\in\chi(M)$ such that

$$\iota_*U=\widetilde{\varphi}N \tag{3}$$

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(since $\widetilde{\varphi}N$ is a vector field tangent to M). Let g be the induced metric $g = \iota^*\widetilde{g}$. Define also the 1-form u on M by

$$u(X) = g(U,X)\ , \quad \forall\ X \in \chi(M)\ . \tag{4}$$

Consider the distribution

$$H(M) = \{X \in \chi(M)\ :\ \eta(X) = 0\ ,\ u(X) = 0\}$$

and the endomorphism $J : H(M) \longrightarrow H(M)$ given by the restriction of $\widetilde{\varphi}$ to $H(M)$, which has the property

$$J^2 = -\mathrm{id}_{H(M)}\ . \tag{5}$$

The tangent space of M can be decomposed in the following direct sum

$$T(M) = H(M) \oplus \text{ span } [U] \oplus \text{ span } [\xi]\ . \tag{6}$$

Let us remark that $d\eta$ is non degenerate on $H(M)$.

Lemma 1. We have

$$[\xi, \Gamma(H(M))] \subset \Gamma(H(M)) \tag{7}$$

$$[U, \Gamma(H(M))] \in \text{ ker } \eta, \tag{8}$$

where $\Gamma(H(M))$ is the $C^\infty(M)$-module of smooth sections in $H(M)$.

Proposition 1. The vector fields ξ and U are orthogonal and unitary and verify $[U,\xi] = 0$.

Proposition 2. The distribution $H(M)$ defines a CR-structure on M of CR-codimension 2.

In the next we define on M a tensor field f of type $(1,1)$ as follows

$$\begin{aligned} &f : \chi(M) \longrightarrow \chi(M) \\ &fX = JX \ \text{ for } \ X \in H(M)\ , \quad fU = 0\ ,\ f\xi = 0 \end{aligned} \tag{9}$$

which verifies the condition

$$f^3 + f = 0\ . \tag{10}$$

Proposition 3. The structure $(\varphi,\ \xi,\ U,\ \eta,\ u)$ defined on M is an f structure with complemented frame (see S.I.Goldberg & K.Yano, [**5**]) or, in other terminology, an f structure with parallelizable kernel (f-*pk* structure).

If X and Y are vector fields on M then we have

$$g(X,Y) = d\eta(X, fY) + u(X)u(Y) + \eta(X)\eta(Y)\ . \tag{11}$$

In [**9**], R.Mizner develops analogs of Webster-Tanaka connections for higher codimension almost CR-manifolds. In [**6**] it is defined a linear connection on an almost $\mathcal{S}$-manifold which generalizes the Tanaka-Webster connection for strictly pseudoconvex CR-manifolds of hypersurface type and hence a CR-integrable almost S-structure on a manifold is canonically interpreted as a reductive Cartan geometry, which is torsion free if and only if the almost S-structure is normal. In the following we will construct a torsion free connection on M as being the analogue of Matzeu-Oproiu connection. In codimension one this connection yields to the same Bochner type curvature invariant as that constructed by using Tanaka-Webster connection (see [**7**]).

From now on we will suppose that the inner product

$$U \lrcorner\, du = 0 \tag{12}$$

holds on M; this condition yields to $[U, \Gamma(H(M))] \subset \ker\ u$ and consequently, by virtue of (8) we obtain $[U, \Gamma(H(M))] \subset \Gamma(H(M))$.

The condition above is a weaker condition than the "$\mathcal{S}$" condition $du = \Phi$.

There are some important cases in which this happens, namely totally contact geodesic (TCG), totally contact umbilical (TCU) and pseudoumbilical hypersurface (PUH) (see e.g. [**11**]).

Denote by $\psi = \frac{1}{2}(\mathcal{L}_\xi f)$ and $p = \frac{1}{2}(\mathcal{L}_U f)$ where $\mathcal{L}$ is the Lie derivative. Since $\widetilde{M}$ is Sasakian we give

Proposition 4. We have

1) $\psi \equiv 0$

2) $2pX = (A + fAf)X + u(X)\ \xi + \eta(X)\ U - b\ u(X)\ U$;

p vanishes in cases (TCG), (TCU) and (PUH).

(Here A is the Weingarten operator.)

In the following we are looking for a torsion free connection on M related in a certain way with the structures defined so far. We can state the following theorem.

Theorem 1. *There exists one and only one torsion free connection on M such that*

$$\begin{aligned}
&(\nabla_X \eta)(Y) = d\eta(X, Y)\ , \qquad (\nabla_X u)(Y) = du(X, Y) \\
&\nabla_X d\eta = 0\ , \quad \nabla_X \xi = 0\ , \quad \nabla_X U = 0 \\
&(\nabla_X f)Y = u(X)\,\{(A + fAf)Y + u(Y)\xi + \eta(Y)U - b\ u(Y)U\} - \\
&\qquad\qquad - d\eta(X, fY)\xi + d\eta(AX, fY)U.
\end{aligned} \tag{13}$$

Proof. A similar idea as in the proof of the Levi Civita theorem yields to the expressions for $\eta(\nabla_X Y)$, $u(\nabla_X Y)$ and $d\eta(\nabla_X Y, Z)$, $X, Y, Z \in \chi(M)$ which completely define the connection ∇. ∎

2. Curvature of the torsion free adapted connection

Consider the curvature tensor field R of ∇ defined by

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\ , \quad X,\ Y,\ Z \in \chi(M). \tag{14}$$

We will find some general relations and properties of R and especially for the restriction of R on $H(M)$. Equations (13) imply:

$$R_{XY}\xi = 0\ , \quad R_{XY}U = 0\ , \quad \forall\ X, Y \in \chi(M). \tag{15}$$

Moreover, $R_{XY}Z$ belongs to $\ker\eta$ for all $X, Y, Z \in \chi(M)$ but it is not necessarily a section in $H(M)$. We also have

$$\begin{aligned}
&R_{XY}fZ = fR_{XY}Z + 4du(X, Y)pZ - \\
&\quad -(\nabla_X du)(Y, fZ)\ U + (\nabla_Y du)(X, fZ)\ U\ , \ X, Y, Z \in \Gamma(H(M)).
\end{aligned} \tag{16}$$

Define now a 4 covariant tensor field $\mathcal{R}$ by

$$\mathcal{R}(W,Z,X,Y)=g(W,R_{XY}Z)\ ,\quad X,Y,Z\in\chi(M) \tag{17}$$

($\mathcal{R}$ is a kind of Riemann Christoffel tensor). Remark that, since $R_{XY}Z$ is a section in $H(M)$ we have

$$\mathcal{R}(W,Z,X,Y)=d\eta(W,\ fR_{XY}Z)$$

for all $X,Y,Z\in\chi(M)$.

We are interested now to find some symmetry properties for the tensor field $\mathcal{R}$ similar those for the usual Riemann Christoffel tensor (in Riemannian geometry).

Obviously $\mathcal{R}(W,Z,X,Y)=-\mathcal{R}(W,Z,Y,X)$ and $\underset{(X,Y,Z)}{\mathfrak{S}}\mathcal{R}(W,Z,X,Y)=0$ (due to the first Bianchi identity fulfilled by R). Using (16) we get

$$\mathcal{R}(Z,Z,X,Y)=d\eta(Z,AZ)\ (d\eta(X,AY)-d\eta(Y,AX)) \tag{18}$$

(with $X,Y,Z\in\Gamma(H(M))$) which implies

$$\begin{array}{c}\mathcal{R}(Z,W,X,Y)+\mathcal{R}(W,Z,X,Y)=\\ =(d\eta(Z,AW)+d\eta(W,AZ))(d\eta(X,AY)-d\eta(Y,AX)).\end{array} \tag{19}$$

As consequence we have

Proposition 5. The Riemann Christoffel tensor field $\mathcal{R}$ of the linear connection ∇ satisfies the following equation

$$\begin{array}{c}\mathcal{R}(W,Z,X,Y)-\mathcal{R}(X,Y,W,Z)=\\ =d\eta(X,AY)d\eta(Z,AW)-d\eta(Y,AX)d\eta(W,AZ)+d\eta(W,AX)d\eta(Z,AY)\\ -d\eta(X,AW)d\eta(Y,AZ)+d\eta(X,AZ)d\eta(Y,AW)-d\eta(Z,AX)d\eta(W,AY).\end{array} \tag{20}$$

From the relation above we easily obtain

$$\mathcal{R}(E,Z,E,Y)-\mathcal{R}(E,Y,E,Z)=d\eta(E,AE)\ (d\eta(Z,AY)-d\eta(Y,AZ)) \tag{21}$$

where E,Y,Z are sections in $H(M)$. Now, replacing E by fE and taking into account that $d\eta(fE,AfE)=-d\eta(E,AE)$ we get

$$\mathcal{R}(E,Y,E,Z)+\mathcal{R}(fE,Y,fE,Z)=\mathcal{R}(E,Z,E,Y)+\mathcal{R}(fE,Z,fE,Y). \tag{22}$$

Remark 1. If M is TCG, TCU or PUH in $\widetilde{M}$ then $\mathcal{R}$ is skew-symmetric in first two arguments and pairs symmetric (i.e. $\mathcal{R}(W,Z,X,Y)=\mathcal{R}(X,Y,W,Z)$).

Consider the two times covariant tensor

$$\rho(R)(Y,Z)=\text{trace }(X\longmapsto R_{XY}Z)\ ,\ X,Y,Z\in\chi(M) \tag{23}$$

(where the trace is made by using the metric g) – the tensor defined above is a kind of Ricci tensor. If we take an orthonormal basis of the form $\{E_i,fE_i,\xi,U\}_{i=\overline{1,n}}$ on M, the Ricci tensor can be written as

$$\rho(R)(Y,Z)=\sum_{i=1}^{n}\{\mathcal{R}(E_i,Z,E_i,Y)+\mathcal{R}(fE_i,Z,fE_i,Y)\}\,.$$

As consequence of the relation (22) we have the symmetry of the Ricci tensor, namely

$$\rho(R)(Y,Z)=\rho(R)(Z,Y)\ ,\ Y,Z\in\Gamma(H(M)). \tag{24}$$

Moreover, we have

$$\rho(R)(fY,fZ)-\rho(R)(Y,Z)=4(du(pY,fZ)+du(pZ,fY)). \tag{25}$$

Examples

In this section we will give some examples. Let consider first as ambient manifolds $\mathbf{R}^5$ with (global) coordinates $x,\ y,\ v,\ w,\ z$ with the usual Sasakian structure (see for details **[2]**).

Example 1. (TCG) Consider M the hyperplane (passing by the origin and being parallel with z axis) defined by $f(x,y,z,v,w) = ax + by + cv + dw \equiv 0$ where $a,\ b,\ c,\ d \in \mathbf{R}$ with $a^2+b^2+c^2+d^2 = \mu^2 \not\equiv 0$. We have

$$U = \frac{2}{\mu}\ \{-b\partial_x + a\partial_y - d\partial_v + c\partial_w - (by+dw)\ \partial_z\}$$

and $\quad X_1 = (c^2+d^2)\ \partial_x - (ac+bd)\ \partial_v - (ad-bc)\ \partial_w + [(c^2+d^2)y - (ac+bd)w]\ \partial_z$; X_1 and $X_2 = JX_1$ belong to $\Gamma(H(M))$ and satisfy $[U,X_1] = 0$, $[U,JX_1] = 0$.

Since the 2-form du vanishes identically we obtain a flat connection.

Example 2. (TCU) Let M be defined by $f(x,y,v,w,z) = x^2+y^2+v^2+w^2-1 \equiv 0$ (a hyper cylinder $S^3\times\mathbf{R}$ in $\mathbf{R}^5$). The vector field U is given by

$$U = 2\{x\partial_y - y(\partial_x + y\partial_z) + v\partial_w - w(\partial_v + w\partial_z)\}\ .$$

The vector fields X_1 and $X_2 = JX_1$ form a basis in $\Gamma(H(M))$ where

$$X_1 = 4\{(v^2+w^2)\partial_x - (xv+yw)\partial_v - (xw-yv)\partial_w + v(yv-xw)\partial_z\}$$

and verify the relations

$$[U,X_1] = -2X_2 \quad , \quad [U,X_2] = 2X_1.$$

Moreover, since $du = \frac{1}{2}\ (dx\wedge dy + dv\wedge dw)$ we get $U \lrcorner\ du = 0$. We have also

$$[X_1,X_2] = -16(v^2+w^2)\ U - 8(v^2+w^2)\ \xi.$$

Now we are able to write the expression of the connection ∇. One obtains

$$\begin{cases} \nabla_{X_1}X_1 = -4xX_1 + 4yX_2 \quad , \quad \nabla_{X_2}X_2 = 4xX_1 - 4yX_2 \\ \nabla_{X_1}X_2 = -4yX_1 - 4xX_2 - 8(v^2+w^2)\ U - 4(v^2+w^2)\ \xi \\ \nabla_U X_1 = -2X_2 \quad , \quad \nabla_U X_2 = 2X_1\ . \end{cases}$$

Computing the curvature tensor of ∇ we get

$$R_{X_1X_2}X_1 = -64(v^2+w^2)X_2 \quad , \quad R_{X_1X_2}X_2 = 64(v^2+w^2)X_1$$

other components being zero. It follows

$$\rho(R)(X_1,X_1) = \rho(R)(X_2,X_2) = 256(v^2+w^2)$$

and $\rho(R)(X_1,X_2) = 0$.

The next example is inspired from the following theorem (**[11]**, Th. 5.2, p. 185): *Let M be a compact orientable pseudo-umbilical hypersurface of S^{2n+1} $(n \geqslant 2)$. Then M is*

$$S^{2n-1}(r_1)\times S^1(r_2)\ , \qquad r_1^2 + r_2^2 = 1.$$

Example 4. (PUH) Let $M = S^3(r_1)\times S^1(r_2)$ with $r_1^2+r_2^2 = 1$ be a pseudo-umbilical hypersurface in S^5 $(\subset \mathbf{R}^6)$ as a Sasakian space form. On $\mathbf{R}^6$ consider global coordinates

x, y, v, w, s, t so, on M we have $|p_1| = r_1$ and $|p_2| = r_2$ where $p_1 = (x, y, v, w)$, $p_2 = (s, t)$ ans $|\cdot|$ denotes the usual Euclidean norm. Consider

$$\xi_1 = \frac{1}{r_1}(-y, x, -w, v)\ ,\ X_1 = \frac{1}{r_1}(-v, w, x, -y)\ ,\ X_2 = \frac{1}{r_1}(-w, -v, y, x)$$

which form an orthonormal frame on $S^3(r_1)$ and $\xi_2 = \frac{1}{r_2}(-s, t) \in \chi(S^1(r_2))$. Consider also the following contact forms on $S^3(r_1)$ and $S^1(r_2)$ respectively

$$\eta_1 = \frac{1}{r_1}(-y\ dx + x\ dy - w\ dv + v\ dw)\ ,\ \eta_2 = \frac{1}{r_2}(-s\ dt + t\ ds).$$

With these notations, the (almost) contact structure on S^5 is given by

$$\xi = r_1\xi_1 + r_2\xi_2\ ,\ \eta = r_1\eta_1 + r_2\eta_2\ ,\ \varphi X_1 = X_2, \varphi X_2 = -X_1.$$

The unit normal vector field on M is $N = -\frac{r_2}{r_1}p_1 + \frac{r_1}{r_2}p_2$ thus $U = -r_2\xi_1 + r_1\xi_2$. We have obtained a global frame on M satisfying

$$\begin{cases} [X_1, \xi] = \ 2X_2\ ,\ [X_1, U] = -\frac{2r_2}{r_1}\ X_2 \\ [X_2, \xi] = -2X_1\ ,\ [X_2, U] = \ \frac{2r_2}{r_1}\ X_1 \\ [X_1, X_2] = -2\xi + \frac{2r_2}{r_1}\ U. \end{cases}$$

Moreover, on M we have $du = -\frac{r_2}{r_1}\ d\eta$. Computing the torsion free adapted connection we obtain:

$$\nabla_{X_1}X_1 = 0\ ,\ \nabla_{X_1}X_2 = -\xi + \frac{r_2}{r_1}\ U\ , \nabla_{X_2}X_2 = 0$$

the other expressions are easy deducible from the relations above. The non-vanishing components of the curvature tensor are

$$R_{X_1X_2}X_1 = -\frac{4}{r_1^2}\ X_2\ ,\ R_{X_1X_2}X_2 = \frac{4}{r_1^2}\ X_1$$

and hence

$$\rho(R)(X_1, X_1) = \rho(R)(X_2, X_2) = \frac{4}{r_1^2}\ ,\ \rho(R)(X_1, X_2) = 0.$$

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References

[1] D.E.Blair, *Contact manifolds in Riemannian Geometry,* Lect. Notes in Math., **509**, Springer-Verlag, 1976.

[2] D.E.Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in Mathematics, **203**, Birkhäuser, 2001.

[3] A.Čap & G.Schmalz, *Partially Integrable Almost CR-manifolds of CR Dimension and Codimension Two*, Lie groups, geometric structures and differential equations – one hundred years after Sophus Lie (Kyoto/Nara, 1999) 45-77, Adv. Stud. Pure Math., **37** (2002), Math. Soc. Japan, Tokyo.

[4] V.V.Ezhov & A.V.Isaev & G.Schmalz, *Invariants of elliptic and hyperbolic CR-structures of codimension* 2, to appear in Int. J. Math.

[5] S.I.Goldberg & K.Yano, *On normal globally f-manifold*, Tôhoku Math. J., 22 (1970), 362-370.

[6] A.Lotta & A.M.Pastore, *The Tanaka-Webster connection for almost S-manifolds and Cartan geometry*, Archivum Mathematicum (Brno), 40 (2004), 47-61.

[7] P.Matzeu & V.Oproiu, *The Bochner type curvature tensor of pseudoconvex CR structures,* SUT J. of Math., **31**, 1 (1995), 1-16.

[8] P.Matzeu & V.Oproiu, *The Bochner type curvature tensor of pseudo-convex CR structures on real hypersurfaces in complex space forms,* J. of Geometry, **63**, (1998), 134-146.

[9] R.I.Mizner, kn:Almost CR-structures, f-structures, almost product structures and associated connections, Rocky Mt. J. Math., **23** (1993) 4, 1337-1359.

[10] G.Schmalz & J.Slovák, *The geometry of hyperbolic and elliptic CR-manifolds of codimension two*, Asian J.Math., **4** (2000), 3, 565-597.

[11] K.Yano & M.Kon, *CR-submanifolds of Kählerian and Sasakian manifolds,* Progress in Mathematics, **30**, Birkhäuser, Boston, Basel Stuttgart, 1983.

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