Doubly Warped Product CR-Submanifolds in Locally Conformal Kähler Manifolds

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Abstract

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1 Introduction

Let $(\tilde{M}, J, \tilde{g})$ be a Hermitian manifold of dimension $2m$. Let Ω be the Kaehler 2-form associated with J and $\tilde{g}$, i.e. $\Omega(X, Y) = \tilde{g}(X, JY)$ for all $X, Y \in \chi(\tilde{M})$. The manifold $\tilde{M}$ is called a *locally conformal Kaehler* (l.c.K.) manifold if there is a closed 1-form ω, globally defined on $\tilde{M}$, such that

$$d\Omega = \omega \wedge \Omega \tag{1}$$

(see [8] for more details). The closed 1-form ω is called the *Lee form* of the l.c.K. manifold $\tilde{M}$. Also $(\tilde{M}, J, \tilde{g})$ is *globally conformal Kaehler* (g.c.K.) (respectively Kaehler) if the Lee form ω is exact (respectively $\omega = 0$). Note that any simply connected l.c.K. manifold is g.c.K.

For a l.c.K. manifold $(\tilde{M}, J, \tilde{g})$ we define the *Lee vector field* $B = \omega^{\#}$, where $\#$ means the rising of the indices with respect to $\tilde{g}$, namely $\tilde{g}(X, B) = \omega(X)$, for all $X \in \chi(\tilde{M})$. If $\tilde{\nabla}$ denotes the Levi Civita connection of $(\tilde{M}, \tilde{g})$ then we have

$$(\tilde{\nabla}_X J)Y = \frac{1}{2}\left(\theta(Y)X - \omega(Y)JX - \tilde{g}(X, Y)A - \Omega(X, Y)B\right) \tag{2}$$

for any $X, Y \in \chi(\tilde{M})$. Here $\theta = \omega \circ J$ and $A = -JB$ are the *anti-Lee form* and the *anti-Lee vector field*, respectively (cf. [8]).

A Riemannian manifold M, isometrically immersed in a l.c.K. manifold $(\tilde{M}, J, \tilde{g})$ is called a *CR-submanifold* if there exists on M a differentiable holomorphic distribution $\mathcal{D}$, i.e. $J_x\mathcal{D}_x = \mathcal{D}_x$ for all $x \in M$, whose orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is a totally real distribution on M, i.e. $J_x\mathcal{D}^\perp_x \subset T(M)^\perp_x$ for all $x \in M$. Here $T(M)^\perp \longrightarrow M$ is the normal bundle of the submanifold M.

A CR-submanifold is called *holomorphic* submanifold if $\dim \mathcal{D}^\perp_x = 0$, *totally real* if $\dim \mathcal{D}_x = 0$ and *proper* if it is neither holomorphic nor totally real. Throughout this paper, we denote by s

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the complex rank of the distribution $\mathcal{D}$ and by p the rank of the totally real distribution $\mathcal{D}^\perp$ for CR-submanifolds.

A CR-submanifold of a (l.c.) Kaehler manifold $\tilde{M}$ is called a CR-product if it is a Riemannian product of a holomorphic submanifold $N^\top$ and a totally real submanifold $N^\perp$ of $\tilde{M}$. The notion of CR-products in Kaehler manifolds was introduced by B.Y. Chen in [5]. Many interesting results on CR-products are proved since then by various authors. Recently, B.Y. Chen introduced in [6], [7] the notion of CR-warped products in Kaehlerian manifolds. The theory of CR-products and CR-warped products was developed including ambient spaces such as locally conformal Kaehler manifold (cf. e.g. [3], [4]) or Sasakian manifolds (cf. [9], [10]).

In the present article we study another class of CR-submanifolds in l.c.K. manifolds; namely, tha class of CR-submanifolds which are doubly warped products of holomorphic submanifolds and totally real submanifdolds.

2 Preliminaries

Singly warped products or simply *warped products* were first defined by Bishop & O'Neill in [1] in order to construct Riemannian manifolds with negative sectional curvature. In general, *doubly warped products* can be considered as generalization of singly warped products. Let (B, g_B) and (F, g_F) be Riemannian manifolds and let $b : B \longrightarrow (0,\infty)$ and $f : F \longrightarrow (0,\infty)$ be smooth functions. The doubly warped product $M =_f B \times_b F$ is the product manifold $B \times F$ endowed with the metric $g = f^2 g_B \oplus b^2 g_F$. More precisely, if $\pi : B \times F \longrightarrow B$ and $\tau : B \times F \longrightarrow F$ are natural projections, the metric g is defined by

$$g = (f \circ \tau)^2 \pi^* g_B + (b \circ \pi)^2 \tau^* g_F. \tag{3}$$

The functions b and f are called *warping functions*. If either $f \equiv 1$ or $b \equiv 1$, but not both, then we obtain a (singly) warped product. If both $f \equiv 1$ and $b \equiv 1$, then we have a product manifold. If neither b nor f is constant, then we have a *non-trivial doubly warped product*. Throughout this paper we use the *natural product coordinate system* on the product manifold $M = B \times F$. This means that, if $(p_0, q_0) \in M$, then there are coordinate charts (U, x) and (V, y) on B and F respectively, such that $p_0 \in U$ and $q_0 \in V$. Then we define coordinate chart (W, z) on M such that W is an open subset in M, contained in $U \times V$ and $(p_0, q_0) \in W$. Then, for all $(p, q) \in W$, $z(p, q) = (x(p), y(q))$. Clearly, the set of all (W, z) defines an atlas on $B \times F$.

If ∇^B and ∇^F are the Levi Civita connections of the Riemannian metrics g_B and g_F respectively, we express the Levi Civita connection ∇, of the doubly warped product M as

$$\begin{cases} \nabla_X Y = \nabla^B_X Y - \frac{f^2}{b^2}\, g_B(X,Y) \nabla^F(\ln f), \\ \\ \nabla_X Z = Z(\ln f) X + X(\ln b) Z \end{cases} \tag{4}$$

for all X, Y tangent to B and Z tangent to F. Here $\nabla^F(\ln f)$ denotes the gradient of $\ln f$ with respect to the metric g_F. Similar formulas hold for $\nabla_Z W$ and $\nabla_Z X$, with W tangent to F (see [11] for details). Remark that $B \times \{q\}$ and $\{p\} \times F$ are totally umbilical in M. Moreover, $B \times \{q\}$ is totally geodesic in M if and only if $\nabla^F(f)_{|_q} = 0$. In the same manner $\{p\} \times F$ are totally geodesic in M if and only if $\nabla^B(b)_{|_p} = 0$.

For later use we give the *Gauss and Weingarten formulas*

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{5}$$

for any $X, Y \in \chi(M)$ and $N \in \Gamma^{\infty}(T(M)^{\perp})$ (smooth sections in the normal bundle). Here h denotes the second fundamental form (of the immersion $M \hookrightarrow \tilde{M}$), $A_N X$ is the shape operator (corresponding to the normal section N) and $\nabla^{\perp}$ is the normal connection. Then

$$g(A_N X, Y) = \tilde{g}(N, h(X,Y)), \tag{6}$$

for all $X, Y \in \chi(M)$ and $N \in \Gamma^{\infty}(T(M)^{\perp})$.

The *equation of Gauss* is given by

$$\begin{aligned} \tilde{g}(W, \tilde{R}_{XY} Z) = \quad & g(W, R_{XY} Z) + \tilde{g}(h(X,Z), h(Y,W)) \\ & -\tilde{g}(h(X,W), h(Y,Z)), \end{aligned} \tag{7}$$

for X, Y, Z, W tangent to M, where R and $\tilde{R}$ denote the curvature tensors of M and $\tilde{M}$, respectively. (We put $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.)

For the second fundamental form h, we define its covariant derivative ∇h, by

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{8}$$

The *equation of Codazzi* is

$$(\tilde{R}_{XY} Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{9}$$

for X, Y, Z tangent to M, where $(\tilde{R}_{XY} Z)^{\perp}$ denotes the normal component of $\tilde{R}_{XY} Z$.

For a CR-submanifold M in a locally conformal Kaehler manifold $\tilde{M}$, we denote by ν the orthogonal complement of $J\mathcal{D}^{\perp}$ in $T(M)\perp$. Hence we have the following orthogonal decomposition of the normal bundle:

$$T(M)^{\perp} = J\mathcal{D}^{\perp} \oplus \nu, \qquad J\mathcal{D}^{\perp} \perp \nu. \tag{10}$$

Set

$$PX = tan(JX), \ FX = nor(JX), \ \text{for } X \in \chi(M), \tag{11}$$

$$tN = tan(JN), \ fN = nor(JN), \ \text{for } N \in \Gamma^{\infty}(T(M)^{\perp}). \tag{12}$$

Here tan_x and nor_x are the natural projection associated with the orthogonal direct sum decomposition

$$T_x(\tilde{M}) = T_x(M) \oplus T(M)^{\perp}_x \tag{13}$$

for any $x \in M$. Then the following identities hold

$$P^2 = -I - tF, \ FP + fF = 0, \ Pt + tf = 0, \ f^2 = -I - Ft \tag{14}$$

where I is the identity transformation.

We recall that the anti-invariant distribution $\mathcal{D}^{\perp}$ is always integrable (cf. e.g. [8], Th. 12.5, p.167). We give now the following

Proposition 2.1 *Let M be a CR-submanifold in a l.c.K. manifold $\tilde{M}$. The holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$h(X, JY) - h(JX, Y) + \Omega(X,Y) nor(B) = 0, \ \forall X, Y \in \mathcal{D} \tag{15}$$

or, equivalently,

$$\tilde{g}(h(X, JY) - h(JX, Y) + \Omega(X,Y)B, JZ) = 0, \ \forall X, Y \in \mathcal{D}, \ \forall Z \in \mathcal{D}^{\perp}. \tag{16}$$

Proof. Using (2) one has

$$\begin{cases} \nabla_X(JY) & = & P\nabla_X Y + t\ h(X,Y) + \frac{1}{2}\ (\theta(Y)X - \omega(Y)JX- \\ & & -g(X,Y)\ tan(A) - \Omega(X,Y)\ tan(B)) \\ h(X,JY) & = & F\nabla_X Y + f\ h(X,Y) - \frac{1}{2}\ (g(X,Y)\ nor(A) + \Omega(X,Y)\ nor(B) \end{cases} \tag{17}$$

and hence

$$\begin{cases} \nabla_X(JY) - \nabla_Y(JX) + \Omega(X,Y)\ tan(B) = P[X,Y], \\ h(X,JY) - h(JX,Y) + \Omega(X,Y)\ nor(B) = F[X,Y] \end{cases} \tag{18}$$

for all $X, Y \in \mathcal{D}$.
If we require for $\mathcal{D}$ to be integrable ($[X,Y] \in \mathcal{D}$ for all $X, Y \in \mathcal{D}$), we obtain (15).
The converse follows from (18).
To end the proof, remark that the ν-part of

$$h(X,JY) - h(JX,Y) + \Omega(X,Y)\ B$$

vanishes for all $X, Y \in \mathcal{D}$. ■

We end this section by giving

Lemma 2.2 *Let M be a CR-submanifold in a l.c.K. manifold $\tilde{M}$. Then we have*
(1) $g(\nabla_U Z, X) = \tilde{g}(JA_{JZ}U, X)-$
$-\frac{1}{2}\ (\theta(Z)g(U,JX) - \omega(Z)g(U,X) + \omega(X)g(U,Z)),$
(2) $A_{JW}Z - A_{JZ}W = \frac{1}{2}\ (\theta(Z)W - \theta(W)Z),$
(3) $A_{J\mu}(JX) - A_\mu X = \frac{1}{2}\ (\omega(\mu)X - \theta(\mu)JX)$
for all X, Y in $\mathcal{D}$, Z, W in $\mathcal{D}^\perp$, μ in ν and U tangent to M.

Proof. The statements follows from (2) and the fact $J\nu = \nu$. ■

3 Doubly warped products ${}_fN^\top \times {}_bN^\perp$ in l.c.K. manifolds

The purpose of this section is to study CR-submanifolds M in a locally conformal Kaehler manifold $\tilde{M}$ which are doubly warped products of the form $M = {}_fN^\top \times {}_bN^\perp$, where $N^\top$ is a holomorphic submanifold and $N^\perp$ is a totally real submanifold in $\tilde{M}$. Remark that $N^\top$ and $N^\perp$ are both totally umbilical in M. If we denote by $\sigma^\top$ and $\sigma^\perp$ the second fundamental forms of $N^\top$ and $N^\perp$ respectively in $\tilde{M}$, then

$$\begin{aligned} \sigma^\top(X,Y) &= -\tfrac{f^2}{b^2}\ g_{N^\top}(X,Y)\nabla^{N^\perp}(\ln f), \\ \sigma^\perp(Z,W) &= -\tfrac{b^2}{f^2}\ g_{N^\perp}(Z,W)\nabla^{N^\top}(\ln b) \end{aligned} \tag{19}$$

for all X, Y tangent to $N^\top$ and Z, W tangent to $N^\perp$. Denote also by $h^\top$ and $h^\perp$ the second fundamental forms of $N^\top$ and $N^\perp$ respectively, in $\tilde{M}$.

Lemma 3.1 *For a doubly warped product CR-submanifold $M = {}_fN^\top \times {}_bN^\perp$ in a l.c.K. manifold $\tilde{M}$, we have*
(1) $\tilde{g}(h(X,Y), JZ) = -\frac{1}{2}\ g(X,Y)\theta(Z), \quad X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp$
(2) $B^{\mathcal{D}^\perp} = \frac{2}{b^2}\ \nabla^{N^\perp}(\ln f),$
(3) $\tilde{g}(h(\mathcal{D},\mathcal{D}), J\mathcal{D}^\perp) = 0$*, whenever the Lee vector field B is tangent to M.*

PROOF. Since $\tilde{M}$ is l.c.K. we get immediately

$$\tilde{g}(h(X,Y),JZ) = g(JY,\nabla_X Z) - \frac{1}{2}\, g(X,Y)\theta(Z) - \frac{1}{2}\, \Omega(X,Y)\omega(Z)$$

for any vector fields X, Y tangent to $N^\top$ and Z tangent to $N\perp$. Combining with (4) one obtains

$$\tilde{g}(h(X,Y),JZ) = -\frac{1}{2}\, g(X,Y)\theta(Z) + \Omega(X,Y)\left(Z(\ln f) - \frac{1}{2}\,\omega(Z)\right).$$

Now, using the symmetry of h and the definition of the gradient, i.e.

$$Z(\ln f) = g_{N^\perp}(Z, \nabla^{N^\perp}(\ln f))$$

we get statements (1) and (2). The last statement is a direct consequence of (1).

Lemma 3.2 *For a doubly warped product CR-submanifold $M = {}_fN^\top \times {}_bN^\perp$ in a l.c.K. manifold $\tilde{M}$, we have*

$$\tilde{g}(h(JX,Z),JW) = \left(X(\ln b) - \frac{1}{2}\,\omega(X)\right) g(Z,W) \tag{20}$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

PROOF. Using (2) one gets

$$\tilde{g}(h(JX,Z),JW) = g(\nabla_X Z, W) - \frac{1}{2}\,\omega(X) g(Z,W),$$

for all X tangent to $N^\top$ and Z, W tangent to $N^\perp$. Since M is a doubly warped product, from (4) we obtain (20). ■

Remark 3.3 From the above Lemma we conclude that

$$\tilde{g}(h(\mathcal{D}, \mathcal{D}^\perp, J\mathcal{D}^\perp) = 0 \iff \omega_{|_{N^\top}} = 2d(\ln b).$$

4 A general inequality for doubly warped product CR-submanifolds

For a doubly warped product CR-submanifold in a locally conformal Kaehler manifold we have the following

Theorem 4.1 *Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product CR-submanifold in a l.c.K. manifold $\tilde{M}$. Then, the norm of the second fundamental form of M satisfies*

$$||h||^2 \geq \frac{s}{2}\, ||B^{J\mathcal{D}^\perp}||^2 + \frac{p}{f^2}\left[||\nabla^{N^\top}(\ln b)||^2_{N^\top} + \frac{f^2}{4}\, ||B^{\mathcal{D}}||^2 - \omega(\nabla^{N^\top}(\ln b))\right] \tag{21}$$

If the equality sign of (21) *holds identically, then $N^\top$ and $N^\perp$ are both totally umbilical submanifolds in $\tilde{M}$.*

Proof. We know

$$||h||^2 = ||h(\mathcal{D},\mathcal{D})||^2 + 2||h(\mathcal{D},\mathcal{D}^\perp)||^2 + ||h(\mathcal{D}^\perp,\mathcal{D}^\perp)||^2 \tag{22}$$

and

$$||h(U,V)||^2 = ||h_{J\mathcal{D}^\perp}(U,V)||^2 + ||h_\nu(U,V)||^2, \tag{23}$$

for all U, V tangent to M.
From (1) in Lemma 3.1 we have $\tilde{g}(h(JX,JY),JZ) = \tilde{g}(h(X,Y),JZ)$ and

$$\begin{aligned}||h_{J\mathcal{D}^\perp}(X,Y)||^2 &= \tilde{g}(h(X,Y),h_{J\mathcal{D}^\perp}(X,Y)) = -\tfrac{1}{2}\,g(X,Y)\tilde{g}(B,h_{J\mathcal{D}^\perp}(X,Y)) = \\ &= -\tfrac{1}{2}\,g(X,Y)\tilde{g}(h(X,Y),B^{J\mathcal{D}^\perp}) = \tfrac{1}{4}\,g(X,Y)^2\tilde{g}(B,B^{J\mathcal{D}^\perp}) = \\ &= \tfrac{1}{4}\,g(X,Y)^2||B^{J\mathcal{D}^\perp}||^2,\end{aligned}$$

for all $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

In order to evaluate $||h||^2$, we will choose an adapted orthonormal basis $\{X_i, JX_i\}_{i=1,s}$ on $N^\top$ and an orthonormal basis $\{Z_\alpha\}_{\alpha=1,p}$ on $N^\perp$. We obtain an orthonormal basis on M, namely

$$\left\{\frac{1}{f}\,X_i, \frac{1}{f}\,JX_i, \frac{1}{b}\,Z_\alpha\right\}.$$

Consequently, we find

$$\begin{aligned}||h_{J\mathcal{D}^\perp}(\mathcal{D},\mathcal{D})||^2 &= \tfrac{2}{f^4}\sum_{i,j=1}^{s}||h_{J\mathcal{D}^\perp}(X_i,X_j)||^2 = \tfrac{2}{f^4}\cdot\tfrac{1}{4}\,||B^{J\mathcal{D}^\perp}||^2\sum_{i,j=1}^{s}g(X_i,X_j)^2 \\ &= \tfrac{s}{2}\,||B^{J\mathcal{D}^\perp}||^2.\end{aligned}$$

In a similar way, by using Lemma 3.2, we get

$$\begin{aligned}||h_{J\mathcal{D}^\perp}(JX,Z)||^2 &= \tilde{g}(h(JX,Z),h_{J\mathcal{D}^\perp}(JX,Z) = \\ &= \big(X(\ln b) - \tfrac{1}{2}\,\omega(X)\big)\,\tilde{g}(JZ,h_{J\mathcal{D}^\perp}(JX,Z)) = \\ &= \big(X(\ln b) - \tfrac{1}{2}\,\omega(X)\big)^2 g(Z,Z)\end{aligned}$$

for all $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.
Consequently, we find

$$\begin{aligned}||h_{J\mathcal{D}^\perp}(\mathcal{D},\mathcal{D}^\perp)||^2 &= \tfrac{1}{f^2b^2}\sum_{i=1}^{s}\sum_{\alpha=1}^{p}\big(||h_{J\mathcal{D}^\perp}(X_i,Z_\alpha)||^2 + ||h_{J\mathcal{D}^\perp}(JX_i,Z_\alpha)||^2\big) = \\ &= \tfrac{p}{f^2}\sum_{i=1}^{s}\big((X_i(\ln b) - \tfrac{1}{2}\omega(X_i))^2 + (JX_i(\ln b) - \tfrac{1}{2}\omega(JX_i))^2\big) = \\ &= \tfrac{p}{f^2}\Big(||\nabla^{N^\top}(\ln b)||^2_{N^\top} + \tfrac{f^2}{4}\,||B^{\mathcal{D}}||^2 - \omega(\nabla^{N^\top}(\ln b))\Big).\end{aligned}$$

Hence we get the inequality of the Theorem.

Now, if we require to have the equality sign in (21), one obtains that

$$h(\mathcal{D},\mathcal{D}) \subseteq J\mathcal{D}^\perp, \quad h(\mathcal{D},\mathcal{D}^\perp) \subseteq J\mathcal{D}^\perp, \quad h(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0. \tag{24}$$

Since $h^\top(X,Y) = \sigma^\top(X,Y) + h(X,Y)$, by virtue of relation (4), (1) in Lemma 3.1 and (refeq24) we deduce

$$h^\top(X,Y) = -g_{N^\top}(X,Y)f^2\left(\frac{1}{2}\,B^{J\mathcal{D}^\perp} + \frac{1}{b^2}\,\nabla^{N^\perp}(\ln f)\right),$$

which means that $N^\top$ is a totally umbilical submanifold in $\tilde{M}$. On the other hand, $N^\perp$ is totally umbilical in M. Therefore, the third condition in (24) implies that $N^\perp$ is also totally umbilical in $\tilde{M}$.

This ends the proof. ∎

Corollary 4.2 *Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product CR-submanifold and totally geodesic in a l.c.K. manifold $\tilde{M}$. Then M is generic, i.e. $J_x\mathcal{D}_x^\perp = T(M)_x^\perp$, M is tangent to the Lee vector field and $\omega_{|_{N^\top}} = 2d\ln b$.* (Moreover, both sides in (21) vanish.)

Proof. Since M is totally geodesic, $h \equiv 0$. Then, using (21), we get the conclusion. ■

Remark 4.3 If M is tangent to the Lee vector field B, the inequality (21) becomes

$$||h||^2 \geq \frac{p}{4}\ ||B^{\mathcal{D}}||^2 + \frac{p}{f^2}\ \left[||\nabla^{N^\top}(\ln b)||^2_{N^\top} - \omega(\nabla^{N^\top}(\ln b))\right].$$

Moreover, if the equality sign in the above inequality holds, then M is a minimal submanifold in $\tilde{M}$.

In order to obtain a converse of the previous result, we give first

Lemma 4.4 *Let $M = {}_fN^\top \times_b N^\perp$ be a doubly warped product CR-submanifold in a l.c.K. manifold $\tilde{M}$. Then*

$$\tilde{g}(h(Z,W), JV) = \tilde{g}(h(Z,V), JW) - \frac{1}{2}\ g(Z,W)\theta(V) + \frac{1}{2}\ g(Z,V)\theta(W) \tag{25}$$

for any $Z, V, W \in \mathcal{D}^\perp$.

Proof. Easy computations. ■

Suppose now that M is generic, i.e. $J_x\mathcal{D}_x^\perp = T(M)_x^\perp$, $\forall x \in M$.
We state the following

Theorem 4.5 *Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product, generic CR-submanifold in a l.c.K. manifold $\tilde{M}$, such that $\dim N^\perp \geq 2$ and $N^\perp$ is totally umbilical in $\tilde{M}$. Then we have the equality sign in (21).*

Proof. Since $N^\perp$ is totally umbilical in $\tilde{M}$, then there exists $\zeta \in \chi(\tilde{M})$, normal to $N^\perp$ and such that

$$h^\perp(Z,W) = g_{N^\perp}(Z,W)\ \zeta, \tag{26}$$

for all Z, W tangent to $N^\perp$. (ζ does not depend on Z and W.)
The vector field ζ can be decomposed as

$$\zeta = \zeta^{\mathcal{D}} + JV$$

where $\zeta^{\mathcal{D}}$ is the $\mathcal{D}$-component of ζ and $V \in \mathcal{D}^\perp$. Since $h^\perp(Z,W) = \sigma^\perp(Z,W) + h(Z,W)$, the relation (26) implies

$$h(Z,W) = g_{N^\perp}(Z,W)\left(\zeta + \frac{b^2}{f^2}\ \nabla^{N^\top}(\ln b)\right)$$

for any $Z, W \in \mathcal{D}^\perp$.
By using Lemma 4.4 we have $\quad ||JV||^2 = \tilde{g}(h(Z,W), JV) =$

$$= b^2\left(g_{N^\perp}(Z,V)g_{N^\perp}(W,V) - \frac{1}{2}\ g_{N^\perp}(Z,W)\theta(V) + \frac{1}{2}\ g_{N^\perp}(Z,V)\theta(W)\right)$$

for any $Z, W \in \mathcal{D}^\perp$. If we choose, for example, $W = V$ and Z orthogonal to V (this can be done since $\dim \mathcal{D}^\perp \geq 2$) we obtain $V = 0$, that is $\zeta = -\frac{b^2}{f^2} \nabla^{N^\top}(\ln b)$ (and so, it is tangent to $N^\top$).

We proved that if $N^\perp$ is a totally umbilical submanifold in $\tilde{M}$, then $h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0$ and thus the equality sign holds in (21). ∎

What's happens when $p = 1$?

In this case M is a hypersurface in $\tilde{M}$ and let N be a normal vector field on M, such that $Z = JN$ (which is tangent to $N^\perp$) is of unit length (w.r.t. $g_{N^\perp}$). Of course, Z generates $\mathcal{D}^\perp$. We give

Theorem 4.6 *Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product, generic CR-submanifold of hypersurface type in a l.c.K. manifold $\tilde{M}$. Then the equality sign of* (21) *holds if and only if $A_N Z$ belongs to the holomorphic distribution $\mathcal{D}$.*

Proof. The third condition in (24) can be written as $h(Z, Z) = 0$. Since M is generic of hypersurface type, this is equivalent to $\tilde{g}(h(Z, Z), JZ) = 0$, i.e. $A_N Z$ belongs to $\mathcal{D}$. ∎

Remark 4.7 If M is as above, the Lee vector field decomposes as $B = B^{\mathcal{D}} + \omega(Z)Z + \theta(Z)JZ$, and thus $||B^{J\mathcal{D}^\perp}||^2 = \theta(Z)^2$ and $||B^{\mathcal{D}}||^2 = ||B||^2 + \left(\omega(Z)^2 + \theta(Z)^2\right)(b^2 - 2)$.

Proposition 4.8 *Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product, generic CR-submanifold of hypersurface type in a l.c.K. manifold $\tilde{M}$ and tangent to the Lee vector field B. Then the equality sign of* (21) *holds if and only if M is minimal.*

Proof. The statement follows from (3) in Lemma 3.1. ∎

5 Examples

1. On $\mathbf{R}^4$ consider global coordinates x, y, u, v and the usual complex structure J defined by $J\partial_x = \partial_y$, $J\partial_y = -\partial_x$, $J\partial_u = \partial_v$ and $J\partial_v = -\partial_u$ ($\partial_x = \frac{\partial}{\partial x}$, etc.). Let b and f be two smooth, positive and non constant functions such that b depends on x and y, while f depends on u. Denote by B the 2-plane (x, y) and by F the x-axis. On B and F consider the Riemannian metrics $g_B = b(dx^2 + dy^2)$ and $g_F = f du^2$, respectively. Finally, consider on $\mathbf{R}^4$ the Riemannian metric $\tilde{g} = bfg_0$, where $g_0 = dx^2 + dy^2 + du^2 + dv^2$ is the Euclidean metric. It is easy to see that $(M = {}_fB \times {}_bF, g = fg_B + bg_F)$ is a (real) hypersurface isometrically embedded in $(\mathbf{R}^4, \tilde{g})$, which is a globally conformal Kaehler manifold with the Lee form

$$\omega = \frac{1}{b}\left(\frac{\partial b}{\partial x}\, dx + \frac{\partial b}{\partial y}\, dy\right) + \frac{f'}{f}\, du.$$

Consequently, the Lee vector field is $\beta = \frac{1}{bf}\left(\frac{1}{b}\frac{\partial b}{\partial x}\partial_x + \frac{1}{b}\frac{\partial b}{\partial y}\partial_y + \frac{1}{f}\, f'\partial_u\right)$. It is easy to check that both B and F are totally umbilical, while M is totally geodesic in $\mathbf{R}^4$. Thus, the hypothesis in Corollary 4.2 are satisfied and consequently both sides in (21) vanish.

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