

Cheeger Gromoll type metrics on the tangent bundle

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Abstract

In this paper we study a Riemannian metric on the tangent bundle $T(M)$ of a Riemannian manifold M which generalizes the Cheeger Gromoll metric and a compatible almost complex structure which together with the metric confers to $T(M)$ a structure of locally conformal almost Kählerian manifold. We found conditions under which $T(M)$ is almost Kählerian, locally conformal Kählerian or Kählerian or when $T(M)$ has constant sectional curvature or constant scalar curvature.

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1 Preliminaries

Given a Riemannian manifold (M, g) one can define several Riemannian metrics on the tangent bundle $T(M)$ of M . Maybe the best known example is the Sasaki metric g_S introduced in [20]. Although the Sasaki metric is *naturally* defined, it is very *rigid*. For example, the Sasaki metric is not, generally, Einstein. Or, the tangent bundle $T(M)$ with the Sasaki metric is never locally symmetric unless the metric g on the base manifold is flat (see [12]). E.Musso & F.Tricerri [15] have proved that the Sasaki metric has constant scalar curvature if and only if (M, g) is locally Euclidian. In the same paper, they have given an explicit expression of a positive definite Riemannian metric introduced by J.Cheeger and D.Gromoll in [9] and called this metric *the Cheeger-Gromoll metric*. M.Sekizawa (see [21]), computed geometric objects related to this metric. Later, S.Gudmundson and E.Kappos in [11], have completed these results and have shown that the scalar curvature of the Cheeger Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, M.T.K.Abbassi & M.Sarih have proved that $T(M)$ with the Cheeger Gromoll metric is never a space of constant sectional curvature (cf. [2]). It is also known that the tangent bundle $T(M)$ of a Riemannian manifold (M, g) can be organized as an almost Kählerian manifold (see [10]) by using the decomposition of the tangent bundle to $T(M)$ into the vertical and horizontal distributions, VTM and HTM respectively (the last one being defined by the Levi Civita connection on M), the Sasaki metric and an almost complex structure defined by the above splitting. A more general metric is given by M.Anastasiu in [6] which generalizes both of the Sasaki and Cheeger Gromoll metrics: it preserves the orthogonality of the vertical

and horizontal distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasaki and the Cheeger Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and hence $T(M)$ becomes a locally conformal almost Kählerian manifold. On the other hand, V.Oproiu and his collaborators constructed a family of Riemannian metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties (cf. [16, 17, 18, 19]) (for example, the scalar curvature of $T(M)$ can be constant also for a non-flat base manifold with constant sectional curvature). Then M.T.K.Abbassi & M.Sarih proved in [3] that the considered metrics by Oproiu form a particular subclass of the so-called *g-natural metrics* on the tangent bundle (see also [1, 3, 4, 5, 13]).

In this paper we described a family g_a of Riemannian metrics of Cheeger Gromoll type, on the tangent bundle $T(M)$ of the Riemannian manifold (M, g) and a compatible almost complex structure J_a which bestow to $T(M)$ a structure of locally conformal almost Kählerian manifold. We found an almost Kählerian structure on $T(M)$ and we proved that there is no Cheeger Gromoll type structure on $T(M)$ such that the manifold $(T(M), g_a, J_a)$ is Kählerian. We studied the possibility for the sectional curvature on $T(M)$ to be constant and we found a flat metric on $T(M)$ (of course of Cheeger Gromoll type). Finally, if M is a real space form, we were interested to find when $T(M)$ endowed with the metric g_a has constant scalar curvature.

2 On the Geometry of the Tangent Bundle $T(M)$

Let (M, g) be a Riemannian manifold and let ∇ be its Levi Civita connection. Let $\tau : T(M) \rightarrow M$ be the tangent bundle. If $\mathbf{u} \in T(M)$ it is well known the following decomposition of the tangent space $T_{\mathbf{u}}T(M)$ (in \mathbf{u} at $T(M)$)

$$T_{\mathbf{u}}T(M) = V_{\mathbf{u}}T(M) \oplus H_{\mathbf{u}}T(M)$$

where $V_{\mathbf{u}}T(M) = \ker \tau_{*, \mathbf{u}}$ is the vertical space and $H_{\mathbf{u}}T(M)$ is the horizontal space in \mathbf{u} obtained by using ∇ . (A curve $\tilde{\gamma} : I \rightarrow T(M)$, $t \mapsto (\gamma(t), V(t))$ is *horizontal* if the vector field $V(t)$ is parallel along $\gamma = \tilde{\gamma} \circ \tau$. A vector on $T(M)$ is *horizontal* if it is tangent to an horizontal curve and *vertical* if it is tangent to a fiber. Locally, if (U, x^i) , $i = 1, \dots, m$, $m = \dim M$, is a local chart in $p \in M$, consider $(\tau^{-1}(U), x^i, y^i)$ a local chart on $T(M)$. If $\Gamma_{ij}^k(x)$ are the Christoffel symbols, then $\delta_i = \frac{\partial}{\partial x^i} - \Gamma_{ij}^k(x)y^j \frac{\partial}{\partial y^k}$ in \mathbf{u} , $i = 1, \dots, m$ span the space $H_{\mathbf{u}}T(M)$, while $\frac{\partial}{\partial y^i}$, $i = 1, \dots, m$ span the vertical space $V_{\mathbf{u}}T(M)$.) We have obtained the horizontal (vertical) distribution *HTM* (*VTM*) and a direct sum decomposition

$$TTM = HTM \oplus VTM$$

of the tangent bundle of $T(M)$. If $X \in \chi(M)$, denote by X^H (and X^V , respectively) the horizontal lift (and the vertical lift, respectively) of X to $T(M)$.

If $\mathbf{u} \in T(M)$ then we consider the energy density in \mathbf{u} on $T(M)$, namely $t = \frac{1}{2} g_{\tau(\mathbf{u})}(\mathbf{u}, \mathbf{u})$.

2.1 The Cheeger-Gromoll Structure

The Cheeger-Gromoll metric on $T(M)$ is given by

$$\begin{cases} g_{CG(p,u)}(X^H, Y^H) = g_p(X, Y), & g_{CG(p,u)}(X^H, Y^V) = 0 \\ g_{CG(p,u)}(X^V, Y^V) = \frac{1}{1+2t} (g_p(X, Y) + g_p(X, u)g_p(Y, u)) \end{cases} \quad (2.1)$$

for any vectors X and Y tangent to M . Moreover, an almost complex structure J_{CG} , compatible with the Cheeger-Gromoll metric, can be defined by the formulas

$$\begin{cases} J_{CG}X_{(p,u)}^H = \tau X^V - \frac{1}{1+\tau} g_p(X, u)u^V \\ J_{CG}X_{(p,u)}^V = -\frac{1}{\tau} X^H - \frac{1}{\tau(1+\tau)} g_p(X, u)u^H \end{cases} \quad (2.2)$$

where $\tau = \sqrt{1+2t}$ and $X \in T_p(M)$. Remark that $J_{CG}u^H = u^V$ and $J_{CG}u^V = -u^H$. We get an almost Hermitian manifold $(T(M), J_{CG}, g_{CG})$. If we denote by Ω_{CG} the Kaehler 2-form (namely $\Omega_{CG}(U, V) = g_{CG}(U, J_{CG}V)$, $\forall U, V \in \chi(T(M))$) one can prove the following

Proposition 2.1 *We have*

$$d\Omega_{CG} = \omega \wedge \Omega_{CG}, \quad (2.3)$$

where $\omega \in \Lambda^1(T(M))$ is defined by

$$\omega_{(p,u)}(X^H) = 0 \text{ and } \omega_{(p,u)}(X^V) = -\left(\frac{1}{\tau^2} + \frac{1}{1+\tau}\right) g_p(X, u), X \in T_p(M).$$

PROOF. A simple computation gives the differential of Ω_{CG} :

$$\begin{aligned} d\Omega_{CG}(X^H, Y^H, Z^H) &= d\Omega_{CG}(X^H, Y^H, Z^V) = d\Omega_{CG}(X^V, Y^V, Z^V) = 0 \\ d\Omega_{CG}(X^H, Y^V, Z^V) &= \frac{1}{\tau} \left(\frac{1}{\tau^2} + \frac{1}{1+\tau}\right) [g(X, Y)g(Z, u) - g(X, Z)g(Y, u)] \end{aligned}$$

for any $X, Y, Z \in \chi(M)$. Hence the statement. \blacksquare

Remark 2.2 The almost Hermitian manifold $(T(M), J_{CG}, g_{CG})$ is never almost Kaehlerian (i.e. $d\Omega_{CG} \neq 0$).

Finally, a necessary condition for the integrability of J_{CG} is that the base manifold (M, g) is locally Euclidian.

2.2 The Cheeger Gromoll Type Structure

A general metric, let's call it g_a , is in fact a family of Riemannian metrics, depending on a parameter a , and the Cheeger-Gromoll metric is obtained by taking $a(t) = \frac{1}{1+2t}$. It is defined by the following formulas (see also [6])

$$\begin{cases} g_{a(p,u)}(X^H, Y^H) = g_p(X, Y) \\ g_{a(p,u)}(X^H, Y^V) = 0 \\ g_{a(p,u)}(X^V, Y^V) = a(t)(g_p(X, Y) + g_p(X, u)g_p(Y, u)), \end{cases} \quad (2.4)$$

for all $X, Y \in \chi(M)$, where $a : [0, +\infty) \longrightarrow (0, +\infty)$.

Proposition 2.3 (see also [14]) *The metric defined above can be construct by using the method described by Musso and Tricerri in [15].*

We intend to find an almost complex structure on $T(M)$, call it J_a , compatible with the metric g_a . Inspired from the previous cases we look for the almost complex structure J_a in the following way

$$\begin{cases} J_a X_{(p,u)}^H = \alpha X^V + \beta g_p(X, u) u^V \\ J_a X_{(p,u)}^V = \gamma X^H + \rho g_p(X, u) u^H \end{cases} \quad (2.5)$$

where $X \in \chi(M)$ and α, β, γ and ρ are smooth functions on $T(M)$ which will be determined from $J_a^2 = -I$ and from the compatibility conditions with the metric g_a . Following the computations made in [6] we get first $\alpha = \pm \frac{1}{\sqrt{a}}$ and $\gamma = \mp \sqrt{a}$.

Without lost of the generality we can take $\alpha = \frac{1}{\sqrt{a}}$ and $\gamma = -\sqrt{a}$. Then one obtains $\beta = -\frac{1}{2t} \left(\frac{1}{\sqrt{a}} + \epsilon \frac{1}{\sqrt{a+2bt}} \right)$ and $\rho = \frac{1}{2t} (\sqrt{a} + \epsilon \sqrt{a+2bt})$ where $\epsilon = \pm 1$.

Remark 2.4 In this general case J_a is defined on $T(M) \setminus 0$ (the bundle of non zero tangent vectors), but if we consider $\epsilon = -1$ the previous relations define J_a on all $T(M)$. From now on we will work with $\epsilon = -1$.

We have the almost complex structure J_a

$$\begin{cases} J_a X^H = \frac{1}{\sqrt{a}} \left(X^V - \frac{1}{\tau(1+\tau)} g(X, u) u^V \right) \\ J_a X^V = -\sqrt{a} \left(X^H + \frac{1}{1+\tau} g(X, u) u^H \right). \end{cases} \quad (2.6)$$

One obtains an almost Hermitian manifold $(T(M), g_a, J_a)$.

If we denote by Ω_a the Kähler 2-form, $\Omega_a(U, V) = g_a(U, J_a V)$, $\forall U, V \in \chi(T(M))$ one obtains

Proposition 2.5 (see also [6]) *The almost Hermitian manifold $(T(M), g_a, J_a)$ is locally conformal almost Kählerian, that is*

$$d\Omega_a = \omega \wedge \Omega_a \quad (2.7)$$

where ω is a closed and globally defined 1-form on $T(M)$ given by

$$\omega(X^H) = 0 \quad \text{and} \quad \omega(X^V) = \left(\frac{a'}{a} - \frac{1}{1+\tau} \right) g(X, u).$$

As consequence one can state the following

Theorem 2.6 *The almost Hermitian manifold $(T(M), g_a, J_a)$ is almost Kählerian if and only if*

$$a(t) = \text{const} \cdot \frac{e^{\sqrt{1+2t}}}{1+\sqrt{1+2t}}. \quad (2.8)$$

PROOF. The result is obtained by integrating the equation $\frac{a'}{a} = \frac{1}{1+\tau}$. ■

We will take $a(\tau) = \frac{2e^{\tau-1}}{1+\tau}$ if we ask $a(0) = 1$.

2.3 The Integrability of J_a .

In order to have an integrable structure J_a on $T(M)$ we have to compute the Nijenhuis tensor N_{J_a} of J_a and to ask that it vanishes identically.

For the integrability tensor N_{J_a} we have the following relations

$$\begin{aligned} N_{J_a}(X^H, Y^H) &= \frac{2a - (1 + \tau)a'}{2a^2\tau(1 + \tau)} (g(X, \mathbf{u})Y - g(Y, \mathbf{u})X)^V + (R_{XY}\mathbf{u})^V \\ N_{J_a}(X^V, Y^V) &= \left(-aR_{XY}\mathbf{u} - \frac{a}{1+\tau} g(Y, \mathbf{u})R_{X\mathbf{u}} + \frac{a}{1+\tau} g(X, \mathbf{u})R_{Y\mathbf{u}} \right)^V - \\ &\quad - \left(\frac{a'}{2a} - \frac{1}{1+\tau} \right) (g(Y, \mathbf{u})X - g(X, \mathbf{u})Y)^V. \end{aligned} \quad (2.9)$$

The expression for $N_{J_a}(X^H, Y^V)$ is very complicated. Thus if J_a is integrable then

$$R_{XY}\mathbf{u} = \frac{2a - (1 + \tau)a'}{2a^2\tau(1 + \tau)} (g(Y, \mathbf{u})X - g(X, \mathbf{u})Y)$$

for every $X, Y \in \chi(M)$ and for every point $\mathbf{u} \in T(M)$. It follows that M is a space form $M(c)$ (c is the constant sectional curvature of M). Consequently,

$$a(\tau) = \frac{e^{2\tau}}{(1 + \tau)(ce^{2\tau}(\tau - 1) + k(1 + \tau))}$$

with k a positive real constant and c must be nonnegative.

Question: Can $(T(M), g_a, J_a)$ be a Kähler manifold?

If this happens then we have to find an appropriate constant in (2.8) such that the expression $\frac{2a - (1 + \tau)a'}{2a^2\tau(1 + \tau)}$ is also a constant.

Theorem 2.7 *There is no Cheeger Gromoll type structure on $T(M)$ such that the manifold $(T(M), g_a, J_a)$ is Kählerian.*

Now we give

Proposition 2.8 *Let (M, g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the metric g_a . Then, the corresponding Levi Civita connection $\tilde{\nabla}^a$ satisfies the following relations:*

$$\left\{ \begin{array}{l} \tilde{\nabla}_{X^H}^a Y^H = (\nabla_X Y)^H - \frac{1}{2} (R_{XY}\mathbf{u})^V \\ \tilde{\nabla}_{X^H}^a Y^V = (\nabla_X Y)^V + \frac{a}{2} (R_{\mathbf{u}Y}X)^H \\ \tilde{\nabla}_{X^V}^a Y^H = \frac{a}{2} (R_{\mathbf{u}X}Y)^H \\ \tilde{\nabla}_{X^V}^a Y^V = \mathbf{L} (g(X, \mathbf{u})Y^V + g(Y, \mathbf{u})X^V) + \frac{1-L}{\tau^2} g(X, Y)\mathbf{u}^V - \\ \quad - \frac{L}{\tau^2} g(X, \mathbf{u})g(Y, \mathbf{u})\mathbf{u}^V, \end{array} \right. \quad (2.10)$$

where $\mathbf{L} = \frac{a'(t)}{2a(t)}$.

PROOF. The statement follows from Koszul formula making usual computations. \blacksquare

Having determined Levi Civita connection, we can compute now the Riemannian curvature tensor \tilde{R}^a on $T(M)$. We give

Proposition 2.9 *The curvature tensor is given by*

$$\begin{aligned}
\tilde{R}_{X^H Y^H}^a Z^H &= (R_{XY}Z)^H + \frac{a}{4} [R_{uR_{XZ}u}Y - R_{uR_{YZ}u}X + 2R_{uR_{XY}u}Z]^H + \\
&\quad + \frac{1}{2} [(\nabla_Z R)_{XY}u]^V \\
\tilde{R}_{X^H Y^H}^a Z^V &= [R_{XY}Z + \frac{a}{4} (R_{YR_{uZ}X}u - R_{XR_{uZ}Y}u)]^V + \text{Lg}(Z, u)(R_{XY}u)^V + \\
&\quad + \frac{1-L}{r^2} g(R_{XY}u, Z)u^V + \frac{a}{2} [(\nabla_X R)_{uZ}Y - (\nabla_Y R)_{uZ}X]^H \\
\tilde{R}_{X^H Y^V}^a Z^H &= \frac{a}{2} [(\nabla_X R)_{uY}Z]^H + \\
&\quad + \frac{1}{2} [R_{XZ}Y - \frac{a}{2} R_{XR_{uY}Z}u + \text{Lg}(Y, u)R_{XZ}u + \frac{1-L}{r^2} g(R_{XZ}u, Y)u]^V \\
\tilde{R}_{X^H Y^V}^a Z^V &= -\frac{a}{2} (R_{YZ}X)^H - \frac{a^2}{4} (R_{uY}R_{uZ}X)^H + \\
&\quad + \frac{a'}{4} [g(Z, u)(R_{uY}X)^H - g(Y, u)(R_{uZ}X)^H] \\
\tilde{R}_{X^V Y^V}^a Z^H &= a(R_{XY}Z)^H + \frac{a'}{2} [g(X, u)R_{uY}Z - g(Y, u)R_{uX}Z]^H \\
&\quad + \frac{a^2}{4} [R_{uX}R_{uY}Z - R_{uY}R_{uX}Z]^H \tag{2.11}
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_{X^V Y^V}^a Z^V &= F_1(t)g(Z, u) [g(X, u)Y^V - g(Y, u)X^V] + \\
&\quad + F_2(t) [g(X, Z)Y^V - g(Y, Z)X^V] + \\
&\quad + F_3(t) [g(X, Z)g(Y, u) - g(Y, Z)g(X, u)]u^V,
\end{aligned}$$

where $F_1 = L' + \frac{1(1-L)}{r^2}$, $F_2 = L^2 - \frac{(1-L)^2}{r^2}$ and $F_3 = \frac{L'-L^2}{r^2} + \frac{1-L}{r^4}$.

In the following let $\tilde{Q}^a(U, V)$ denote the square of the area of the parallelogram with sides U and V for $U, V \in \chi(T(M))$,

$$\tilde{Q}^a(U, V) = g_a(U, U)g_a(V, V) - g_a(U, V)^2.$$

We have

Lemma 2.10 *Let $X, Y \in T_p M$ be two orthonormal vectors. Then*

$$\begin{cases} \tilde{Q}^a(X^H, Y^H) = 1, & \tilde{Q}^a(X^H, Y^V) = a(t)(1 + g(Y, u)^2) \\ \tilde{Q}^a(X^V, Y^V) = a(t)^2(1 + g(X, u)^2 + g(Y, u)^2). \end{cases} \tag{2.12}$$

We compute now the sectional curvature of the Riemannian manifold $(T(M), g_a)$, namely $\tilde{K}^a(U, V) = \frac{g_a(\tilde{R}_{uV}^a V, U)}{\tilde{Q}^a(U, V)}$ for $U, V \in \chi(T(M))$.

Denote by $T_0(M) = T(M) \setminus \mathbf{0}$ the tangent bundle of non-zero vectors tangent to M .

For a given point $(p, u) \in T_0(M)$ consider an orthonormal basis $\{e_i\}_{i=\overline{1, m}}$ for the tangent space $T_p(M)$ of M such that $e_1 = \frac{u}{|u|}$. Consider on $T_{(p, u)}T(M)$ the following vectors

$$E_i = e_i^H, \quad i = \overline{1, m}, \quad E_{m+1} = \frac{1}{r\sqrt{a}} e_1^V, \quad E_{m+k} = \frac{1}{\sqrt{a}} e_k^V, \quad k = \overline{2, m}. \tag{2.13}$$

It is easy to check that $\{E_1, \dots, E_{2m}\}$ is an orthonormal basis in $T_{(p,u)}T(M)$ (with respect to the metric g_a). We will write the expressions of the sectional curvature \tilde{K}^a in terms of this basis. We have

$$\begin{cases} \tilde{K}^a(E_i, E_j) = K(e_i, e_j) - \frac{3a(t)}{4} |R_{e_i e_j} \mathbf{u}|^2, \\ \tilde{K}^a(E_i, E_{m+1}) = 0, \\ \tilde{K}^a(E_i, E_{m+k}) = \frac{1}{4} |R_{u e_k} e_i|^2, \\ \tilde{K}^a(E_{m+1} E_{m+k}) = -\frac{F_2 + 2tF_3}{a(t)}, \\ \tilde{K}^a(E_{m+k} E_{m+l}) = -\frac{F_2}{a(t)}, \quad i, j = 1, \dots, m; \quad k, l = 2, \dots, m. \end{cases} \quad (2.14)$$

Here $|\cdot|$ denotes the norm of the vector with respect to the metric g (in a point).

Question: *Can we have constant sectional curvature \tilde{c} on $T(M)$?*

If this happens, then it must be 0, so $T(M)$ is flat. One gets easily that M is locally Euclidean. Then, we should also have $F_2(t) = 0$. It follows, $F_3(t) = 0$ and $F_1(t) = 0$. On the other hand an ordinary differential equation occurs:

$$\frac{a'(t)}{a(t)} = \frac{2}{1 + \sqrt{1+2t}}.$$

A simple computation shows that

$$a(t) = a_0 \frac{e^{2\sqrt{1+2t}}}{(1 + \sqrt{1+2t})^2}, \quad a_0 > 0. \quad (2.15)$$

Remark 2.11 The manifold $T(M)$ equipped with the Cheeger Gromoll has non constant sectional curvature.

Putting a_0 , such that $a(0) = 1$ we can state the following

Theorem 2.12 *Consider g_1 on $T(M)$ given by*

$$\begin{cases} g_1(X^H, Y^H) = g(X, Y), \quad g_1(X^H, Y^V) = 0 \\ g_1(X^V, Y^V) = \frac{4e^{2(\tau-1)}}{(1+\tau)^2} (g(X, Y) + g(X, \mathbf{u})g(Y, \mathbf{u})) \end{cases} \quad (2.16)$$

The manifold $(T(M), g_1)$ is flat.

Let us now compare the scalar curvatures of (M, g) and $(T(M), g_a)$.

Proposition 2.13 *Let (M, g) be a Riemannian manifold and endow the tangent bundle $T(M)$ with the metric g_a . Let scal and $\widetilde{\text{scal}}^a$ be the scalar curvatures of g and g_a respectively. The following relation holds:*

$$\widetilde{\text{scal}}^a = \text{scal} + \frac{2-3a}{2} \sum_{i < j} |R_{e_i e_j} \mathbf{u}|^2 + \frac{1-m}{a} (mF_2 + 4tF_3), \quad (2.17)$$

where $\{e_i\}_{i=1, \dots, m}$ is a local orthonormal frame on $T(M)$.

PROOF. Using that $\text{scal} = \sum_{i \neq j} K(e_i, e_j)$ and the formula

$$\sum_{i,j=1}^m |R_{e_i u} e_j|^2 = \sum_{i,j=1}^m |R_{e_i e_j} u|^2$$

we get the conclusion. ■

Consider M a real space form with c the constant sectional curvature.

Question: *Could we find functions a such that $T(M)$ equipped with the metric g_a has constant scalar curvature?*

Then $(T(M), g_a)$ has constant scalar curvature if and only if a satisfies the following ODE:

$$\begin{aligned} & -\frac{1}{2(1+2t)^2 a(t)^3} \\ & (-2(m+2(-2+m)t)a(t)^2 - 4t(c+2ct)^2 a(t)^3 + \\ & + 6t(c+2ct)^2 a(t)^4 + (-6+m)t(1+2t)a'(t)^2 + \\ & + 2a(t)((m+2(-1+m)t)a'(t) + 2t(1+2t)a''(t))) = \text{const.} \end{aligned}$$

which seems to be very complicated to solve.

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