A new Tzitzeica hypersurface and cubic Finslerian metrics of Berwald type

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Abstract

A new hypersurface of Tzitzeica type is obtained in all three forms: parametric, implicit and explicit. To a two-parameters family of cubic Tzitzeica surfaces we associate a cubic Finsler function for which the regularity is expressed as the non-flatness of the Tzitzeica indicatrix. A natural relationship is obtained between cubic Tzitzeica surfaces and three-dimensional Berwald spaces with cubic fundamental Finsler function.

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1 Tzitzeica hypersurfaces and a new example

The first centroaffine invariant was introduced in 1907 by G. Tzitzeica in the classical theory of surfaces, [14]. Namely, if $M^2$ is such a (non-flat) surface embedded in $\mathbb{R}^3$ the Tzitzeica invariant is $Tzitzeica(M^2) = K d^4$, where $K$ is the Gaussian curvature and $d$ is the distance from the origin of $\mathbb{R}^3$ to the tangent plane in an arbitrary point of $M^2$, see also [15].

Since then, the class of surfaces with a constant $Tzitzeica$ was the subject of several fruitful research, see for example [1] and the surveys [5], [13] and [16]. For example, they are called $Tzitzeica$ surfaces by Romanian geometers, affine spheres by Blaschke and projective spheres by Wilczynski. One interesting direction of study is the generalization to higher dimension: a hypersurface $M^n$ of $\mathbb{R}^{n+1}$ was called $Tzitzeica$ hypersurface if $Tzitzeica(M^n) = \frac{K}{d^{4n+2}}$ is a real constant.
The classical examples of Tzitzeica surfaces are:
1) the quadrics with center, particularly spheres;
2) Tzitzeica himself obtains the surface $M^2_1 : xyz = 1$ which was generalized
by Calabi to $M^n_1 : x^n = \cdots \cdot x^{n+1} = 1$ in [4].

The aim of this section is to derive the $n(\geq 3)$-dimensional version of
another well-known Tzitzeica surface, [14]:

$$M^2_2 : z(x^2 + y^2) = 1. \tag{1.0}$$

The importance of this surface is pointed out by the first theorem of [6]
which is a classification of 3D affine spheres; see the Remarks 1.2 below.

Inspired by [3] we search for $M^n_2$ being a hypersurface of rotation in the
parametrical form:

$$M^n : x_1 = \frac{2u^n u_1}{\Delta}, \ldots, x^{n-1} = \frac{2u^n u^{n-1}}{\Delta}, x^n = \frac{u^n (\Delta - 2)}{\Delta}, x^{n+1} = f(u^n)$$

with $f : (0, +\infty) \to \mathbb{R}$ and $\Delta = (u^1)^2 + \ldots + (u^{n-1})^2 + 1$. We consider the
parameters $u^1, \ldots, u^{n-1} \in \mathbb{R}$ and $u^n > 0$.

For easy computations we derive an implicit equation of $M^n$. More
precisely, from the above equations we have: $(u^n)^2 = (x^1)^2 + \ldots + (x^n)^2$ and then
$M^n : x^{n+1} = f(\sqrt{(x^1)^2 + \ldots + (x^n)^2})$ i.e.:

$$M^n : F(x^1, \ldots, x^{n+1}) := f(\sqrt{(x^1)^2 + \ldots + (x^n)^2}) - x^{n+1} = 0. \tag{1.1}$$

Recall that choosing the normal $N = -\frac{\nabla F}{\parallel \nabla F \parallel}$, the Gaussian curvature of
$M^n$ is:

$$K = -\left| \begin{array}{cc} F_{ij} & F_i \\ F_j & 0 \end{array} \right| \parallel \nabla F \parallel^{n+2}.$$ 

For $p = (x^i) \in M^n$ the tangent hyperplane $T_p M$ has the equation
$F_i(p)(X^i - x^i) = 0$ and then:

$$d = \left| \frac{F_i x^i}{\nabla F} \right|.$$ 

From the last two relations the Tzitzeica condition reads:

$$\left| \begin{array}{cc} F_{ij} & F_i \\ F_j & 0 \end{array} \right| = -Tzitzeica(M^n) |F_i x^i|^{n+2}. \tag{1.2}$$
With $F$ of (1.1) we obtain that the LHS of (1.2) is \( \frac{f''(f')^{n-1}}{\delta^{\frac{n+2}{2}}} \) while the RHS of (1.2) is \(-T_{\text{zitzeica}}(M^n)|f'|\sqrt{\delta} - f|^{n+2}\), where $\delta = (x^1)^2 + ... + (x^n)^2$. Therefore, denoting $\sqrt{\delta} = t > 0$ and considering $f = f(t)$ we have:

\[
\frac{f''(f')^{n-1}}{\delta^{\frac{n+2}{2}}} = -T_{\text{zitzeica}}(M^n)t^{n-1}|tf' - f|^{n+2}
\]

for which we search $f(t) = t^a$ with $a \in \mathbb{R}$. The comparison of degrees of $t$ gives:

\[
a - 2 + (a - 1)(n - 1) = n - 1 + a(n + 2)
\]

with solution $a = -n$. Hence:

**Theorem 1.1** The hypersurface of $\mathbb{R}^{n+1}$:

\[
M^n_2 : x^{n+1}[\{(x^1)^2 + ... + (x^n)^2\}]^\frac{\bar{n}}{2} = 1
\]

is a Tzitzeica one with:

\[
T_{\text{zitzeica}}(M^n_2) = \frac{(-n)^n}{(n + 1)^{n+1}}.
\]

**Remarks 1.2**

1) If in computing $K$ we choose the opposite normal $-N$ then we obtain $-T_{\text{zitzeica}}(M^n)$. For example in [17, p. 137] is $M^n_2$ with $T_{\text{zitzeica}}(M^n_2) = -\frac{4}{\pi^n}$.

2) The main theorem of [6] gives the 4-dimensional affine locally strongly convex hypersurface: \((y^2 - z^2 - w^2)^3x^2 = 1\) as generalization of (1.0). The authors continue their study in [7] where Theorem 1 states the 5-dimensional version \((y^2 - z^2 - w^2 - t^2)^2x = 1\). It results that Tzitzeica hypersurfaces are not affine hyperspheres.

The following Matlab picture of $M^n_2$ shows for $z = k = \text{constant}$ the circles $x^2 + y^2 = \frac{1}{k}$.
The Calabi-Tzitzeica hypersurface $M^n_1 : x_1 ... x^{n+1} = 1$ has:

$$T_{\text{zitzeica}}(M^n_1) = \frac{1}{(n + 1)^{n+1}}$$  \hspace{1cm} (1.5)

and the Matlab picture of $M^2_1$ put in evidence the hyperbolas $z = \text{constant}$:

2 Cubic Finsler functions for a class of cubic Tzitzeica surfaces

The aim of this section is to provide a class of Finsler cubic-Minkowski metrics, connected to a family of 2-parameters Tzitzeica surfaces. The geo-
metric meaning of the regularity of these cubic metrics is expressed by the Tzitzeica condition \( Tzitzeica(M^2) \neq 0 \).

Firstly, we can give an unified formula for \( M^2_1 \) and \( M^2_2 \). With the transformation:

\[
\begin{align*}
  x &= \frac{1}{\sqrt{2}}(\tilde{x} + \tilde{y}) \\
  y &= \frac{1}{\sqrt{2}}(\tilde{x} - \tilde{y}) \\
  z &= \tilde{z}
\end{align*}
\]

which is a rotation of angle \( \frac{\pi}{4} \) in the \((\tilde{x}, \tilde{y})\)-plane, we have: \( M^2_1 : (\tilde{x}^2 - \tilde{y}^2)\tilde{z} = 2 \) and \( M^2_2 : (\tilde{x}^2 + \tilde{y}^2)\tilde{z} = 1 \). Let \( \varepsilon \in \{1, 2\} \), then:

\[
M^2_\varepsilon : [\tilde{x}^2 + (-1)^\varepsilon \tilde{y}^2] \tilde{z} = \frac{2}{\varepsilon}
\]

with:

\[
Tzitzeica(M^2_\varepsilon) = \frac{(-1)^{\varepsilon+1}\varepsilon^2}{27}.
\]

The above considerations leads to:

**Definition 2.1** The Tzitzeica surface \( xyz = 1 \) can be called *vertical hyperbolic-cubic Tzitzeica surface* while the surface \( z(x^2 + y^2) = 1 \) can be called *vertical elliptic-cubic Tzitzeica surface*.

A first natural problem is to search about a parabolic version of these surfaces:

**Proposition 2.2** In the class of surfaces \( M_{p, \rho} : z(y^2 - 2px) = \rho \) with \( \rho \neq 0 \) there are no Tzitzeica surfaces.

**Proof** The Tzitzeica condition for \( M_{p, \rho} \) reads:

\[
8pp(2p + \rho) = -Tzitzeica(M_{p, \rho}) \cdot [\rho + z(y^2 + 1)]
\]

which is impossible due to the presence of \( z \) and \( y \). \( \square \)

A second natural problem is about the general class:

\[
M^2_{\alpha, \beta, \gamma} : z(\alpha x^2 + \beta y^2 + \gamma z^2) = 1
\]

which is solved by:

**Proposition 2.3** The Tzitzeica surfaces of the class \( M^2_{\alpha, \beta, \gamma} \) are characterized by \( \gamma = 0 \).

**Proof** The equation (1.2) for \( M^2_{\alpha, \beta, \gamma} \) is:

\[
4\alpha \beta(3 - 12\gamma z^3) = -Tzitzeica(M^2_{\alpha, \beta, \gamma}) \cdot 3^4
\]
which yields $\gamma = 0$ and:

$$Tzitzeica(M_{a,\beta,0}^2) = -\frac{4\alpha\beta}{3^3}. \quad (2.4)$$

For $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$ we recover $M_1^2$ while for $\alpha = \beta = 1$ it results $M_2^2$. \quad \square

Inspired by [2, p. 23] who associates to $M_n^1$ the Berwald-Moór Finslerian function $F(y) = \sqrt[3]{y^1...y^n}$ we study in this section a Finsler metric naturally associated to $M_{a,\beta,0}^2$. Let us consider the 3-dimensional manifold $\mathbb{R}^3$ with global coordinates $x = (x^1, x^2, x^3)$ and for $x \in \mathbb{R}^3$ denote on the tangent space $T_x\mathbb{R}^3$ the coordinates $(y^1, y^2, y^3)$.

In the tangent bundle $T\mathbb{R}^3$ let us fix the domain

$$D = \{(y^1, y^2, y^3); [\alpha(y^1)^2 + \beta(y^2)^2]y^3 > 0\}$$

and the function

$$F(y^1, y^2, y^3) = \frac{3}{\sqrt[3]{[\alpha(y^1)^2 + \beta(y^2)^2]y^3}}. \quad (2.5)$$

$F$ is a Finsler fundamental function of cubic-Minkowski type having as indicatrix $F = 1$ the Tzitzeica surface provided by the above Proposition.

For $F$ we use the theory from [10] and [11], so we put the expression $F = \sqrt[3]{a_{i_1...i_3}y^{i_1}y^{i_2}y^{i_3}}$ with:

$$\begin{align*}
a_{113} &= a_{131} = a_{311} = \frac{\alpha}{3} \\
a_{223} &= a_{232} = a_{322} = \frac{\beta}{3}.
\end{align*} \quad (2.6)$$

The Matsumoto-Numata tensors:

$$F^2 \cdot a_i = a_{ijk}y^jy^k, \quad F \cdot a_{ij} = a_{ijk}y^k$$

are for our example (2.5):

$$\begin{align*}
a_1 &= \frac{2\alpha y^1y^3}{3F^2}, a_2 = \frac{2\beta y^2y^3}{3F^2}, a_3 = \frac{\alpha(y^1)^2 + \beta(y^2)^2}{3F^2} = \frac{F}{3y^3} \\
a_{11} &= \frac{\alpha y^3}{3F}, a_{22} = \frac{\beta y^3}{3F}, a_{33} = a_{12} = 0, a_{13} = \frac{\alpha y^1}{3F}, a_{23} = \frac{\beta y^2}{3F}.
\end{align*} \quad (2.7)$$

From [11, p. 94] the Finsler metric $g_{ij}$ is:

$$g_{ij} = (3 - 1)a_{ij} - (3 - 2)a_i a_j \quad (2.9)$$

which yields:

$$\begin{align*}
g_{11} &= \frac{2\alpha(y^1)^2}{3F^2} [\alpha(y^1)^2 + 3\beta(y^2)^2] \\
g_{22} &= \frac{2\beta(y^2)^2}{3F^2} [3\alpha(y^1)^2 + \beta(y^2)^2] \\
g_{33} &= -\frac{(\frac{F}{3y^3})^2}{y^1(y^1)^2 + (y^2)^2} = -\frac{(\frac{F}{3y^3})^2}{(y^1)^2 + (y^2)^2} \quad (2.10)
\end{align*}$$
\[ g_{12} = \frac{-4\alpha\beta y_1 y_2 (y_3^2)}{9F^4}; \quad g_{13} = \frac{4\alpha y_1}{9F}; \quad g_{23} = \frac{4\beta y_2}{9F} \].

(2.11)

The determinant of the basic tensor \((a_{ij})\) is:

\[ \det(a_{ij}) = -\frac{\alpha\beta}{27} = \frac{1}{4} T \text{zitzeica}(M_{a,\beta,0}) \]  

(2.11)

and then, after [11, p. 94], the function \(F\) of (2.5) is regular if and only if \(\alpha\beta \neq 0\). In other words, the regularity of \(F\) is in correspondence with the non-flatness of the Tzitzeica surface \(M_{a,\beta,0}^2 : (\alpha x^2 + \beta y^2)z = 1\).

3 A relationship between cubic Tzitzeica surfaces and cubic three-dimensional Berwald metrics

Recall that one of the most important geometrical object in Finsler geometry is the Berwald connection who lives on the tangent bundle. A simplified condition in that connection is to depend only to base coordinates, [12, p. 723], which yields:

**Definition 3.1** A Finsler manifold is a Berwald space if there exists a symmetric affine connection \(\Gamma\) such that the parallel transport with respect to this connection preserves the function \(F\).

For example, any Riemannian metric is Berwald and the associated connection is the Levi-Civita connection; therefore Berwald spaces are close to Riemannian ones. The aim of this section is to show that \(M_{1}^2\) and \(M_{2}^2\) correspond to Berwald metrics.

Let a three-dimensional manifold \(M\) with coordinates \((x^i) = (x, y, z)\) and tangent bundle coordinates \((y^i) = (p, q, r)\). In [12, p. 886] it is proved that three-dimensional Berwald spaces with cubic metric are conformally with: \(F_1 = (pqr)^{1/3}\) and \(F_2 = (p^3 + q^3 + r^3 - 3pqr)^{1/3}\), the conformal factor being an arbitrary function of \((x, y, z)\). Obviously, \(F_1\) corresponds to the Tzitzeica surface \(M_{1}^2\) from the indicatrices point of view and we perform a change of coordinates in \(M\) in order to associate \(F_2\) to \(M_{2}^2\).

This coordinates change is inspired by [12, p. 887], \((x, y, z) \rightarrow (u, v, w)\):

\[
\begin{align*}
    u &= z - \frac{x}{2} - \frac{y}{2} \\
    v &= \frac{\sqrt{3}}{2}(x - y) \\
    w &= x + y + z.
\end{align*}
\]  

(3.1)
In these new coordinates we have: $F_2(\dot{u}, \dot{v}, \dot{w}) = (\dot{u}^2 + \dot{v}^2)\dot{w}$ and we obtain the Finslerian function corresponding to $M_2^2$. Let us remark that a picture of $M_2^2$ in the coordinates $(p, q, r)$ appears in [9, p. 6] where is called Appell sphere or ternary unit sphere and as is pointed out in [8, p. 168] this surface is the ternary analogue of the circle $S^1$. In conclusion, there is a natural relationship between cubic three-dimensional Berwald spaces and cubic Tzitzeica surfaces.

We close with another natural problem, namely to find cubic Tzitzeica surfaces inspired by the expression of $F_2$:

\[ M_{c_1,c_2,c_3,b}^2 : c_1x^3 + c_2y^3 + c_3z^3 - 6bxyz = 1. \]  

(3.2)

**Proposition 3.2** The Tzitzeica surfaces of the class $M_{c_1,c_2,c_3,b}^2$ are characterized by $c_1c_2c_3 = 8b^3$.

**Proof** A straightforward computation gives:

\[ 3^4 \cdot 4[b^2 + (8b^3 - c_1c_2c_3)xyz] = -Tzitzeica(M_{c_1,c_2,c_3,b}^2) \cdot 3^4 \]

which gives the conclusion and:

\[ Tzitzeica(M_{c_1,c_2,c_3,b}^2) = -(2b)^2. \]  

(3.3)

We get $M_{2b,2b,2b,b}^2 : 2b(x^3 + y^3 + z^3 - 3xyz) = 1$ and then for $b = \frac{1}{2}$ we recover $M_2^2$ in the $(p, q, r)$-expression above. \[ \square \]

If the transformation (3.1) is performed for $F_1$ we get $F_1(\dot{u}, \dot{v}, \dot{w}) = \frac{1}{27}(2\dot{u}^3 + \dot{v}^3 - 3\dot{u}^2\dot{w} - 6\dot{u}\dot{v}^2 - 3\dot{v}^2\dot{w})$; then, with a factor of $\frac{1}{3}$ we obtain the Tzitzeica surface:

\[ \widetilde{M}_1^2 : 2x^3 + z^3 - 3x^2z - 6xy^2 - 3y^2z = 1 \]  

(3.4)

with:

\[ Tzitzeica(\widetilde{M}_1^2) = 4. \]  

(3.5)

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8
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