Tema 10. Ecuaţii diferenţiale liniare cu coeficienţi constanţi

§1. Ecuaţii omogene

Exerciţiul 1.1. *Aflaţi soluţia generală a următoarelor ecuaţii diferenţiale liniare omogene*

a) $x''' + x = 0,$ $\lambda_1 = -1, \lambda_{2,3} = \dots$

b) $x''' - x = 0,$ $\lambda_1 = 1, \lambda_{2,3} = \dots$

c) $x''' - 3x'' + 3x' - x = 0,$ $\lambda_{1,2,3} = 1$

d) $x^{\mathrm{iv}} + 8x'' + 16x = 0,$ $\lambda_{1,2} = \lambda_{3,4} = \pm 2i$

e) $x^{\mathrm{iv}} - 6x'' + 9x = 0,$ $\lambda_{1,2} = \lambda_{3,4} = \pm\sqrt{3};$

f) $x^{\mathrm{iv}} + a^2 x'' = 0, \quad a \in \mathbb{R},$ $\lambda_1 = \lambda_2 = 0, \lambda_{3,4} = \pm ai$

g) $x^{\mathrm{iv}} - a^4 x = 0, \quad a \in \mathbb{R},$ $\lambda_1 = a, \lambda_2 = -a, \lambda_{3,4} = \pm ai$

h) $x^{\mathrm{iv}} + 2x''' + 4x'' - 2x' - 5x = 0,$ $\lambda_1 = 1, \lambda_2 = -1, \lambda_{3,4} = -1 \pm 2i$

i) $x^{\mathrm{v}} + 4x^{\mathrm{iv}} + 5x''' - 6x' - 4x = 0,$ $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2, \lambda_{4,5} = -1 \pm i$

j) $x^{\mathrm{v}} = 0,$ $\lambda_{1,\dots,5} = 0$

k) $x^{\mathrm{v}} - x = 0,$ $\lambda_k = \dots$

l) $x^{\mathrm{v}} + x = 0,$ $\lambda_k = \dots$

m) $x''' - 7x'' + 14x' - 8x = 0,$ $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 4$

n) $x^{(6)} - x^{(5)} - 4x^{(4)} + 2x''' + 5x'' - x' - 2x = 0,$ $\lambda_{1,2} = 1, \lambda_{3,4,5} = -1, \lambda_6 = 2$

o) $x''' - 6x'' + 11x' - 6x = 0,$ $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

p) $x''' - 3x'' + 9x' + 13x = 0,$ $\lambda_1 = -1, \lambda_{2,3} = 2 \pm 3i$

r) $x^{(7)} - x^{(6)} + x^{(5)} - x^{(4)} = 0,$ $\lambda_{1,2,3,4} = 0, \lambda_5 = 1, \lambda_{6,7} = \pm i$

s) $x^{\mathrm{iv}} + 2x''' + 3x'' + 2x' + x = 0,$ $\lambda_{1,2} = \lambda_{3,4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

§2. Ecuaţii neomogene

Exerciţiul 2.1. *Aflaţi soluţia generală a următoarelor ecuaţii diferenţiale liniare neomogene*:

a) $x^{\text{iv}} - 2x''' + x'' = t + e^t,$ $\qquad \lambda_{1,2} = 0, \lambda_{3,4} = 1, \tilde{x}_1 = \dfrac{t^3}{6} + t^2, \tilde{x}_2 = \dfrac{t^2}{2}e^t$

b) $x''' - x = t^3 - 1,$ $\qquad \lambda_1 = 1, \lambda_{3,4} = \dots, \tilde{x} = -t^3 - 5$

c) $x^{\text{iv}} + x''' = \cos 4t,$ $\qquad \lambda_{1,2,3} = 0, \lambda_4 = -1, \tilde{x} = \frac{1}{1088}(4\cos 4t - \sin 4t)$

d) $x''' + x'' = t^2 + 1 + 3te^t,$ $\qquad \tilde{x}_1 = \frac{3}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{12}t, \tilde{x}_2 = e^t(\frac{3}{2}t - \frac{15}{4})$

e) $x''' + x'' + x' + x = te^t,$ $\qquad \tilde{x} = e^t(\frac{1}{4}t - \frac{3}{8})$

f) $x''' + x' = \frac{\sin t}{\cos^2 t},$ $\qquad \tilde{x} = \sec t + \cos t \ln \cos t - \operatorname{tg} t \sin t + t \sin t$

g) $x'' + x = \sin t - \cos t,$ $\qquad \tilde{x} = -\frac{1}{2}(\cos t + \sin t)$

h) $x^{\text{iv}} + 2x''' + 5x'' + 8x' + 4x = \cos t,$ $\qquad \lambda_{1,2} = -1, \lambda_{3,4} = \pm 2i, \tilde{x} = \frac{1}{6}\sin t$

i) $x'' + 4x = \frac{1}{\cos 2t},$ $\qquad \tilde{x} = \dfrac{t}{2}\sin 2t + \dfrac{1}{4}\cos 2t \ln \cos 2t$

j) $x'' + x = \operatorname{tg} t,$ $\qquad \tilde{x} = -\cos t \ln \operatorname{tg}(\frac{t}{2} + \frac{\pi}{4})$

k) $x'' - x = \dfrac{e^t}{e^t - 1},$ $\qquad \tilde{x} = -te^t - 1 - (e^t + e^{-t}) \ln |e^t - 1|$

l) $x'' - 2x' + x = \dfrac{e^t}{t^2 + 1},$ $\qquad \tilde{x} = e^t(t \operatorname{arctg} t - \frac{1}{2}\ln(t^2 + 1))$

§3. Ecuaţii de tip Euler

Exerciţiul 3.1. *Integraţi ecuaţiile următoare şi comparaţi soluţia găsită cu cea indicată:*

a) $t^2x'' + tx' - x = 0,$ $\qquad x = c_1t + c_2t^{-1}$

b) $t^2x'' + 3tx' + x = 0,$ $\qquad x = t^{-1}(c_1 + c_2 \ln t)$

c) $t^2x'' + tx' = 0,$ $\qquad x = c_1 + c_2 \ln t$

d) $t^3x''' - 3t^2x'' + 6tx' - 6x = 0,$ $\qquad x = c_1t + c_2t^2 + c_3t^3$

e) $t^2x''' - 3tx'' + 3x' = 0,$ $\qquad x = c_1 + c_2t^2 + c_3t^4$

f) $(t+1)^2x'' - 2(t+1)x' + 2x' = 0,$ $\qquad x = c_1(t+1) + c_2(t+1)^2$

g) $t^2x'' - tx' + x = 8t^3,$ $\qquad x = t(c_1 + c_2 \ln t) + 2t^3$

h) $t^2x'' - 2x = \sin \ln t,$ $\qquad x = c_1t^2 + c_2t^{-1} + \frac{1}{10}\cos \ln t - \frac{3}{10}\sin \ln t$