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# Classification of almost contact structures associated with a strongly pseudo-convex *CR*-structure (**)

## 1 - Introduction

A *CR*-structure of hypersurface type on a $(2n+1)$-dimensional manifold $M$ is defined by a 1-codimensional subbundle $H(M)$ of the tangent bundle together with a complex structure $J$ on $H(M)$ satisfying certain integrability conditions.

Many authors dedicated their attention to the study of almost contact structures associated with a pseudo-convex *CR*-structure. One of the main problems concerning these structures is to find the geometric properties belonging to all almost contact structures associated with the same *CR*-structure, i.e. invariant under gauge transformations (see for example [**7**], [**10**], [**11**], [**12**]).

On the other hand, in the case of 3-dimensional manifolds, F. Belgun recently obtained in [**1**] a complete classification of Sasakian structures associated with the same *CR*-structure while P. Gauduchon and L. Ornea found a condition such that the gauge transformations carry Sasakian structures into Sasakian structures [**5**].

The main purpose of this paper is to classify the almost contact metric structures associated with a strongly pseudo-convex *CR*-structure, in the light of the results of D. Chinea and C. Gonzales in [**3**], where a complete classification of almost contact metric structures in 12 different classes has been found. We remark

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also that in [4] D. Chinea and J. C. Marrero studied this classification on the viewpoint of conformal geometry.

Applying this classification to the $CR$-manifolds, we analyse in particular the properties for the gauge transformations under which it is possible to obtain different types of almost contact metric structures associated with the same $CR$-structure. Some conditions for remarkable structures are given.

As interesting examples, we consider our results on the unit tangent bundle of a Riemannian manifold of constant sectional curvature and on the Heisenberg group $H_3$; in $H_3$ we also construct the gauge transformations convenient to obtain different almost contact structures.

The outline of the paper is as follows. Sections **2** and **3** are devoted to general results on pseudo-convex $CR$-structures, gauge transformations and to the classification of almost contact stuctures respectively [3]. In section **4** we apply this classification to almost contact metric structures associated with a same strongly pseudo-convex $CR$-structure and finally in the last section we describe in detail the cited examples.

## 2 - Preliminaries

Let $M$ be an orientable $C^\infty$ m-dimensional manifold; a $CR$-structure $(M, H(M))$ on $M$, is defined by a complex vector subbundle $H(M)$ in the complexification $T^c M$ of the tangent bundle of $M$ so that:

(a) $A(M) \cap H(M) = \{0\}$ where $A(M) = \overline{H(M)}$.

(b) $H(M)$ is complex involutive, i.e. for two $H(M)$-valued complex vector fields $Z, W$, the bracket $[Z, W]$ is $H(M)$-valued too.

Denoted now by $H(M)$ also the decomplexification of the complex subbundle, let $J$ be the operator on $H(M)$ corresponding to the multiplication by $i$; then the condition of complex involutivity can be expressed by:

$$(2.1) \qquad \begin{cases} \text{(i)} \ [X, Y] - [JX, JY] \in \Gamma(H(M)) \\ \text{(ii)} \ N_J(X, Y) = [JX, JY] - [X, Y] - J\{[JX, Y] + [X, JY]\} = 0 \end{cases}$$

for every $X, Y$ belonging to $\Gamma(H(M))$, $\Gamma(H(M))$ being the $C^\infty(M)$-module of cross-sections on $H(M)$.

From now on, we shall suppose that $dim\, M = 2n+1$, $codim\, H(M) = 1$ and that the Levi form of $(M, H(M))$ is nondegenerate, i.e. we shall consider only pseudo-convex $CR$-structures of hypersurface type. Then, if we denote by $\eta$ the local 1-form having $H(M)$ as null bundle, the property of pseudoconvexity of

$(M, H(M))$ assures that $\eta \wedge (d\eta)^n \neq 0$ and $\eta$ is a contact form on $M$. Notice that, if we consider $M$ globally oriented, then $\eta$ is globally defined.

Then for a pseudo-convex structure we have $TM = \mathrm{span}[\xi] \oplus H(M)$, where $\xi$ is the Reeb vector field defined by $\eta(\xi) = 1$, $i_\xi d\eta = 0$; moreover if $\phi$ is the (1,1)-tensor field given by

$$\phi X = J(X - \eta(X)\xi), \qquad \forall X \in \chi(M) \tag{2.2}$$

the following relations hold

$$\eta \circ \phi = 0, \qquad \phi\xi = 0, \qquad \phi^2 = -I + \eta \otimes \xi;$$

hence $(\phi, \xi, \eta)$ defines an almost contact structure on $M$ which is called associated with the pseudo-convex $CR$-structure $(M, H(M))$ (see [**2**], [**11**]).

Consider now the new 1-form $\tilde{\eta} = \varepsilon e^{\sigma}\eta$, where $\sigma \in C^{\infty}(M)$ and $\varepsilon = \pm 1$; it is trivial that $\tilde{\eta}$ defines the same distribution $H(M)$ as $\eta$. Examining the relations between the associated almost contact structures $(\phi, \xi, \eta)$ and $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ respectively induced by $\eta$ and $\tilde{\eta}$ the following proposition follows

Proposition 1 [**10**]. *Two almost contact structures $(\phi, \xi, \eta)$, $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ are subordinated to the same pseudoconvex CR-structure if and only if there exists a function $\sigma \in C^{\infty}(M)$ such that*:

$$\begin{cases} \tilde{\eta} = \varepsilon e^{\sigma}\eta, & d\tilde{\eta} = \varepsilon e^{\sigma}(d\eta + d\sigma \wedge \eta) \\ \tilde{\xi} = \varepsilon e^{-\sigma}(\xi + \phi A), & \tilde{\phi} = \phi + \eta \otimes A \end{cases} \tag{2.3}$$

*where, assuming $\varepsilon = 1$ and denoting by $h$ the projection operator on $H(M)$, $A$ is a vector field defined by the conditions*:

$$\eta(A) = 0, \qquad d\eta(\phi A, X) = d\sigma(hX) = hX(\sigma).$$

It is an important geometric property that the complex involutivity is invariant under gauge transformations [**7**].

Remark 2. We shall consider from now on $\varepsilon = 1$ only, the case where $\varepsilon = -1$ being completely similar.

If we suppose the $CR$-structure strongly pseudo-convex, then the metric $g$ defined for all $X, Y \in \Gamma(H(M))$ by the equations

$$g(X, Y) = d\eta(X, \phi Y), \qquad g(X, \xi) = \eta(X)$$

is positively defined and satisfies the following compatibility conditions with respect to $(\phi, \xi, \eta)$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In the sequel, note that $d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta[X, Y]$.

After a gauge transformation, imposing the compatibility conditions with respect to the new structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$, we obtain from $g$ a new Riemannian metric $\tilde{g}$ on $M$ which generally doesn't satisfy the equation $\tilde{g}(X, Y) = d\tilde{\eta}(X, \tilde{\phi} Y)$, with $X, Y \in \Gamma(H(M))$. But, if we require that the restrictions of $g$ and $\tilde{g}$ are related by a conformal transformation on $H(M)$, then we get the following relation between $g$ and $\tilde{g}$ (see also [**12**])

$$\text{(2.4)} \quad \begin{cases} \tilde{g}(X, Y) = e^{2\sigma}\{ g(X, Y) - \eta(X)\, g(\phi A, Y) - \eta(Y)\, g(\phi A, X) \\ \qquad\qquad + g(A, A)\, \eta(X)\, \eta(Y)\} \qquad \forall X, Y \in \chi(M); \end{cases}$$

and the equality

$$\tilde{g}(X, Y) = e^{\sigma} d\tilde{\eta}(X, \tilde{\phi} Y)$$

holds for all $X, Y \in \Gamma(H(M))$.

## 3 - The 12 classes

It is known that the existence of an almost contact metric structure on $M$ is equivalent to the existence of a reduction of the structural group $\mathcal{O}(2n+1)$ to $\mathcal{U}(n) \times 1$. If we denote by $\Phi$ the fundamental 2-form of $(M, \phi, \xi, \eta, g)$ defined by $\Phi(X, Y) = g(X, \phi Y)$ and by $\nabla$ the Riemannian connection of $g$, the covariant derivative $\nabla\Phi$ is a covariant tensor of degree 3 which has various symmetry proprieties.

Let $V$ be a real vector space of dimension $2n+1$ endowed with an almost contact structure $(\phi, \xi, \eta)$ and a compatible inner product $\langle , \rangle$ and $\mathcal{C}(V)$ the vector space of 3-forms on $V$ having the same symmetries of $\nabla\Phi$, i.e.

$$\mathcal{C}(V) = \{\alpha \in \otimes_3^0 V \,|\, \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \phi y, \phi z) + \eta(y)\, \alpha(x, \xi, z) + \eta(z)\, \alpha(x, y, \xi)\}.$$

In [3] the authors have been obtained the following decomposition of $\mathcal{C}(V)$ into twelve components $\mathcal{C}_i(V)$ which are mutually orthogonal, irreducible and invariant subspaces under the action of $\mathcal{U}(n)\times 1$:

$$\mathcal{C}(V) = \bigoplus_{i=1,\dots,12} \mathcal{C}_i(V), \tag{3.1}$$

where

$$\mathcal{C}_1(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, x, y) = \alpha(x, y, \xi) = 0\},$$

$$\mathcal{C}_2(V) = \{\alpha \in \mathcal{C}(V)\,|\,\mathfrak{S}_{(x,y,z)}\,\alpha(x, y, z) = 0,\ \alpha(x, y, \xi) = 0\},$$

$$\mathcal{C}_3(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) - \alpha(\phi x, \phi y, z) = 0,\ c_{12}\alpha = 0\},$$

$$\mathcal{C}_4(V) = \Big\{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \frac{1}{2(n-1)}[(\langle x, y\rangle - \eta(x)\eta(y))\,c_{12}\alpha(z) - (\langle x, z\rangle - \eta(x)\eta(z))\,c_{12}\alpha(y) - \langle x, \phi y\rangle\,c_{12}\alpha(\phi z) + \langle x, \phi z\rangle\,c_{12}\alpha(\phi y)],\ c_{12}\alpha(\xi) = 0\Big\},$$

$$\mathcal{C}_5(V) = \Big\{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \frac{1}{2n}[\langle x, \phi z\rangle\,\eta(y)\,\bar{c}_{12}\alpha(\xi) - \langle x, \phi y\rangle\,\eta(z)\,\bar{c}_{12}\alpha(\xi)]\Big\},$$

$$\mathcal{C}_6(V) = \Big\{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \frac{1}{2n}[\langle x, y\rangle\,\eta(z)\,c_{12}\alpha(\xi) - \langle x, z\rangle\,\eta(y)\,c_{12}\alpha(\xi)]\Big\},$$

$$\mathcal{C}_7(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \eta(z)\,\alpha(y, x, \xi) - \eta(y)\,\alpha(\phi x, \phi z, \xi),\quad c_{12}\alpha(\xi) = 0\},$$

$$\mathcal{C}_8(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = -\eta(z)\,\alpha(y, x, \xi) - \eta(y)\,\alpha(\phi x, \phi z, \xi),\quad \bar{c}_{12}\alpha(\xi) = 0\},$$

$$\mathcal{C}_9(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \eta(z)\,\alpha(y, x, \xi) + \eta(y)\,\alpha(\phi x, \phi z, \xi)\},$$

$$\mathcal{C}_{10}(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = -\eta(z)\,\alpha(y, x, \xi) + \eta(y)\,\alpha(\phi x, \phi z, \xi)\},$$

$$\mathcal{C}_{11}(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = -\eta(x)\,\alpha(\xi, \phi y, \phi z)\},$$

$$\mathcal{C}_{12}(V) = \{\alpha \in \mathcal{C}(V)\,|\,\alpha(x, y, z) = \eta(x)\,\eta(y)\,\alpha(\xi, \xi, z) + \eta(x)\,\eta(z)\,\alpha(\xi, y, \xi)\}.$$

Here, if $\{e_i\}$, $i = 1, 2, \ldots, 2n+1$ denotes an arbitrary orthonormal basis we have

$$(3.2) \qquad \begin{cases} c_{12}\,\alpha(x) = \sum \alpha(e_i, e_i, x) \\ \bar{c}_{12}\,\alpha(x) = \sum \alpha(e_i, \phi e_i, x), \qquad \text{for all } x \in V. \end{cases}$$

Applying this algebraic decomposition to the geometry of almost contact structures, for each invariant subspace we obtain a different class of almost contact metric manifolds; more precisely, we shall say $M$ of class $\mathcal{C}_k$, $k = 1, \ldots, 12$, if, for every $p \in M$, the 3-form $(\nabla\Phi)_p$ of the vector space $(T_pM, \phi_p, \xi_p, \eta_p, g_p)$ belongs to $\mathcal{C}_k(T_pM)$.

For example, $\mathcal{C}_6$ corresponds to the class of $\alpha$ −Sasakian manifolds, $\mathcal{C}_2 \oplus \mathcal{C}_9$ to the class of almost cosymplectic manifolds, $\mathcal{C}_3 \oplus \ldots \oplus \mathcal{C}_8$ to that one of normal manifolds (for an extensive study of these structures see [3]).

## 4 - Classification of gauge transformations

Let $M$ be an $(2n+1)$-dimensional manifold endowed with an almost contact metric structure associated with a pseudo-convex $CR$-structure $(M, H(M))$ of hypersurface type.

Theorem 3. *$M$ is of class $\mathcal{C}_6 \oplus \mathcal{C}_9$.*

Proof. Following [3] we split the space $\mathcal{C}(T_pM)$, $p \in M$, into the direct sum

$$(4.1) \qquad \mathcal{C}(T_pM) = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3,$$

where

$$(4.2) \qquad \begin{cases} \mathcal{D}_1 = \{\alpha \in \mathcal{C}(V) \,|\, \alpha(\xi, x, y) = \alpha(x, \xi, y) = 0\} \\ \mathcal{D}_2 = \{\alpha \in \mathcal{C}(V) \,|\, \alpha(x, y, z) = \eta(x)\,\alpha(\xi, y, z) + \eta(y)\,\alpha(x, \xi, z) + \eta(z)\,\alpha(x, y, \xi)\} \\ \mathcal{D}_3 = \{\alpha \in \mathcal{C}(V) \,|\, \alpha(x, y, z) = \eta(x)\,\eta(y)\,\alpha(\xi, \xi, z) + \eta(x)\,\eta(z)\,\alpha(\xi, y, \xi)\} \end{cases}$$

obtaining

$$(4.3) \qquad \begin{cases} \mathcal{D}_1 = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_4 \\ \mathcal{D}_2 = \mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{11} \\ \mathcal{D}_3 = \mathcal{C}_{12}. \end{cases}$$

As a consequence of (4.3) we can consider $(\nabla\Phi)_p$ as the sum of three components $\alpha_k \in \mathcal{D}_k,\ k = 1, 2, 3$:

$$(\nabla \Phi)_p = \alpha_1 + \alpha_2 + \alpha_3. \tag{4.4}$$

On the other hand, a straightfoward computation proves that, for all $X, Y, Z \in \Gamma(H(M))$, the involutivity conditions (2.1) imply the equations:

$$\begin{cases} (\nabla_X \Phi)(Y, Z) = \dfrac{1}{2}\eta([[\phi Z, \phi Y] - \phi[\phi Z, Y] - \phi[Z, \phi Y] - [Z, Y], X]) = \\ \qquad = \dfrac{1}{2}\eta([N_\phi(Z, Y), X]) = 0\,, \end{cases} \tag{4.5}$$

$$\nabla_\xi \Phi = 0\,, \tag{4.6}$$

(in the following, as in (4.5) and (4.6), to simplify the notations, we shall omit indicating the point p).

From (4.5) and (4.6) we deduce that $\nabla\Phi$ has not component in $\mathcal{D}_1 = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_4$ as well as in $\mathcal{D}_3 = \mathcal{C}_{12}$; therefore $\nabla\Phi$ reduces to the only component $\alpha_2 \in \mathcal{D}_2$.

Now comparing the equalities

$$\begin{cases} \bar{c}_{12}(\nabla\Phi)(\xi) = 0\,, \\ c_{12}(\nabla\Phi)(\xi) = n\,, \end{cases} \tag{4.7}$$

with (3.1) we immediately obtain that $\nabla\Phi$ has not component in $\mathcal{C}_5$ too, and that the component in $\mathcal{C}_6$ is different from zero.

The non-existence of components for $\nabla\Phi$ in $\mathcal{C}_7 \oplus \mathcal{C}_8$ follows from the relation

$$(\nabla_X \Phi)(\xi, Z) = -(\nabla_{\phi X}\Phi)(\xi, \phi Z) - g(X, Z) \tag{4.8}$$

true for all $X, Z \in \Gamma(H(M))$.

Computing now directly from (3.1) the components of $\nabla\Phi$ in $\mathcal{C}_9 \oplus \mathcal{C}_{10}$, applying (2.1), we find that $\nabla\Phi$ has a component different from zero in $\mathcal{C}_9$; for $X, Z \in \Gamma(H(M))$ and $Y = \xi$ its expression is: $\dfrac{1}{2} g((\mathcal{L}_\xi \phi) Z, X)$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of $\phi$ with respect to $\xi$.

Finally, a simple computation proves that the component in $\mathcal{C}_{11}$ vanishes.
This completes the proof. ■

According to [3] we obtain

Corollary 4. *$M$ is of class $\mathcal{C}_6$ if and only if the almost contact structure $(\phi, \xi, \eta)$ is normal.*

Proof. From the previous theorem we have that the component in $\mathcal{C}_9$ is zero iff $\mathcal{L}_\xi \phi = 0$, and this relation is always satisfied when the almost contact structure is normal, i.e. when the (1,2)-tensor field $N$ given by

$$N = N_\phi + d\eta \otimes \xi$$

vanishes.

On the other hand, in [7] it has been also proved that if $(M, H(M))$ satisfies the involutivity conditions and $\mathcal{L}_\xi \phi = 0$, then the almost contact structure $(\phi, \xi, \eta)$ is normal. ■

Let $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be now the new almost contact metric structure on $M$ obtained from $(\phi, \xi, \eta, g)$ by a gauge transformation (2.3) and (2.4); this means that both almost contact structures are associated to the same strongly pseudo-convex structure $CR$-structure $(M, H(M))$ of $M$.

If $\tilde{\nabla}$ and $\tilde{\Phi}$ denote the Levi-Civita connection and the fundamental 2-form of $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ respectively, taking into account (2.3) and (2.4), an easy computation gives

$$\tilde{\Phi}(X,Y) = e^{2\sigma}\{\Phi(X,Y) - \eta(X)\, g(A,Y) + \eta(Y)\, g(A,X)\} \qquad \text{for all } X, Y \in \chi(M)\,; \tag{4.9}$$

furthermore it will be useful for us to remark that the following formula holds:

$$\begin{aligned} \mathcal{L}_{\tilde{\xi}}\tilde{\phi}(X) = e^{-\sigma}\{&\mathcal{L}_\xi \phi(X) + (\phi X(\sigma) + \eta(X)\, A(\sigma))(\xi + \phi A) + [\phi A, \phi X] \\ &- \phi[\phi A, X] + hX(\sigma)\, A + \eta(X)[\xi + \phi A, A]\}\,. \end{aligned} \tag{4.10}$$

Theorem 5. *If dimension of $M$ is $2n+1$, $n \geq 2$, $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is of class $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$. When $n = 1$ then $M$ has dimension 3 and $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is of class $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$.*

Proof. Taking into account previous formulas and definitions, after lengthy straightforward computation, it is possible to prove the following relations between $\tilde{\nabla}\tilde{\Phi}$ and $\nabla\Phi$

$$(4.11)\quad \begin{aligned}&(\tilde{\nabla}_X\tilde{\Phi})(Y, Z) = e^{2\sigma}(\nabla_X\Phi)(Y, Z) + \\ &+\frac{e^{2\sigma}}{2}\{Z(\sigma)g(X, \varphi Y) - Y(\sigma)\, g(X, \varphi Z) + \varphi Z(\sigma)\, g(X, Y) - \varphi Y(\sigma)\, g(X, Z)\},\end{aligned}$$

$$(4.12)\quad \begin{aligned}(\tilde{\nabla}_X\tilde{\Phi})(\xi, Z) = e^{\sigma}(\nabla_X\Phi)(\xi, Z) - \frac{e^{\sigma}}{2}\{\xi(\sigma)\, g(X, \varphi Z) - \varphi Z(\sigma)\, g(\varphi A, X) \\ -g([\varphi A, \varphi Z), X) - g([\varphi A, Z], \varphi X) - Z(\sigma)\, g(A, X)\},\end{aligned}$$

$$(4.13)\quad \tilde{\nabla}_\xi\tilde{\Phi} = 0\,.$$

Suppose $n \geq 2$ and, as above, consider $\tilde{\nabla}\tilde{\Phi}$ as the sum of three components $\alpha_k \in \mathcal{D}_k$, $k = 1, 2, 3$:

$$(4.14)\quad \tilde{\nabla}\tilde{\Phi} = \alpha_1 + \alpha_2 + \alpha_3.$$

The vanishing of $\alpha_3$ follows easily from the equations (4.6) and (4.13); as a consequence $\tilde{\nabla}\tilde{\Phi}$ has no component in $\mathcal{C}_{12}$.

With regard to $\alpha_2$, Theorem 3, (4.10) and (4.12) imply that we have only three components different from zero in $\mathcal{C}_5$, $\mathcal{C}_6$ and $\mathcal{C}_9$ given respectively by

$$(4.15)\quad -\frac{1}{2}e^{\sigma}\xi(\sigma)\, g(X, \phi Z), \qquad -\frac{1}{2}e^{\sigma}g(X, Z), \qquad \frac{1}{2}\tilde{g}((\mathcal{L}_\xi\tilde{\phi})\, Z, X),$$

for every $X, Z \in \Gamma(H(M))$ and $Y = \xi$.

Supposing at the end $X, Y, Z \in \Gamma(H(M))$ we can compute the component in $\mathcal{D}_1$. As the restriction to $H(M)$ of our structure reduces to an almost Hermitian structure, applying [**6**] and comparing with (4.5) and (4.11) we find for $(\tilde{\nabla}_X\tilde{\Phi})(Y, Z)$ the only following component in $\mathcal{C}_4$:

$$(4.16)\quad \begin{cases}\frac{1}{2}e^{\sigma}(Z(\sigma)\, g(X, \phi Y) - Y(\sigma)\, g(X, \phi Z)) + \\ +\frac{1}{2}e^{\sigma}((\phi Z)(\sigma)\, g(X, \phi Y) - (\phi Y)(\sigma)\, g(X, \phi Z)).\end{cases}$$

The case $n = 1$ directly follows from [**3**] and the above considerations. ■

Corollary 6. *Supposing $M$ of dimension $2n+1 \geq 5$, we have*:

(i) $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is of class* $\mathcal{C}_4 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$ *iff* $\xi(\sigma) = 0$;

(ii) $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is of class* $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$ *iff* $X(\sigma) = 0$, $\forall X \in \Gamma(H(M))$;

(iii) $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *is of class* $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$ *iff* $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ *is normal, i.e. iff* (4.14) *holds.*

Remark 7. *From Corollary* 4 *and Corollary* 6, (iii), *we deduce that the normality of the structure is preserved iff*

$$[\phi A, \phi X] - \phi[\phi A, X] = -\phi X(\sigma)(\xi + \phi A) + hX(\sigma) A .$$

Then we can state

Corollary 8. *If* $(M, \phi, \xi, \eta, g)$ *is Sasakian and* $\dim M = 3$ *then* $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *obtained by* (2.3) *with* $\sigma$ *not constant is Sasakian iff*

(a) $\xi(\sigma) = 0$;

(b) $[\phi A, A] = -A(\sigma)(\xi + \phi A)$.

## 5 - Examples

### The unit tangent bundle

Let $(M, g)$ be an $(n+1)$-dimensional Riemannian manifold, $n \geq 2$; we denote by $TM$ the tangent bundle of the manifold $M$ and by $\bar{\pi}: TM \to M$ the canonical projection. If $(x^1, \ldots, x^{n+1})$ are local coordinates on $M$, then $(x^1, \ldots, x^{n+1})$ and the fibre coordinates $(y^1, \ldots, y^{n+1})$ define together a system of local coordinates on $TM$. The Levi-Civita connection $D$ of $g$ determines a decomposition of $TTM$ in the direct sum of the vertical distribution $VTM$ and the horizontal distribution $HTM$, i.e. $TTM = VTM \oplus HTM$. Then the well known almost complex structure on $TM$ is defined by:

$$JX^H = X^V, \qquad JX^V = -X^H \qquad X \in \chi(M) \tag{5.1}$$

where $X^H$, $X^V$ are the horizontal and vertical lifts of $X$ with respect to $D$ respectively. Furthermore the Sasaki metric $\dot{g}$ on $TM$ is given by

$$\dot{g}(X^V, Y^V) = g(X, Y), \quad \dot{g}(X^H, Y^H) = g(X, Y), \quad \dot{g}(X^V, Y^H) = 0 \qquad X, Y \in \chi(M).$$

Let $T_1 M$ be the unit tangent bundle of $M$; then, we have $v \in T_1 M$ iff $v \in TM$ and $g(v, v) = 1$. If $v = y^i \dfrac{\partial}{\partial x^i}$, we conclude that the unit tangent bundle

$\pi : T_1 M \rightarrow M$ is a hypersurface in $TM$, given in the local coordinates by the equation:

$$g_{ij}(x)\, y^i y^j - 1 = 0\,. \tag{5.2}$$

It is possible to prove that, as hypersurface of the almost Kaehlerian manifold $(TM, J, \dot{g})$, $T_1 M$ has a natural almost contact metric structure which defines a pseudo-convex $CR$-structure $(T_1 M, H(T_1 M))$ iff the base manifold $M$ has constant sectional curvature $c$ (see [8], [9], [13]).

Moreover, if we consider a generator system for $H(T_1 M))$ given by the following vector fields: $Y_i = (\delta_i^j - g_{i0}\, y^j)\, \dfrac{\partial}{\partial y^j}$ and $X_i = (\delta_i^j - g_{i0}\, y^j)\, \dfrac{\delta}{\delta x^j}$, where $g_{i0} = g_{ik}\, y^k$, and we still denote by $\dot{g}$ the metric induced from $TM$ on $T_1 M$, the almost contact structure $(\phi, \xi, \eta, \dot{g})$ associated with the $CR$-structure $(T_1 M, H(T_1 M))$ satisfies the following relations:

$$\begin{cases} \eta = g_{i0}\, dx^i, \qquad \xi = y^i\, \dfrac{\delta}{\delta x^i} \\ \phi X_i = Y_i, \qquad \phi Y_i = -X_i, \qquad \phi\xi = 0 \qquad\qquad i, j = 1, \ldots, n+1, \end{cases} \tag{5.3}$$

where $\dfrac{\delta}{\delta x^i} = \left(\dfrac{\partial}{\partial x^i}\right)^H = \dfrac{\partial}{\partial x^i} - \Gamma_{i0}^j\, \dfrac{\partial}{\partial y^j}$, $\Gamma_{i0}^j = y^k\, \Gamma_{ik}^j$, where $\Gamma_{ik}^j$ are the Christoffel symbols corresponding to the connection $D$.

Computing now the Levi-Civita connection $\dot{\nabla}$ of the metric $\dot{g}$ on the vector fields $Y_i, X_i, \xi$ we find:

$$\begin{cases} \dot{\nabla}_{Y_i} Y_j = -g_{j0}\, Y_i, \qquad \dot{\nabla}_{X_i} Y_j = (\Gamma_{ij}^k - g_{i0}\, \Gamma_{j0}^k)\, Y_k + \dfrac{c}{2}\, h_{ij}\, \xi \\ \dot{\nabla}_{Y_i} X_j = -g_{j0}\, X_i + \dfrac{c-2}{2}\, h_{ij}\, \xi\,, \qquad \dot{\nabla}_{X_i} X_j = (\Gamma_{ij}^k - g_{i0}\, \Gamma_{j0}^k)\, X_k \\ \dot{\nabla}_{Y_i} \xi = -\dfrac{c-2}{2}\, X_i, \qquad \dot{\nabla}_{X_i} \xi = -\dfrac{c}{2}\, Y_i, \qquad \dot{\nabla}_{\xi} \xi = 0 \\ \dot{\nabla}_{\xi} Y_i = \Gamma_{i0}^k\, Y_k - \dfrac{c}{2}\, X_i, \qquad \dot{\nabla}_{\xi} X_i = \Gamma_{i0}^k\, X_k + \dfrac{c}{2}\, Y_i, \qquad i, j, k = 1, \ldots, n+1\,, \end{cases} \tag{5.4}$$

where

$$h_{ij} = g_{ij} - g_{i0}\, g_{j0}\,. \tag{5.5}$$

Then we easily can write the expressions of the following Lie brackets:

$$(5.6)\qquad \begin{cases} [Y_i, Y_j] = g_{i0} Y_j - g_{j0} Y_i, \qquad [X_i, X_j] = (g_{j0} \Gamma^k_{i0} - g_{i0} \Gamma^k_{j0}) X_k \\ [Y_i, X_j] = -g_{j0} X_i - (\Gamma^k_{ij} - g_{j0} \Gamma^k_{i0}) Y_k - h_{ij} \xi \\ [Y_i, \xi] = X_i - \Gamma^k_{i0} Y_k, \qquad [X_i, \xi] = -cY_i - \Gamma^k_{i0} X_k. \end{cases}$$

From the previous formulas, we obtain that the covariant derivative $\dot{\nabla} \Phi$ of the fundamental 2-form $\Phi(X, Y) = \dot{g}(X, \phi Y) = -d\eta(X, Y)$ of $(\phi, \xi, \eta, \dot{g})$ is not vanishing only in the following cases

$$(5.7)\qquad \begin{cases} (\dot{\nabla}_{Y_i} \Phi)(Y_j, \xi) = -(\dot{\nabla}_{Y_i} \Phi)(\xi, Y_j) = \dfrac{c-2}{2} h_{ij} \\ (\dot{\nabla}_{X_i} \Phi)(X_j, \xi) = -(\dot{\nabla}_{X_i} \Phi)(\xi, X_j) = -\dfrac{c}{2} h_{ij}, \end{cases}$$

and finally, from formulas (5.6), we have that the following equations hold

$$(5.8)\qquad (\mathcal{L}_\xi \phi) X_i = (c-1) X_i, \qquad (\mathcal{L}_\xi \phi) Y_i = (1-c) Y_i \qquad i = 1, \ldots, n+1 .$$

As a consequence, taking into account Theorem 3 and Corollary 4, we can state

Proposition 9. *$(T_1 M, \phi, \xi, \eta, \dot{g})$ is of class $\mathcal{C}_6 \oplus \mathcal{C}_9$. In particular, $(T_1 M, \phi, \xi, \eta, \dot{g})$ belongs to $\mathcal{C}_6$ iff $c = 1$.*

Apply now the gauge transformation (2.3) to $(\phi, \xi, \eta)$, obtaining $\tilde{\eta} = e^\sigma g_{i0}\, dx^i$; furthermore the vector field $A \in H(M)$ can be expressed by means of $\{Y_i, X_i\}$ as

$$(5.9)\qquad A = \lambda^i Y_i + \mu^i X_i, \qquad \text{where } \lambda^i, \mu^i \in C^\infty(T_1 M).$$

Moreover, taking into account (2.4), we obtain for the new metric $\tilde{g}$ the relations:

$$(5.10)\qquad \begin{cases} \tilde{g}(Y_i, Y_j) = \tilde{g}(X_i, X_j) = e^{2\sigma} h_{ij}, \quad \tilde{g}(X_i, Y_j) = 0 \\ \tilde{g}(X_i, \xi) = e^{2\sigma} Y_i(\sigma), \qquad \tilde{g}(Y_i, \xi) = -e^{2\sigma} X_i(\sigma) \\ \tilde{g}(\xi, \xi) = e^{2\sigma}(1 + \|A\|^2), \qquad \tilde{g}(\tilde{\xi}, \tilde{\xi}) = 1, \qquad \tilde{g}(\xi, \tilde{\xi}) = e^\sigma, \end{cases}$$

where $\|A\|^2 = \lambda^i Y_i(\sigma) + \mu^i X_i(\sigma)$.

Then, considering the covariant derivative $\tilde{\nabla} \tilde{\Phi}$ of the fundamental 2-form

$\widetilde{\Phi}(X, Y) = \tilde{g}(X, \tilde{\phi} Y)$ of the new structure, we obtain:

$$
(5.11)\quad \begin{cases}
(\widetilde{\nabla}_{Y_i}\widetilde{\Phi})(Y_j, Y_k) = -(\widetilde{\nabla}_{Y_i}\widetilde{\Phi})(X_j, X_k) = \\
\qquad = (\widetilde{\nabla}_{X_i}\widetilde{\Phi})(Y_j, X_k) = \dfrac{e^{2\sigma}}{2}(X_j(\sigma)\, h_{ik} - X_k(\sigma)\, h_{ij}) \\
(\widetilde{\nabla}_{X_i}\widetilde{\Phi})(Y_j, Y_k) = -(\widetilde{\nabla}_{X_i}\widetilde{\Phi})(X_j, X_k) = \\
\qquad = -(\widetilde{\nabla}_{Y_i}\widetilde{\Phi})(Y_j, X_k) = \dfrac{e^{2\sigma}}{2}(Y_j(\sigma)\, h_{ik} - Y_k(\sigma)\, h_{ij}) \\
(\widetilde{\nabla}_{Y_i}\widetilde{\Phi})(Y_j, \tilde{\xi}) = \dfrac{e^{\sigma}}{2}(c-2)\, h_{ij} - \dfrac{e^{\sigma}}{2} g_{k0}\lambda^k h_{ij} - \dfrac{e^{\sigma}}{2}\mu^k(\Gamma^l_{jk} - g_{j0}\Gamma^l_{k0})\, h_{li} + \\
\qquad + \dfrac{e^{\sigma}}{2}\big(X_i(\sigma)\, X_j(\sigma) - Y_i(\sigma)\, Y_j(\sigma)\big) + \dfrac{e^{\sigma}}{2}\big(Y_i(\lambda^k)\, h_{jk} - X_j(\mu^k)\, h_{ik}\big) \\
(\widetilde{\nabla}_{Y_i}\widetilde{\Phi})(X_j, \tilde{\xi}) = e^{\sigma}\xi(\sigma)\, h_{ij} - e^{\sigma} g_{k0}\mu^k h_{ij} - \\
\qquad - \dfrac{e^{\sigma}}{2}\big(X_i(\sigma)\, Y_j(\sigma) + Y_i(\sigma)\, X_j(\sigma)\big) + \dfrac{e^{\sigma}}{2}\big(Y_i(\mu^k)\, h_{jk} + Y_j(\mu^k)\, h_{ik}\big) \\
(\widetilde{\nabla}_{X_i}\widetilde{\Phi})(Y_j, \tilde{\xi}) = -e^{\sigma}\xi(\sigma)\, h_{ij} + \dfrac{e^{\sigma}}{2}\lambda^k \dfrac{\partial}{\partial x^k}(h_{ij}) - \\
\qquad - \dfrac{e^{\sigma}}{2}(X_i(\sigma)\, Y_j(\sigma) + Y_i(\sigma)\, X_j(\sigma)) + \dfrac{e^{\sigma}}{2}(X_i(\lambda^k)\, h_{jk} + X_j(\lambda^k)\, h_{ik}) \\
(\widetilde{\nabla}_{X_i}\widetilde{\Phi})(X_j, \tilde{\xi}) = -\dfrac{e^{\sigma}}{2} c h_{ij} + \dfrac{e^{\sigma}}{2} g_{k0}\lambda^k h_{ij} + \dfrac{e^{\sigma}}{2}\mu^k(\Gamma^l_{ik} - g_{i0}\Gamma^l_{k0})\, h_{lj} + \\
\qquad + \dfrac{e^{\sigma}}{2}(Y_i(\sigma)\, Y_j(\sigma) - X_i(\sigma)\, X_j(\sigma)) + \dfrac{e^{\sigma}}{2}(X_i(\mu^k)\, h_{jk} - Y_j(\lambda^k)\, h_{ik})
\end{cases}
$$

and, as in the general case, $\widetilde{\nabla}_{\tilde{\xi}}\widetilde{\Phi} = 0$.

Finally, after a straightforward computation, we find that the new structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ is not normal and Theorem 5 and Corollary 6 imply that $(T_1 M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ belongs to $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$.

Every component of $(T_1 M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ with respect to the basis $\{X_i, Y_i, \xi\}$ can be explicitely written by means of (5.11).

### The Heisenberg group

As it is well known (see for example [**14**]), the Heisenberg Lie group $H_3$ is the subgroup of $GL(3, \mathbb{R})$ given by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbb{R} \right\} \tag{5.12}$$

with the usual matrix multiplication.

Then it is easy to see that

$$ds^2 = dx^2 + dz^2 + (dy - xdz)^2 \tag{5.13}$$

is a left invariant metric on $H_3$ as well as the following vector fields:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial y}. \tag{5.14}$$

If we consider $H(H_3)$ generated by $X_1$ and $X_2$, we have that $(H_3, H(H_3))$ is a pseudo-convex $CR$-structure on the Heisenberg group with associated almost contact metric structure defined by the formulas:

$$\begin{cases} \eta = x\,dz - dy & \xi = -X_3 \\ \phi X_1 = X_2, & \phi X_2 = -X_1, \qquad \phi\xi = 0, \end{cases} \tag{5.15}$$

while the equation (5.13) gives the associated metric $g$.

Let $\nabla$ be the Levi-Civita connection of $g$ and $\Phi$ the fundamental 2-form defined as usual. Then, the only cases where the covariant derivative is different from zero are the following:

$$(\nabla_{X_1}\Phi)(X_1, \xi) = (\nabla_{X_2}\Phi)(X_2, \xi) = \frac{1}{2},$$

and $(H_3, \phi, \eta, \xi, g) \in \mathcal{C}_6$.

Put now $A = \mu X_1 + \lambda X_2$, $\lambda, \mu \in C^\infty(H_3)$; after the gauge transformation we have

$$\mu = -X_1(\sigma), \qquad \lambda = -X_2(\sigma),$$

and the components of the new covariant derivative are:

$$(5.16)\quad \begin{cases} (\tilde{\nabla}_{X_1}\tilde{\Phi})(X_1, \tilde{\xi}) = \dfrac{e^{\sigma}}{2}(X_1(\mu) - X_2(\lambda) - \lambda^2 + \mu^2 + 1) \\ (\tilde{\nabla}_{X_2}\tilde{\Phi})(X_2, \tilde{\xi}) = \dfrac{e^{\sigma}}{2}(X_2(\lambda) - X_1(\mu) - \mu^2 + \lambda^2 + 1) \\ (\tilde{\nabla}_{X_1}\tilde{\Phi})(X_2, \tilde{\xi}) = e^{\sigma}(-\xi(\sigma) + X_1(\lambda) + \mu\lambda) \\ (\tilde{\nabla}_{X_2}\tilde{\Phi})(X_1, \tilde{\xi}) = e^{\sigma}(\xi(\sigma) + X_2(\mu) + \mu\lambda). \end{cases}$$

Formulas (5.16) and Theorem 5 imply that $(H_3, \tilde{\varphi}, \tilde{\eta}, \tilde{\xi}, \tilde{g}) \in \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$. In particular taking into account Corollary 8, after a straightforward computation, we can state

Proposition 10. *$(H_3, \tilde{\varphi}, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ is of class $\mathcal{C}_6$ iff*

$$\sigma(x, y, z) = -ln((x-\alpha)^2 + (z-\beta)^2 + \gamma) + \varepsilon,$$

*with $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$ and $\gamma > 0$.*

Remark 11. We remark that, from Corollary 8, for every $\sigma = \sigma(y)$ a not constant function one obtains an almost contact metric structure associated with $(H_3, H(H_3))$ belonging to $\mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$. We have also for

$$\sigma(x, y, z) = -ln\,((x-\alpha)^2 + a(z-\beta)^2 + \gamma) + \varepsilon$$

with $\alpha, \beta, \gamma, \varepsilon, a \in \mathbb{R}$ and $\gamma, a > 0$, $a \neq 1$ an almost contact metric structure belonging to $\mathcal{C}_6 \oplus \mathcal{C}_9$.

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## Abstract

*In this paper gauge transformations of almost contact metric structures associated with strongly pseudo-convex CR-structures are studied from an algebraic point of view and some examples are given.*

* * *