Some results on the $CR-$structures on 3-dimensional manifolds

Marian-Ioan Munteanu *
University 'Al.I.Cuza' of Iaşi, Faculty of Mathematics,
Bd. Carol I, no.11, 6600-Iaşi, Romania
e-mail: munteanu@uaic.ro

Abstract

Let M be a connected, smooth contact manifold of dimension 3 and let $H(M)$ be the contact bundle. The Bochner type tensor field for the pseudoconvex $CR-$structure $(M, H(M))$ vanishes identically. Some other results are given in particular cases.

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1. Introduction

There are some tensor fields, invariant under certain transformations, that are obtained from the curvature tensor fields of linear connections on real or complex manifolds. The conformal Weyl tensor field is obtained from the curvature tensor field of a Riemannian connection as an invariant under the conformal transformations of the metric on the Riemannian manifold. The Weyl projective tensor field is obtained from the curvature tensor field of a torsion free linear connection on a real manifold as an invariant under the projective transformations of the connection. As a formal Kählerian analogue of the conformal Weyl curvature tensor, in [2] it is introduced the so-called Bochner curvature tensor on a Kähler manifold which is related, in a certain way, to the conformal changes of metric. For a complex structure there exists an invariant tensor field called the H-projective curvature tensor field and in the case of a normal almost contact structure there is the $C-$projective curvature tensor field. There exists also Chern pseudoconformal curvature tensor which is invariant under certain transformations. For a pseudo-convex $CR-$structure it is defined the Bochner type curvature tensor field which is invariant under gauge transformation of the almost contact structure associated to the $CR-$structure.

The case where the dimension is equal to 3 is rather exceptional. In [13] (for example) it is proved that the Weyl conformal curvature tensor field W vanishes identically. It is the same case of Chern's pseudoconformal curvature tensor which vanishes identically in dimension 3 (see for example the remark in [3]). On the other hand, in [5] it is considered a Kähler manifold M of dimension 2n and

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an open set U in M for which it is defined a $CR-$structure on $M(U) = I\!R \times U$. The Bochner tensor B of U coincides with the fourth order Chern Moser tensor of the strictly pseudo-convex $CR-$manifold $M(U)$; B is zero when $n = 1$.

In this paper we will prove that if the manifold M is 3 dimensional and $(M, H(M))$ is a pseudo-convex $CR-$structure on M, then the Bochner type curvature tensor field vanishes identically.

At the end of the paper we compute the adapted conection and the associated almost contact structure in some particular cases, like the unit tangent bundle of a Riemannian manifold of dimension 2, the Heisenberg group, the sphere S^3 and the group $PL(1, I\!R)$.

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2. Preliminaries

A $CR-$structure on the differentiable manifold M is defined by a complex vector subbundle $H(M)$ in the complexification $T^C M$ of the tangent bundle TM of M satisfying the following conditions:

(a) $A(M) \cap H(M) = \{0\}$ where $A(M) = \overline{H(M)}$.

(b) $H(M)$ is involutive, i.e. if Z, W are two sections in $H(M)$ then the bracket $[Z, W]$ is also a section in $H(M)$.

Denote also by $H(M)$ the decomplexification of $H(M)$ and let J be the linear operator on $H(M)$ (real) corresponding to the multiplication by the imaginary unit i on $H(M)$ (complex). Thus, J defines an almost complex structure on $H(M)$.

The complex involutivity condition is equivalent to to the following two conditions expressed in terms of the decomplexification $H(M)$:

$$(2.1) \qquad \begin{cases} (i) \;\; [X, Y] - [JX, JY] \in \Gamma(H(M)) \\ (ii) \;\; N_J(X, Y) = [JX, JY] - [X, Y] - J\{[JX, Y] + [X, JY]\} = 0 \end{cases}$$

for every X, Y belonging to $\Gamma(H(M))$, $\Gamma(H(M))$ being the $C^\infty(M)$-module of cross-sections on $H(M)$.

Let M be an orientable manifold of odd dimension 2n+1. Suppose that the dimension of $H(M)$ is 2n. Then the $CR-$ structure is called *of hypersurface type.*

Consider η a 1-form on M such that

$$(2.2) \qquad \Gamma(H(M)) = \{X \in \chi(M) \mid \eta(X) = 0\} \ .$$

Due the orientability of M the 1-form η can be chosen globally defined on M. The $CR-$structure $(M, H(M))$ of hypersurface type is said to be *pseudo-convex* if η is a contact form, i.e. $\eta \wedge (d\eta)^n \neq 0$.

Let $\xi \in \chi(M)$ be the Reeb vector field defined by

$$(2.3) \qquad \eta(\xi) = 1, \qquad i_\xi d\eta = 0.$$

The associated almost contact structure is defined by the 1-form η, the vector field ξ (as above) and the (1,1) tensor field φ given by the condition

$$\varphi X = JhX, \tag{2.4}$$

where h is the operator of projection on $H(M)$ obtained from the decomposition

$$TM = H(M) \oplus \mathrm{span}[\xi].$$

Consider the following metric g on $H(M)$ given by

$$g(X,Y) = d\eta(\varphi X, Y); \quad X, Y \in \Gamma(H(M)). \tag{2.5}$$

By taking $g(\xi,\xi) = 1$, and $g(X,\xi) = 0$, $\forall X \in \Gamma(H(M))$, one has an almost contact metric structure on M.

Recall that in the theory of the contact metric manifolds in addition to the basic structure (φ, ξ, η, g), there is another tensor field that plays a fundamental role, $\psi = \frac{1}{2}\mathcal{L}_\xi \varphi$ ($\mathcal{L}$ denotes Lie differentiation). ψ anti-commutes with φ, $\psi\xi = 0$ and ψ vanishes if and only if ξ is Killing (see [1]). In the case of pseudo-convex $CR-$structure ψ vanishes if and only if the associated almost contact structure is normal (see [7]).

Remark that the associated almost contact structure defined is not unique. If we consider the 1-form $\eta' = \varepsilon e^f \eta$, $\varepsilon = \pm 1$ and $f \in C^\infty(M)$ one gets:

$$\left\{ \begin{array}{l} d\eta' = \epsilon e^f (d\eta + df \wedge \eta) \\ \xi' = \epsilon e^{-f}(\xi + \varphi A) \\ \varphi' = \varphi + \eta \otimes A \end{array} \right. \tag{2.6}$$

where, when $\varepsilon = 1$, A is a vector field defined by the conditions:

$$\eta(A) = 0 \ , \ d\eta(\varphi A, X) = df(hX). \tag{2.7}$$

These changes are called *gauge transformations.*

In [7] it is proved the following

Theorem 1. *If (φ, ξ, η) is an almost contact structure subordinated to the pseudoconvex $CR-$structure $(M, H(M))$, then there exists a unique torsion free connection ∇ so that*

$$\left\{ \begin{array}{l} (\nabla_X \eta)(Y) = \frac{1}{2} d\eta(X,Y) \ , \ \nabla_X d\eta = 0 \ , \ \nabla_X \xi = 0 \\ (\nabla_X \varphi)Y = 2\eta(X)\psi Y - \frac{1}{2} d\eta(X, \varphi Y)\xi \ \ , \ \forall X, Y \in \chi(M). \end{array} \right. \tag{2.8}$$

The connection above is called *the torsion free canonical connection,* associated with the almost contact structure (φ, ξ, η). It is determined by the formulas:

$$
(2.9) \qquad \begin{cases} 2\eta(\nabla_X Y) = 2X(\eta(Y)) - d\eta(X, Y) \\ \\ 2d\eta(\nabla_X Y, Z) = 2\eta(X)d\eta(\varphi Y, \psi Z) + 2\eta(Y)d\eta(\varphi X, \psi Z) + \\ \quad +\varphi Z(d\eta(X, \varphi Y)) + d\eta([X, \varphi Z], \varphi Y) + d\eta([Y, \varphi Z], \varphi X) + \\ \quad +X(d\eta(Y, Z)) + Y(d\eta(X, Z)) + d\eta([X, Y], Z). \end{cases}
$$

Under a gauge transformations, the torsion free connection ∇ and the corresponding curvature tensor field R change to ∇' and R' respectively (see [7]).

By using the metric g defined above we define the Ricci tensor field of R as usual by $\rho(R)(Y, Z) = \mathrm{trace}(X \mapsto R_{XY} Z)$, $X, Y, Z \in \Gamma(H(M))$ and the function $\tau(R) = \mathrm{trace}(\rho(R))$.

It is also proved

Theorem 2. **([7])** *Let (φ, ξ, η) be an almost contact structure on M subordinated to the pseudo-convex $CR-$structure $(M, H(M))$. Then the tensor field*

$$
(2.10) \qquad \begin{aligned} B(R)_{XY} Z = R_{XY} Z &+ L(X, Z)Y - L(Y, Z)X + L(Y, \varphi Z)\varphi X - \\ &-L(X, \varphi Z)\varphi Y - \{L(X, \varphi Y) - L(Y, \varphi X)\}\varphi Z - \\ &-d\eta(X, Y)KZ - \tfrac{1}{2}d\eta(X, Z)KY + \tfrac{1}{2}d\eta(Y, Z)KX + \\ &+\tfrac{1}{2}d\eta(X, \varphi Z)\varphi KY - \tfrac{1}{2}d\eta(Y, \varphi Z)\varphi KX, \\ &\qquad X, Y, Z \in \Gamma(R(M)) \end{aligned}
$$

is invariant under the action of gauge transformation, where

$$
(2.11) \qquad \begin{aligned} L(X, Y) = &\tfrac{1}{2(n+2)}\{\rho(R)(X, Y) + 2d\eta(\varphi X, \psi Y)\} + \\ &+\tfrac{1}{8(n+1)(n+2)}\tau(R)d\eta(X, \varphi Y) \end{aligned}
$$

and

$$
(2.12) \qquad \frac{1}{2}d\eta(KX, Y) = L(X, Y).
$$

The tensor field $B(R)$ is called *the Bochner type curvature tensor field* for the pseudo-convex $CR-$structure $(M, H(M))$.

3. $CR-$structures on 3-dimensional manifolds and Bochner type tensor

Let M be a 3-dimensional, connected, smooth contact manifold with a non-degenerate 1-form η. Denote the contact bundle by $H(M) = \ker \eta$. If (φ, ξ, η) is the almost contact structure on M, let $J : H(M) \to H(M)$ be the restriction of φ to $H(M)$; thus J is a complex structure on $H(M)$ which defines a pseudo-convex $CR-$structure on M.

Let $H(M)$ be generated locally by the vector fields X_1, X_2, where

$$\varphi X_1 = X_2,\ \varphi X_2 = -X_1. \tag{3.1}$$

Consider Lie brackets given by:

$$[X_1, X_2] = aX_1 + bX_2 + c\xi,\ [X_1, \xi] = uX_1 + vX_2,\ [X_2, \xi] = pX_1 + qX_2 \tag{3.2}$$

where a, b, c, u, v, p and q are real functions on M, with $c \neq 0$ in every point (to have a CR-structure it is necessary to have $[X_1, X_2] \neq 0$ (mod X_1, X_2)).

The Jacobi identity gives us the following equations:

$$\begin{cases} q + u = -\frac{1}{c}\xi(c) \\ av - ub = -X_2(v) + X_1(q) + \xi(b) \\ bp - qa = X_1(p) - X_2(u) + \xi(a) \end{cases}. \tag{3.3}$$

Proposition 3. *The operator ψ vanishes if and only if $p + v = 0$ and $\xi(c) + 2uc = 0$.*

Proof. Note that $\psi X_1 = -\frac{1}{2}(p + v)X_1 + \frac{1}{2}(u - q)X_2$. Using the first condition from the relations above one gets the conclusion. ∎

Since

$$d\eta(X_1, X_2) = -c. \tag{3.4}$$

it follows that the auxiliary metric on $H(M)$ is given by

$$g(X_1, X_1) = c,\ g(X_1, X_2) = 0,\ g(X_2, X_2) = c. \tag{3.5}$$

Computing the expression of the torsion free canonical connection one gets

$$\begin{cases} \nabla_{X_1} X_1 = \frac{1}{2c} X_1(c) X_1 - \left(a + \frac{1}{2c} X_2(c)\right) X_2 \\ \nabla_{X_1} X_2 = \left(a + \frac{1}{2c} X_2(c)\right) X_1 + \frac{1}{2c} X_1(c) X_2 + \frac{c}{2}\,\xi \\ \nabla_{X_1} \xi = 0 \\ \\ \nabla_{X_2} X_1 = \frac{1}{2c} X_2(c) X_1 - \left(b - \frac{1}{2c} X_1(c)\right) X_2 - \frac{c}{2}\xi \\ \nabla_{X_2} X_2 = \left(b - \frac{1}{2c} X_1(c)\right) X_1 + \frac{1}{2c} X_2(c) X_2 \\ \nabla_{X_2} \xi = 0 \\ \\ \nabla_\xi X_1 = [\xi, X_1],\ \nabla_\xi X_2 = [\xi, X_2], \nabla_\xi \xi = 0. \end{cases} \tag{3.6}$$

A simple computation shows that the expression of the tensor field $R_{X_1X_2}$ is given by:

$$
(3.7) \qquad \begin{cases} R_{X_1X_2}X_1 = \left(\frac{1}{2}\xi(c) + cu\right) X_1 + (k_0 + cv)X_2 \\ R_{X_1X_2}X_2 = (-k_0 + cp)X_1 - \left(\frac{1}{2}\xi(c) + cu\right) X_2 \\ R_{X_1X_2}\xi = 0 \end{cases}
$$

where

$$
k_0 = (X_2(a) - X_1(b)) - \frac{1}{2c^2}\left(X_1(c)^2 + X_2(c)^2\right) + \frac{1}{2c^2}\left(X_1(X_1(c)) + X_2(X_2(c))\right) +
$$

$$
+a^2 + b^2 + \frac{1}{2c}\left(aX_2(c) - bX_1(c)\right).
$$

We deduce the following relations for the Ricci tensor $\rho(R)$ and for the function $\tau(R)$:

$$
(3.8) \qquad \begin{cases} \rho(R)(X_1, X_1) = -k_0 - cv \\ \rho(R)(X_1, X_2) = \frac{1}{2}\xi(c) + uc \\ \rho(R)(X_2, X_2) = -k_0 + cp, \end{cases}
$$

$$
(3.9) \qquad \tau(R) = (p - v) - \frac{2}{c}k_0.
$$

Proposition 4. *Let ρ_1 and ρ_2 be the eigenvalues of the Ricci tensor (restricted to $H(M)$). Then $\rho_1 = \rho_2$ if and only if (M, φ, ξ, η) is normal.*

Proof. Considering the matrix of the Ricci tensor

$$
\begin{pmatrix} -k_0 - cv & \frac{1}{2}\xi(c) + uc \\ \frac{1}{2}\xi(c) + uc & -k_0 + cp \end{pmatrix}
$$

and imposing the condition $\rho_1 = \rho_2$ one gets $v + p = 0$ and $\xi(c) + 2uc = 0$.

From Proposition 3 and taking into account the integrability conditions for the $CR-$ structure $(M, H(M))$ we obtain the normality condition for the almost contact structure associated (see for example [7]). ■

We want now to compute the Bochner type curvature tensor field. To do this, we get first the additional relations for the tensor fields L and K :

$$
(3.10) \qquad \begin{cases} L(X_1, X_1) = \frac{ck_1}{2} \\ L(X_1, X_2) = \frac{cu}{2} + \frac{1}{4}\xi(c) \\ L(X_2, X_2) = \frac{ck_2}{2} \end{cases}
$$

$$
(3.11) \qquad \begin{cases} KX_1 = -\left(u + \frac{1}{2c}\xi(c)\right) X_1 + \ k_1X_2 \\ KX_2 = -k_2X_1 + \left(\frac{1}{2c}\xi(c) + u\right) X_2. \end{cases}
$$

where,

$$k_1 = -\frac{1}{4c}k_0 - \frac{1}{8}(3p+5v) \text{ and } k_2 = -\frac{1}{4c}k_0 + \frac{1}{8}(3v+5p). \tag{3.12}$$

Thus we have the following

Theorem 5. *For a three dimensional differentiable manifold, on which we have defined a $CR-$structure $H(M)$ the Bochner type tensor field for the canonical connection vanishes identically.*

Proof. The relations (3.7) - (3.12) yield to

$$\begin{cases} B(R)_{X_1X_2}X_1 = 0 \\ B(R)_{X_1X_2}X_2 = 0 \end{cases} \tag{3.13}$$

and hence the vanishing of $B(R)$. ∎

4. Examples

In this section we will present some examples of $CR-$manifold of dimension 3 with the associated almost contact structure.

4.1. The unit tangent bundle of a 2-dimensional Riemannian manifold

Let (M, ds^2) be a 2-dimensional Riemannian manifold, where $ds^2 = Edu_1^2 + 2Fdu_1du_2 + Gdu_2^2$, u_1, u_2 being a system of local coordinates on M. Denote by $\Delta = EG - F^2 \neq 0$ the determinant of the matrix of ds^2.

Let the tangent bundle of M be endowed with the usual almost complex structure J defined by

$$JX^H = X^V \;,\; JX^V = -X^H \qquad X \in \chi(M) \tag{4.1}$$

where X^H, X^V are the horizontal and vertical lifts of X with respect to the Levi Civita connection of ds^2 respectively. Furthermore the Sasaki metric $\dot{g}$ on TM is given by

$$\dot{g}(X^V, Y^V) = g(X,Y) \;,\; \dot{g}(X^H, Y^H) = g(X,Y),\; \dot{g}(X^V, Y^H) = 0 \quad X, Y \in \chi(M). \tag{4.2}$$

The unit tangent bundle T_1M of M is a hypersurface in the tangent bundle of M given by the equation

$$Ev_1^2 + 2Fv_1v_2 + Gv_2^2 = 1, \tag{4.3}$$

where v_1, v_2 are the coordinates of the fibre (see for example [8], [9]).

By using the idea in [9] one obtains the following formulas:

$$N = v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2}, \quad \xi = v_1\frac{\delta}{\delta u_1} + v_2\frac{\delta}{\delta u_2}, \quad \frac{\delta}{\delta u_i} = \frac{\partial}{\partial u_i} - \Gamma^k_{ij}v_j\frac{\partial}{\partial v_k}, \tag{4.4}$$

where N is the unitary vector field normal to T_1M.

The distribution $H(M)$ is generated by the local vector fields

$$Y_0 = \frac{1}{\sqrt{\Delta}}\left(\alpha\frac{\partial}{\partial v_1} + \beta\frac{\partial}{\partial v_2}\right), \qquad X_0 = \frac{1}{\sqrt{\Delta}}\left(\alpha\frac{\delta}{\delta u_1} + \beta\frac{\delta}{\delta u_2}\right), \tag{4.5}$$

where α and β are smooth functions on TM given by the formulas

$$\alpha(u,v) = F(u)v_1 + G(u)v_2,\ \beta(u,v) = -E(u)v_1 - F(u)v_2,\ u = (u_1,u_2), v = (v_1,v_2). \tag{4.6}$$

Remark that $\{X_0, Y_0, \xi\}$ is an orthonormal local frame on T_1M.

The 1-form η whose null bundle is $H(M)$ is given by

$$\eta = \alpha du_2 - \beta du_1. \tag{4.7}$$

Taking $\varphi \in \mathcal{T}^1_1(T_1M)$ such that

$$\varphi X_0 = Y_0, \qquad \varphi Y_0 = -X_0 \tag{4.8}$$

one gets the almost contact structure associated to the pseudoconvex $CR-$structure $(M, H(M))$.

Remark also that the metric $\dot{g}$ is compatible with the almost contact structure. Moreover it satisfies the condition

$$\dot{g}(X,Y) = d\eta(\varphi X, Y). \tag{4.9}$$

Note that X_0 and Y_0 are the eigenvectors of ψ corresponding to the eigenvalues $\pm\frac{k-1}{2}$ respectively.

In this case, the Lie brackets of the vector fields X_0, Y_0 and ξ are given by the formulas

$$[X_0, Y_0] = \xi, \qquad [X_0, \xi] = -kY_0, \qquad [Y_0, \xi] = X_0, \tag{4.10}$$

where k is the Gaussian curvature of M.

Writting down the expression of the canonical connection one obtains

$$\begin{cases} \nabla_{X_0}Y_0 = \frac{1}{2}\xi, & \nabla_{Y_0}X_0 = -\frac{1}{2}\xi \\ \nabla_\xi X_0 = kY_0, & \nabla_\xi Y_0 = -X_0 \end{cases}, \tag{4.11}$$

the other components being 0.

Proposition 6. *The adapted connection ∇ defined on T_1M is never flat.*

Proof. Doing the necessary computations for the curvature tensor one gets $R_{X_0Y_0}X_0 = -kY_0$, $R_{X_0Y_0}Y_0 = X_0$. Thus, it follows that R is never 0. ■

The matrix of the Ricci tensor with respect to the basis $\{X_0, Y_0, \xi\}$ is

$$Ric = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.12}$$

and its eigenvalues are $\rho_1 = k$, $\rho_2 = 1$ and $\rho_3 = 0$.

Proposition 7. *The almost contact metric structure on T_1M is normal if and only if $k = 1$, i.e. $\rho_1 = \rho_2$. In this case T_1M endowed with the almost contact metric structure is Sasakian.*

Proof. Let $\Phi(X,Y) = \dot{g}(X, \varphi Y)$. Remark that $\Phi = d\eta$ and hence the almost contact metric structure is Sasakian ([6]). ■

4.2. $CR-$structure on Lie groups of dimension 3

1. The Heisenberg group (I)

The Heisenberg Lie group H_3 is a subgroup in $GL(3, I\!R)$ defined as follows (see for example [11], [12])

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \; ; \; x, y, z \in I\!R \right\} \tag{4.13}$$

with the usual matrix multiplication operation.

It is the same group with $(I\!R^3, *)$ where the operation $*$ is defined as follows

$$(a,b,c) * (x,y,z) = (a + x, b + y + az, c + z) \tag{4.14}$$

It is known that the following three vector fields are left invariant:

$$X_1 = \frac{\partial}{\partial x} \; , \; X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \; , \; \xi = \frac{\partial}{\partial y}. \tag{4.15}$$

By duality we can consider the 1-forms $\eta^1 = dx$, $\eta^2 = dz$ and $\eta = dy - x\, dz$. The only one contact form is η; thus we define a pseudoconvex $CR-$structure $(H_3, H(H_3))$ as its null bundle. We have also:

$$[X_1, X_2] = \xi, \quad [X_1, \xi] = 0, \quad [X_2, \xi] = 0. \tag{4.16}$$

Taking $\varphi \in \mathcal{T}_1^1(H_3)$ defined by

$$\varphi X_1 = X_2, \quad \varphi X_2 = -X_1, \quad \varphi\xi = 0 \tag{4.17}$$

we have the almost contact structure subordinate to the pseudoconvex $CR-$ structure $(H_3, H(H_3))$. The adapted connection is given by the formulas:

$$\left\{ \begin{array}{lll} \nabla_{X_1} X_1 = 0 & \nabla_{X_1} X_2 = \frac{1}{2}\,\xi & \nabla_{X_1}\xi = 0 \\ \nabla_{X_2} X_1 = -\frac{1}{2}\,\xi & \nabla_{X_2} X_2 = 0 & \nabla_{X_2}\xi = 0 \\ \nabla_{\xi}\, X_1 = 0 & \nabla_{\xi}\, X_2 = 0 & \nabla_{\xi}\, \xi = 0 \end{array} \right. \tag{4.18}$$

and it easily follows that $R = 0$. Thus the torsion free canonical connection is flat. As a remark, the associated almost contact structure is normal.

2. Heisenberg group (II)

Consider $I\!R^3$ with the following group multiplication:

$$(x, y, z) \bullet (x', y', z') = \left(x + x', y + y', z + z' + \frac{k}{2}(xy' - x'y) \right) \tag{4.19}$$

with $k \neq 0$.

Consider now the following left invariant metric:

$$ds^2 = dx^2 + dy^2 + \left(dz + \frac{k}{2}(ydx - xdy) \right)^2 \tag{4.20}$$

As in previous example, we have the following left invariant vector fields:

$$X_1 = \frac{\partial}{\partial x} - \frac{k}{2} y \frac{\partial}{\partial z}\ ,\ X_2 = \frac{\partial}{\partial y} + \frac{k}{2} x \frac{\partial}{\partial z}\ ,\ \xi = \frac{\partial}{\partial z}\ , \tag{4.21}$$

whose Lie brackets are:

$$[X_1, X_2] = k\xi, \quad [X_1, \xi] = 0, \quad [X_2, \xi] = 0. \tag{4.22}$$

The relations bellow give an almost contact metric structure with η the contact form:

$$\left\{ \begin{array}{l} \eta = dz + \frac{k}{2}(ydx - xdy) \\ \varphi X_1 = X_2,\ \varphi X_2 = -X_1, \quad \varphi\xi = 0 \end{array} \right. \tag{4.23}$$

Computing the adapted connection one gets:

$$(4.24)\qquad \begin{cases} \nabla_{X_1}X_1 = 0 & \nabla_{X_1}X_2 = \frac{k}{2}\,\xi, & \nabla_{X_1}\xi = 0 \\ \nabla_{X_2}X_1 = -\frac{k}{2}\,\xi, & \nabla_{X_2}X_2 = 0, & \nabla_{X_2}\xi = 0 \\ \nabla_{\xi}\,X_1 = 0, & \nabla_{\xi}\,X_2 = 0 & \nabla_{\xi}\,\xi = 0 \end{cases}$$

Remark 8. *The adapted connection defined on* $(H_3, \bullet)$ *is also flat and the associated almost contact structure is also normal.*

3. The sphere S^3

Let $S^3 = \{(x,y,u,v) \in I\!R^4 \mid x^2+y^2+u^2+v^2 = 1\}$ with the induced metric of $I\!R^4$.

Denote by N the unit normal vector field to the sphere S^3 and consider the vector field ξ defined as follows:

$$(4.25)\qquad \xi = -J\ N = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\ ,$$

where $J \in \mathcal{T}_1^1(I\!R^4)$ is defined by using the quaternions

$$J : I\!R^4 \equiv I\!H \longrightarrow I\!R^4 \equiv I\!H,\ Jq = iq,\ q = x + yi + uj + vk \in I\!H.$$

We define the $CR-$structure on S^3 as holomorphic space in every point of S^3, i.e.

$$(4.26)\qquad H_q = \{X_q \in T_qS^3 \mid J_qX_q = i\cdot X_q \in T_qS^3\}$$

Thus $H(S^3)$ is generated by the vector fields X_1, X_2 defined by:

$$(4.27)\qquad \begin{cases} X_1 = -u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + x\frac{\partial}{\partial u} - y\frac{\partial}{\partial v} \\ X_2 = -v\frac{\partial}{\partial x} - u\frac{\partial}{\partial y} + y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v} \end{cases}$$

The expressions of the Lie brackets are:

$$(4.28)\qquad [X_1, X_2] = 2\xi,\ [X_1,\xi] = -2X_2,\ [X_2,\xi] = 2X_1.$$

The 1-form

$$(4.29)\qquad \eta = y\ dx - x\ dy + v\ du - u\ dv$$

has the following properties

$$\eta(\xi) = 1,\ i_\xi d\eta = 0,\ \eta \wedge d\eta \neq 0,\ \eta(H(S^3)) = 0. \tag{4.30}$$

It is easy to see that

$$d\eta(X_1, X_2) = -2\ . \tag{4.31}$$

The torsion free canonical connection is given by the relations

$$\begin{cases} \nabla_{X_1} X_1 = 0 & \nabla_{X_1} X_2 = \ \xi & \nabla_{X_1} \xi = 0 \\ \nabla_{X_2} X_1 = -\xi & \nabla_{X_2} X_2 = 0 & \nabla_{X_2} \xi = 0 \\ \nabla_\xi\ X_1 = 2X_2 & \nabla_\xi\ X_2 = -2X_1 & \nabla_\xi\ \xi = 0. \end{cases} \tag{4.32}$$

Proposition 9. *The cannonical connection associated to the pseudoconvex CR-structure is not flat.*

Proof. Making the computations for some components of the curvature tensor field we obtain

$$\begin{cases} R_{X_1,X_2} X_1 = -4X_2 \\ R_{X_1,X_2} X_2 = \ 4X_1. \end{cases} \tag{4.33}$$

Hence, we get the conclusion. ■

Remark 10. *The almost contact structure associated is normal.*

4. The group $PL(1;I\!R)$ The real projective space $I\!RP^1$ is an analitic manifold by considering two local charts (U, φ), (V, ψ) where $U = \{[x^1, x^2] \in I\!RP^1 \mid x^1 \neq 0\}$, $x = \varphi([x^1, x^2]) = \frac{x^2}{x^1}$ and $V = \{[x^1, x^2] \in I\!RP^1 \mid x^2 \neq 0\}$, $y = \psi([x^1, x^2]) = \frac{x^1}{x^2}$.

The group $GL(2, I\!R)$ acting on $I\!R^2$ as usually by

$$\tilde{x}^1 = ax^1 + bx^2,\ \tilde{x}^2 = cx^1 + dx^2,\ ad - bc \neq 0,$$

determines an action on RP^1 given in the local chart (V, ψ) by formula $\tilde{y} = \frac{ay+b}{cy+d}$.

These transformations determines the group $PL(1, I\!R)$ called *the projective group,* which is a Lie group of dimension 3. Working in a neighbourhood of the identity we consider $d = 1$.

The multiplication of group $PL(1, I\!R)$ is given by

$$(a', b', c') \circ (a, b, c) = \left(\frac{a'a + b'c}{bc' + 1}\ ,\ \frac{a'b + b'}{bc' + 1}\ ,\ \frac{ac' + c}{bc' + 1} \right). \tag{4.34}$$

There are known the following left invariant vector fields (see for example [4],[11]):

$$
(4.35)\qquad \begin{cases} X_1 = -ac\frac{\partial}{\partial a} + (a-bc)\frac{\partial}{\partial b} - c^2\frac{\partial}{\partial c} \\ X_2 = b\frac{\partial}{\partial a} + \frac{\partial}{\partial c} \\ \xi = a\frac{\partial}{\partial a} + c\frac{\partial}{\partial c} \end{cases}
$$

whose Lie brackets are

$$
(4.36)\qquad [X_1, X_2] = 2\xi,\ [X_1, \xi] = -X_1,\ [X_2, \xi] = X_2.
$$

The left invariant 1-forms are

$$
(4.37)\qquad \eta^1 = \frac{1}{a-bc}db,\ \eta^2 = \frac{1}{a-bc}(-c\ da + a\ dc),\ \eta = \frac{1}{a-bc}(da + c\ db - b\ dc).
$$

The only one contact form is η. The pseudoconvex $CR-$structure is considered to be as null bundle of η. By taking $\varphi \in \mathcal{T}_1^1(PL(1, I\!R))$ given by

$$
\varphi X_1 = X_2,\ \varphi X_2 = -X_1,\ \varphi\xi = 0
$$

one obtains an almost contact structure which is not normal.

Proposition 11. *The adapted connection ∇ subordinated to the pseudoconvex CR-structure defined on $PL(1, I\!R)$ is not flat.*

Proof. Computing $R_{X_1X_2}$ one gets $R_{X_1X_2}X_1 = -2X_1$, $R_{X_1X_2}X_2 = 2X_2$ which show us that ∇ is not flat. ∎

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