Harmonicity and Gauge Transformations in dimension 3

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Abstract. Let (M,η) be a contact manifold. Consider a particular class of gauge transformations for the contact form η. In this note we study the harmonicity of the identity map of the manifold M, of dimension three, endowed with the Webster metric g and with another metric $\widetilde{g}$ obtained from g after a gauge transformation.

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1. Introduction

Harmonic maps on Riemannian and Kähler manifolds have been studied by many mathematicians (see [5], [7], [9], [12]). In 1964, Eells and Sampson [8] proved that every holomorphic map $f : M \longrightarrow M'$ between two Kähler manifolds is harmonic. In odd dimension, the analogue of almost Hermitian manifold is given by the almost contact manifold (see [2], [3]). Later, in 1995, Ianuş and Pastore [11] proved an analogous result of Eells and Sampson when both M and M' are contact metric manifolds or when M is Sasakian manifold and M' is Kähler. Moreover, by a result of H.Urakawa [21] (cf. also theorem 7 in [4]) any holomorphic CR map f of a compact strongly pseudoconvex CR manifold M into another one M' is harmonic if and only if $f_*\xi = \lambda\xi'$ for some $\lambda \in C^\infty(M)$ with $\xi(\lambda) = 0$ (here the Riemannian manifolds are endowed with the Webster metrics). If (M,η) is a contact manifold denote by $H(M)$ its contact bundle. We have a natural CR structure and the associated almost contact structure. As it is well known, the 1-form defining the distribution $H(M)$ is not unique and consequently the almost contact structure associated to the CR structure is not unique. Such two almost contact structures are related by a *gauge transformation.* On M consider the Webster metric g and after the gauge transformation consider the metric $\widetilde{g}$ obtained by the compatibility conditions with the almost contact structure and such that the restriction on $H(M)$ is given by a conformal transformation (see [13], [20]). It is also known that if

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M and M' are two strongly pseudoconvex CR manifolds with the contact metric structures (ϕ, ξ, η, g) and $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ respectively, then a $(\phi, \widetilde{\phi})$ holomorphic map is a harmonic map (see [10]). We will consider a 3-dimensional manifold and use a particular type of gauge transformations, called *special gauge transformations*, namely those for which the new structure vector field makes a constant angle with the contact distribution. Denoting by $f \in C^\infty(M)$ the function occurring in the gauge transformation, we prove the following **Theorem:** *The identity map* $1_M : (M, g) \longrightarrow (M, \widetilde{g})$ *is harmonic if and only if either* **1.** f *is a constant function, or* **2.** *there exists a local unit vector field* X*, belonging to* $H(M)$*, verifying* $[X, \xi] = v\ \phi X$, $[\phi X, \xi] = 0$, *with* v *constant greater (strictly) than* 2, *and* f *is given by* $df = \pm\sqrt{v-2}\ X^b$ *(the 1-form obtained from* X *by the musical isomorphism).*

The structure of this article is the following. In Section 2 we present some notions we need during this paper. The proof of the theorem is given in Section 3 and in Section 4 we give some examples.

2. Preliminaries

CR manifolds and associated almost contact structures

Let M be an orientable m-dimensional, smooth, manifold. A $CR-$structure $H(M)$ on M, is defined by a complex vector subbundle $H(M)$ of the complexification T^cM of the tangent bundle of M so that:

(a) $H(M) \cap \overline{H(M)} = \{0\}$

(b) $H(M)$ is complex involutive, i.e. for two sections Z, W in $H(M)$, the bracket $[Z, W]$ is also a section in $H(M)$.

Denote also by $H(M)$ the decomplexification of the complex subbundle and let J be the usual almost complex structure on $H(M)$ (real) such that $H(M)$ (complex) is the i eigenspace of J. The condition of complex involutivity can be expressed by:

$$\begin{array}{ll} (i) & [X,Y] - [JX, JY] \in \Gamma(H(M)) \\ (ii) & N_J(X,Y) = [JX, JY] - [X,Y] - J\{[JX,Y] + [X, JY]\} = 0 \end{array} \tag{2.1}$$

for every X, Y belonging to $\Gamma(H(M))$, $\Gamma(H(M))$ being the $C^\infty(M)$-module of cross-sections on $H(M)$.

We suppose that $\dim M = 2n+1$ and codim $H(M) = 1$. Assume also that the Levi form of $(M, H(M))$ is defined, i.e. we consider only strictly pseudo-convex

$CR-$structures of hypersurface type. Then, if we denote by η the local $1-$form having $H(M)$ as null bundle, we have $\eta \wedge (d\eta)^n \neq 0$ i.e. η is a contact form on M. Notice that, since M is oriented, then η is globally defined.

Let ξ be the Reeb vector field defined by

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0. \tag{2.2}$$

Then, the tangent bundle decomposes as $T(M) = \text{span}[\xi] \oplus H(M)$. Moreover if ϕ is the (1,1)-tensor field given by

$$\phi X = J(X - \eta(X)\xi) \ , \quad \forall X \in \chi(M) \tag{2.3}$$

the following relations hold

$$\eta \circ \phi = 0 \ \ , \ \ \phi\xi = 0 \ \ , \ \phi^2 = -I + \eta \otimes \xi;$$

hence (ϕ, ξ, η) defines an almost contact structure on M which is called *associated with the pseudo-convex $CR-$structure* $(M, H(M))$ (see [2], [18]).

If $\dim M = 3$, then the integrability condition is automatically fulfilled.

Gauge transformations.

Consider the new form $\widetilde{\eta} = \varepsilon e^f \eta$, where $f \in C^\infty(M)$ and $\varepsilon = \pm 1$. It is obvious that η and $\widetilde{\eta}$ define the same distribution $H(M)$.

It is known the following result (see for example [17]): *Two almost contact structures (ϕ, ξ, η) and $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta})$ are subordinated to the same pseudoconvex CR-structure if and only if there exists a function $f \in C^\infty(M)$ such that:*

$$\begin{cases} \widetilde{\eta} = \varepsilon e^f \eta \\ \widetilde{\xi} = \varepsilon e^{-f}(\xi + \phi A) \\ \widetilde{\phi} = \phi + \eta \otimes A \end{cases} \tag{2.4}$$

where, assuming $\varepsilon = 1$ and denoting by h the projection operator on $H(M)$, A is a vector field defined by the conditions:

$$\eta(A) = 0 \ , \ d\eta(\phi A, X) = df(hX) = hX(f). \tag{2.5}$$

A direct consequence is the equality $d\widetilde{\eta} = \varepsilon e^f (d\eta + df \wedge \eta)$.

It is an important geometric property that the complex involutivity is invariant under gauge transformations (see [14]). From now on we shall consider $\varepsilon = 1$, the case $\varepsilon = -1$ being completely similar.

After a gauge transformation, imposing the compatibility conditions with respect to the new structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta})$, we obtain from g a new Riemannian metric $\widetilde{g}$ on M which generally does not satisfy the equation $\widetilde{g}(X,Y) = d\widetilde{\eta}(X, \widetilde{\phi}Y)$, with $X, Y \in \Gamma(H(M))$. However, if we require that the restrictions of g and $\widetilde{g}$ on $H(M)$ are related by a conformal transformation, then we get the following relation between g and $\widetilde{g}$ (see also [20])

$$\begin{cases} \widetilde{g}(X,Y) = e^{2f}\{g(X,Y) - \eta(X)g(\phi A, Y) - \eta(Y)g(\phi A, X) + \\ \qquad\qquad + g(A,A)\eta(X)\eta(Y)\} \qquad \forall X, Y \in \chi(M); \end{cases} \tag{2.6}$$

and the equality

$$\widetilde{g}(X,Y) = e^{f} d\widetilde{\eta}(X, \widetilde{\phi}Y) \tag{2.7}$$

holds for all $X, Y \in \Gamma(H(M))$.

Harmonic maps.

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $F \in C^\infty(M, N)$ let τ be the tension field of F defined by

$$\tau(F)(p) = \sum_{i=1}^{m} \left(\widetilde{\nabla}_{e_i} F_* e_i - F_* \nabla_{e_i} e_i\right)(p), \quad p \in M, \tag{2.8}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on M, ∇ is the Levi Civita connection of g and $\widetilde{\nabla}$ is the induced connection on the induced bundle $F^{-1}TN$ (see for details [22]). The map F is harmonic if and only if

$$\tau(F) = 0, \qquad \text{everywhere on } M. \tag{2.9}$$

The condition above is called *the Euler Lagrange equation* of the energy

$$E(F) = \frac{1}{2}\int_M |dF|\, dv_g.$$

If we endow M with two different metrics g and h, then the tension field of the identity map $F = 1_M : (M, g) \longrightarrow (M, h)$ takes the form

$$\sum_{i=1}^{m} \left(\nabla^{(h)}_{e_i} e_i - \nabla^{(g)}_{e_i} e_i\right) = 0, \tag{2.10}$$

where $\nabla^{(g)}$ and $\nabla^{(h)}$ are the Levi Civita connections of g and h respectively.

3. The harmonicity of the identity

Let M be a smooth contact 3 dimensional manifold and let η be the nondegenerate 1-form. Let $H(M)$ be the contact bundle, i.e. $H(M) = \ker \eta$. If (ϕ, ξ, η) is the almost contact structure on M let $J : H(M) \longrightarrow H(M)$ be the restriction of ϕ to $H(M)$; thus, J is a complex structure on $H(M)$. We will assume that $d\eta$ is positive definite, in the following sense: $d\eta(X, JX) \geq 0$, $\forall X \in H(M)$ and $d\eta(X, JX) = 0 \iff X = 0$. Assume also that ξ is the Reeb vector field associated to η; as consequence, if $X \in H(M)$ then $[\xi, X] \in H(M)$.

Choose $X, Y \in H(M)$ such that $Y = \phi X$; they are linearly independent. Consider for the Lie brackets the following expressions

$$\left\{ \begin{array}{lcl} [X,Y] & = & a\,X + b\,Y + c\,\xi \\ [X,\xi] & = & u\,X + v\,Y \\ [Y,\xi] & = & p\,X + q\,Y \end{array} \right. \tag{3.1}$$

where a, b, c, u, v, p, q are differentiable functions on M. From the positivity of $d\eta$ one gets $c < 0$. Assume that $c = -1$. We can do this scaling X, Y by a factor $\frac{1}{\sqrt{-c}}$.

Then the Jacobi's identity yields

$$\left\{ \begin{array}{lcl} q + u & = & 0 \\ av - ub & = & -Y(v) + X(q) + \xi(b) \\ bp - qa & = & X(p) - Y(u) + \xi(a). \end{array} \right. \tag{3.2}$$

Thus we can rewrite 3.1 in the following way

$$\left\{ \begin{array}{lcl} [X,Y] & = & a\,X + b\,Y - \ \xi \\ [X,\xi] & = & u\,X + v\,Y \\ [Y,\xi] & = & p\,X - u\,Y, \end{array} \right. \tag{3.3}$$

where the function coefficients satisfy

$$\begin{array}{l} ub - av = Y(v) + X(u) - \xi(b) \\ bp + au = X(p) - Y(u) + \xi(a). \end{array}$$

Considering the Webster metric associated to the $CR-$structure one has

$$\left\{ \begin{array}{l} g(X,X) = g(Y,Y) = g(\xi,\xi) = 1 \\ g(X,Y) = g(X,\xi) = g(Y,\xi) = 0, \end{array} \right. \tag{3.4}$$

thus $\{X, Y, \xi\}$ determine a local orthonormal basis for $\chi(M)$ with respect to the metric g. Let us remark that, if we consider another orthonormal basis $\{\widetilde{X}, \widetilde{Y}, \xi\}$, $\widetilde{X}, \widetilde{Y} \in \ker \eta$, with the same orientation as the other, the new c is also -1. In order to study the harmonicity of the identity map $1_M : (M,g) \longrightarrow (M,\widetilde{g})$, where $\widetilde{g}$ will be defined, we get, after the computations, the following expressions

$$\nabla_X X = -a\ Y - u\ \xi, \quad \nabla_Y Y = b\ X + u\ \xi, \quad \nabla_\xi \xi = 0. \tag{3.5}$$

Let's do now a gauge transformation as in (2.4). The vector field A, belonging to $H(M)$ can be expressed as

$$A = \mu\ X + \lambda\ Y, \tag{3.6}$$

where μ and λ are differentiable functions on M obtained from (2.5). More precisely,

$$\mu = -X(f) \qquad \text{and} \qquad \lambda = -Y(f). \tag{3.7}$$

Denote by l the length of the vector field A and hence $l^2 = \mu^2 + \lambda^2$.

The metric $\widetilde{g}$ is taken to be compatible with the new almost contact structure and requiring that the restrictions of g and $\widetilde{g}$ are related by a conformal transformation on $H(M)$ (cf. [13]). By using 2.7 one has

$$\begin{cases} \widetilde{g}(X,X) = \widetilde{g}(Y,Y) = e^{2f} \quad \widetilde{g}(X,Y) = 0 \quad \widetilde{g}(X,\xi) = \lambda e^{2f} \\ \widetilde{g}(Y,\xi) = -\mu e^{2f} \quad \widetilde{g}(\xi,\xi) = (l^2+1)e^{2f}\ . \end{cases} \tag{3.8}$$

After the computations one gets

$$\begin{cases} \widetilde{\nabla}_X X = \alpha_1 X + \beta_1 Y + \gamma_1 \xi \\ \widetilde{\nabla}_Y Y = \alpha_2 X + \beta_2 Y + \gamma_2 \xi \\ \widetilde{\nabla}_\xi \xi = \alpha_0 X + \beta_0 Y + \gamma_0 \xi \end{cases} \tag{3.9}$$

where

$$\begin{cases} \gamma_1 = X(\lambda) + \lambda\mu - \xi(f) - u - a\mu \\ \alpha_1 = -\mu - \lambda\gamma_1 \\ \beta_1 = 2\lambda - a + \mu\gamma_1 \end{cases} \tag{3.10}$$

$$\begin{cases} \gamma_2 = -Y(\mu) - \lambda\mu - \xi(f) + u - b\lambda \\ \alpha_2 = 2\mu + b - \lambda\gamma_2 \\ \beta_2 = -\lambda + \mu\gamma_2 \end{cases} \tag{3.11}$$

$$\begin{cases} \gamma_0 = (1-l^2)\xi(f) + \lambda l X(l) - \mu l Y(l) - u\lambda^2 + u\mu^2 + v\lambda\mu + p\lambda\mu \\ \alpha_0 = -\lambda\gamma_0 + \xi(\lambda) + 2\lambda\xi(f) + \mu(l^2+1) - lX(l) + u\lambda - v\mu \\ \beta_0 = \mu\gamma_0 - \xi(\mu) - 2\mu\xi(f) + \lambda(l^2+1) - lY(l) + p\lambda + u\mu. \end{cases} \tag{3.12}$$

We will prove now the following result.

THEOREM. *Let M be a contact 3-dimensional smooth manifold on which consider a special gauge transformation. Then, the identity map $1_M : (M,g) \longrightarrow (M,\widetilde{g})$ is* *harmonic if and only if either f is a constant function, or there exists a local unit vector field X, belonging to the contact distribution $H(M)$, verifying $[X,\xi] = v\,\phi X$, $[\phi X, \xi] = 0$, with v constant greater (strictly) than 2, and f is given by* $df = \pm\sqrt{v-2}\,X^b$. (Here f is the C^∞ function on which the special gauge transformation depends.)

PROOF. The harmonicity condition $\tau(1_M) = 0$ becomes

$$\left\{\begin{array}{rcl} \gamma_1 + \gamma_2 + \gamma_0 &=& 0 \\ \xi(\lambda) + 2\lambda\xi(f) + \mu(l^2+2) - lX(l) + u\lambda - v\mu &=& 0 \\ -\xi(\mu) - 2\mu\xi(f) + \lambda(l^2+2) - lY(l) + p\lambda + u\mu &=& 0. \end{array}\right. \tag{3.13}$$

If we develop the relation $(3.13)_1$ we get

$$X(\lambda) - Y(\mu) + \lambda l X(l) - \mu l Y(l) - (l^2+1)\xi(f) -$$
$$-a\mu - b\lambda - u\lambda^2 + u\mu^2 + (v+p)\lambda\mu = 0. \tag{$*$}$$

On the other hand we have

$$X(\lambda) - Y(\mu) = -[X,Y](f) = a\mu + b\lambda + \xi(f). \tag{3.14}$$

Now we sum $(*)$ with $(3.13)_2$ multiplicated by λ and with $(3.13)_3$ multiplicated by $(-\mu)$ and obtain

$$l\,(\xi(l) + l\xi(f)) = 0.$$

The case $l = 0$ yields to $A = 0$ and consequently $\lambda = \mu = 0$. From (3.14) it follows also that $\xi(f) = 0$. Thus, f is a constant function on M.

Let study now the case $l \neq 0$. We have

$$\xi(l) + l\xi(f) = 0. \tag{3.15}$$

In this moment we restrict to a smaller class of gauge transformations, more precisely we will study the case when the vector field $\widetilde{\xi}$ makes a constant angle with $H(M)$ (with respect to the metric g). If we denote by t this angle (between $\widetilde{\xi}$ and his projection on $H(M)$) we can write

$$\cos t = \frac{l}{\sqrt{1+l^2}}.$$

Thus $l = \text{ctg } t$, ($t \neq 0$ since $\widetilde{\xi}$ never belongs to $H(M)$) and consequently l is constant. Call this gauge transformations *special gauge transformations.*

From (3.15) one obtains $\xi(f) = 0$. We also have

$$\xi(\lambda) = u\lambda - p\mu \quad , \quad \xi(\mu) = -u\mu - v\lambda. \tag{3.16}$$

Returning to (3.13) one gets

$$\begin{cases} 2u\lambda - (p+v)\mu + \mu(l^2+2) = 0 \\ 2u\mu + (p+v)\lambda + \lambda(l^2+2) = 0. \end{cases} \tag{3.17}$$

Also, the relation $(*)$ becomes

$$u\left(\lambda^2 - \mu^2\right) = (v+p)\lambda\mu. \tag{3.18}$$

If $4u^2 + (p+v)^2 \neq (l^2+2)^2$ then $\mu = 0$ and $\lambda = 0$; a contradiction with $l \neq 0$. In order to have a nonconstant function f which assures the harmonicity of the identity we have to consider $l^2 + 2 = \sqrt{4u^2 + (p+v)^2}$. It is obvious that we cannot have $u = 0$ and $p + v = 0$ simultaneously, i.e. the manifold cannot be normal (see for details [15]). Denote by $r = \sqrt{4u^2 + (p+v)^2}$.

Case 1. $u = 0$ (and $p + v \neq 0$).
From (3.18) it follows $\lambda\mu = 0$. We will expose here only the case $\lambda = 0,\ \mu \neq 0$, the other case ($\mu = 0,\ \lambda \neq 0$) can be discussed in a similar way. First of all note that $\mu = \pm l$. From (3.16) we have $p = 0$ and from (3.17) we have $l^2 + 2 = v =$ constant. Remark also that we can have solution iff $v > 2$ and in this situation $l = \sqrt{v-2}$. Hence, the function f is given by the following partial differential equations system

$$Y(f) = 0 \qquad \xi(f) = 0 \qquad X(f) = \pm\sqrt{v-2} = \text{constant}. \tag{3.19}$$

Other compatibility conditions are: $\xi(a) = 0$, $-\xi(b) + av = 0$ obtained from Jacobi's identity. As consequence, we have $\xi(\xi(b)) = 0$.

Case 2. $u \neq 0$ (and $\mu, \lambda \neq 0$).
Denote by $\varepsilon = \text{signum}(u) = \begin{cases} +1, & \text{if } u > 0 \\ -1, & \text{if } u < 0. \end{cases}$

We have $\lambda = -\varepsilon\sqrt{\frac{r-(p+v)}{r+(p+v)}}\,\mu$ and taking into account that $\lambda^2 + \mu^2 = l^2$ one gets $\mu = \pm\sqrt{\frac{(r-2)(r+p+v)}{2r}}$ which has solution iff $r > 2$. We get also $\lambda = \mp\varepsilon\sqrt{\frac{(r-2)(r-(p+v))}{2r}}$.

Thus, the function f verifies the following partial differential equations system

$$X(f) = -\mu, \qquad Y(f) = -\lambda, \qquad \xi(f) = 0. \tag{3.20}$$

It is more convenient to make a change of the local basis. Let's take $X' = \alpha\, X + \beta\, Y$ and $Y' = \phi X' = -\beta\, X + \alpha\, Y$, where α, β are differentiable functions on M satisfying $\alpha^2 + \beta^2 = 1$. Then we can write $[X', \xi] = u'X' + v'Y'$ and $[Y', \xi] = p'X' - u'Y'$, where

$$\begin{cases} u' = (\alpha^2 - \beta^2)\ u + \alpha\beta\ (p+v) \\ v' = -2\alpha\beta\ u + \alpha^2 v - \beta^2 p + \beta\ \xi(\alpha) - \alpha\ \xi(\beta) \\ p' = -2\alpha\beta\ u + \alpha^2 p - \beta^2 v + \alpha\ \xi(\beta) - \beta\ \xi(\alpha). \end{cases} \tag{3.21}$$

Let us remark that ($u = 0$ and $p+v = 0$) is equivalent to ($u' = 0$ and $p'+v' = 0$) and, as we saw in [15] it is still equivalent with the normality condition for the almost contact structure. We will check α and β such that $u' = 0$ (and as consequence, $p' + v' \neq 0$).

The condition $u' = 0$ yields to

$$(\alpha^2 - \beta^2)\ u = -\ \alpha\beta\ (p + v). \tag{3.22}$$

Taking into account that $\alpha^2 + \beta^2 = 1$ we obtain $\alpha^2 = \frac{r \pm (p+v)}{2r}$. Take

$$\alpha = \sqrt{\frac{r + (p+v)}{2r}} \qquad \text{and} \qquad \beta = -\varepsilon\sqrt{\frac{r - (p+v)}{2r}}\ . \tag{3.23}$$

Thus, the relation (3.22) is satisfied. Computing $\mu' = -X'(f)$ and $\lambda' = -Y'(f)$ we obtain $\lambda' = 0$ and $\mu' = \pm\sqrt{r-2} =$ constant. Moreover, after the computations, we get $p' = 0$ and $v' = r$. In other words we arrive exactly in the case 1.

4. Examples

The Heisenberg group.

On $\mathbf{R}^3$ consider the global set of coordinates (x, y, s). Let's take the contact form

$$\eta = \frac{1}{l}\ ds - \frac{1}{2}\ \frac{xdy - ydx}{1 + m(x^2 + y^2)} \tag{4.1}$$

where l, m are constants, $l \neq 0$. The Reeb vector field is $\xi = l\frac{\partial}{\partial s}$. Consider the complex distribution

$$H = \text{span}\ \left\{-2i\ (1 + mz\bar{z})\frac{\partial}{\partial z} + \frac{lz}{2}\ \frac{\partial}{\partial s}\right\} \tag{4.2}$$

or, equivalently, consider the real vector fields

$$\begin{aligned} X &= \left(1 + m(x^2 + y^2)\right) \tfrac{\partial}{\partial y} + \tfrac{lx}{2}\ \tfrac{\partial}{\partial s} \\ Y &= \left(1 + m(x^2 + y^2)\right) \tfrac{\partial}{\partial x} - \tfrac{ly}{2}\ \tfrac{\partial}{\partial s} \end{aligned} \tag{4.3}$$

which generate $\ker \eta$ (we have denoted $z = x + iy$). The multiplication by i allows us to consider

$$\phi X = Y, \qquad \phi Y = -X, \qquad \phi\xi = 0. \tag{4.4}$$

Thus we have a $CR-$structure on $\mathbf{R}^3$ and the associated almost contact structure. Computing the Lie brackets one has

$$[X, Y] = 2mx\ X - 2myY - \xi, \quad [X, \xi] = 0, \quad [Y, \xi] = 0. \tag{4.5}$$

The metric g is given by

$$g = \frac{1}{\left(1 + m(x^2 + y^2)\right)^2}\ (dx^2 + dy^2) + \left(\frac{1}{l}\ ds - \frac{1}{2}\ \frac{xdy - ydx}{1 + m(x^2 + y^2)}\right)^2. \tag{4.6}$$

Note that g is the Webster metric corresponding to the almost contact structure. With respect to this metric $\{X, Y, \xi\}$ is an orthonormal basis.

Let's do a special gauge transformation. Thus, the identity map $1_{\mathbf{R}^3} : (\mathbf{R}^3, g) \longrightarrow (\mathbf{R}^3, \widetilde{g})$ is harmonic iff we have *a homothetic transformation,* i.e. the function f is constant.

The sphere S^3.

Let $S^3 = \{(x, y, u, v) \in \mathbf{R}^4\ :\ x^2 + y^2 + u^2 + v^2 = 1\}$. As hypersurface of $\mathbf{R}^4$ we can define on S^3 a $CR-$structure. If N is the position vector of an arbitrary point on the sphere we define

$$\xi = -JN = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} \tag{4.7}$$

where $J \in \mathcal{T}_1^1(\mathbf{R}^4)$ is the usual complex structure on $\mathbf{R}^4$

$$J : \mathbf{R}^4 \longrightarrow \mathbf{R}^4 \qquad J(x, y, u, v) = (-y, x, -v, u).$$

The $CR-$structure on S^3 is obtained by considering the holomorphic spaces in every point of S^3, i.e. $H_p = \{X_p \in T_pS^3\ :\ J_pX_p \in T_pS^3\}$. On $H(S^3)$ we can choose the vector fields X, Y as

$$X = \frac{1}{\sqrt{2}}\ (-v,\ -u,\ y,\ x), \qquad Y = \frac{1}{\sqrt{2}}\ (-u,\ v,\ x,\ -y) \tag{4.8}$$

having the following expression for the Lie brackets

$$[X, Y] = -\xi, \qquad [X, \xi] = 2\ Y, \qquad [Y, \xi] = -2\ X. \tag{4.9}$$

The 1-form $\eta = y\ dx - x\ dy + v\ du - u\ dv$ is a contact form on S^3 having $H(S^3)$ as null bundle. We have also $\eta(\xi) = 1$ and $i_\xi d\eta = 0$.

The Webster metric g is given by $g = \eta^2 + \theta^2 + \omega^2$ where

$$\theta = \sqrt{2}\ (-v\ dx - u\ dy + y\ du + x\ dv) \quad \text{and}$$

$$\omega = \sqrt{2}\,(-u\,dx + v\,dy + x\,du - y\,dv) \quad .$$

After the special gauge transformation, if we require that $1_{S^3} : (S^3, g) \longrightarrow (S^3, \widetilde{g})$ is a harmonic map we obtain again homothetic transformations.

The third example.

On $\mathbf{R}^3$ with global coordinates (x, y, z) consider the contact form

$$\eta = \frac{1}{a}\ \cos az\ dx + \frac{1}{a}\ \sin az\ dy\ , \quad a > 0. \tag{4.10}$$

The Reeb vector field ξ is given by the formula

$$\xi = a\ \cos az\ \frac{\partial}{\partial x} + a\ \sin az\ \frac{\partial}{\partial y} \quad . \tag{4.11}$$

As usual, let $H = \ker \eta$. We have $H = \text{span}\ \{X, Y\}$ where

$$X = \frac{\partial}{\partial z} \qquad Y = -\sin az\ \frac{\partial}{\partial x} + \cos az\ \frac{\partial}{\partial y}\ \cdot \tag{4.12}$$

Computing the Lie brackets one obtains

$$[X, Y] = -\xi, \qquad [X, \xi] = a^2\ Y, \qquad [Y, \xi] = 0. \tag{4.13}$$

Consider the Webster metric g and a special gauge transformation which assures the harmonicity of the identity map. One gets the following PDE system

$$\begin{matrix} -\sin az\ \frac{\partial f}{\partial x} + \cos az\ \frac{\partial f}{\partial y} = 0 \\ a\ \cos az\ \frac{\partial f}{\partial x} + a\ \sin az\ \frac{\partial f}{\partial y} = 0 \end{matrix} \qquad \frac{\partial f}{\partial z} = \pm\sqrt{a^2 - 2}. \tag{4.14}$$

Thus, for $a > \sqrt{2}$ we obtain two families of functions $\{f = \pm\sqrt{a^2 - 2}\ z + \text{const}\}$ which assure that the map $1_{\mathbf{R}^3} : (\mathbf{R}^3, g) \longrightarrow (\mathbf{R}^3, \widetilde{g})$ is harmonic.

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