**Scientific Report on the implementation of the project**
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# Qualitative properties of the solution set of differential inclusions

In the frame of the present project the following activities took place: documentation and research, preparation and writing scientific articles, published in ISI journals, participation at national/international conferences. The proposed objectives were complitely realized. We mention that in 2014 there have been published or accepted for publishing **6 papers in ISI journals**.

The research objective for this year was: *To get new Filippov-Plis type results with applications in invariance, near viability and relaxation problems.* This research objective was realized. In the following we give a presentation of the main results obtained.

Let $X$ be a Banach space, $A : D(A) \subseteq X \rightsquigarrow X$ an m-dissipative operator, generator of a semigroup (possibly nonlinear) $\left\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\right\}$ and $F : [0,T] \times X \rightsquigarrow X$ a nonempty valued multifunction. For $\xi \in \overline{D(A)}$ we considered the Cauchy problem

$$x'(t) \in Ax(t) + F(t, x(t)),\ x(0) = \xi. \tag{1}$$

We say that the continuous function $x : [0,T] \to \overline{D(A)}$ is a solution of the problem (1) if it is an integral solution of the problem

$$x'(t) \in Ax(t) + f(t),\ x(0) = \xi \in \overline{D(A)}, \tag{2}$$

where $f$ is a Bochner integrable function with $f(t) \in F(t, x(t))$ a.e. $t \in [0,T]$. We recall that the continuous function $x(\cdot)$ is an integral solution of the problem (2) on $[0,T]$ if, for any $u \in D(A)$, $v \in Au$, $0 \leq t \leq T$, the following inequality holds:

$$\|x(t) - u\| \leq \|\xi - u\| + \int_0^t [x(\tau) - u, f(\tau) - v]_+ \, d\tau.$$

We mention that we denoted by $[x,y]_+$ the right directional derivative of the norm in $x$ in direction $y$.

**<u>Definition</u>.** A Caratheodory function $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$, integrally bounded on bounded sets, is called Perron function if, for any $T > 0$, the null function is the only solution on $[0,T]$ of the problem

$$r'(t) = \omega(t, r(t)),\ r(0) = 0.$$

An important result on Perron function, used in our study, is the following:

**Lemma**. *Suppose that* $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a Perron function, integrally bounded and* $(a_k)_k$,$(b_k)_k \subset \mathbb{R}_+$ *are two sequences connvergent to* 0. *Let* $r_k(\cdot)$ *be a solution of the problem*

$$r'(t) = \omega(t, r(t)) + a_k,\ r(0) = b_k.$$

*Then,* $\lim_{k\to\infty} r_k(t) = 0$ *uniformly for any* $t \in [0,T]$.

**Definition.** The multifunction $F : [0,T] \times X \rightsquigarrow X$ is called *one-sided Perron* (with respect to the Perron function $\omega$) if, for any $x, y \in X$, a.e. $t \in [0,T]$ $f \in F(t,x)$, there exists $g \in F(t,y)$ such that

$$[x - y, f - g]_+ \leq \omega(t, \|x - y\|).$$

In some papers these multifunctions are called *one-sided Kamke.* The one-sided Perron condition is weaker than the Lipschitz one.

We assumed the following hypotheses:

($H_1$) $X$ is a separable Banach space and the m-disipative operator $A$ generates a compact semigroup $\left\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\right\}$.

($H_2$) The multifunction $F$ is almost lower semicontinuous, nonempty closed valued and has sublinear growth, i.e., there exist two positive Lebesgue integrable functions $a(\cdot)$ and $b(\cdot)$ such that

$$\|v\| \leq a(t) + b(t)\|z\|,$$

a.e. $t \in [0,T]$ and for any $z \in X$ and any $v \in F(t,z)$.

We first poved the following auxiliary rezultat.

**Proposition**. *Let* $G : [0,T] \times X \rightsquigarrow X$ *an almost lower semicontinuous multifunction. Then there exists a sequence* $(\Delta_m)_m$ *of compact pairwise disjoint subsets of* $[0,T]$, *with Lebesgue measure of the reunion* $\cup_m \Delta_m$ *equal to* $T$, *such that, for any* $m \in \mathbb{N}^*$, $G(\cdot,\cdot)$ *is lower semicontinuous on* $\Delta_m \times X$.

We proved the following new Filippov-Plis result, where we weaken the Lipschitz hypothesis on the multifunction $F$, imposing the one-sided Perron condition.

**Theorem**. *Assume* ($H_1$) *and* ($H_2$). *Moreover, assume that* $F : [0,T] \times X \rightsquigarrow X$ *is one-sided Perron (with respect to a Perron function* $\omega$*) and* $f : [0,T] \to \mathbb{R}$ *is a positive, Lebesgue integrable function. Then, for any* $\xi \in \overline{D(A)}$, $\varepsilon > 0$ *and any solution* $x(\cdot)$ *on* $[0,T]$ *of the problem*

$$x'(t) \in Ax(t) + F(t, x(t)) + f(t)B,\ x(0) = \xi,$$

*there exists a solution of the problem* (1) *on* $[0,T]$, $y(\cdot)$, *such that*

$$\|x(t) - y(t)\| \leq r_\varepsilon(t),$$

*for any* $t \in [0,T]$, *where* $r_\varepsilon(\cdot)$ *is the maximal solution of the Cauchy problem*

$$r'(t) = \omega(t, r(t)) + f(t) + \varepsilon,\ r(0) = 0.$$

A similar result was obtained in the paper [O. Cârjă, T. Donchev, V. Postolache, *Nonlinear evolution inclusions with one-sided Perron right-hand side*, J. Dyn. Control Syst., 2013] in thecasewhen the Banach space $X$ has a uniformly convex dual. Moreover, the continuity hypothesis imposed on the multifunction $F$ is upper semicontinuity.

The relaxation problems consist in proving that the solution set of the initial problem (1) is dense in the solution set of the convexified (relaxed) problem:

$$x'(t) \in Ax(t) + \overline{co}F(t, x(t)),\ x(0) = \xi. \tag{3}$$

The existing relaxation rezults for nonlinear differential inclusions impose Lipschitz conditions on the multifunction $F$. Also, these rezults are obtained under the suplimentary hypothesis that the Banach space $X$ is reflexive.

The relaxation results that we have established within this project hold under weaker hypotheses on $F$ than the Lipschitz condition, more exactly we assume conditions of one-sided Perron type. Moreover, we do not impose the reflexivity of the Banach space $X$.

We obtained the following rezult:

**Theorem**. *Let $X$ be a separable Banach. Suppose that $(H_1)$ and $(H_2)$ hold. Moreover, suppose that there exists a Perron function $\omega(\cdot,\cdot)$ such that, for any $x, y \in X$, a.e. $t \in [0,T]$ and any $f \in \overline{co}F(t,x)$, there exists $g \in F(t,y)$ such that*

$$[x-y, f-g]_+ \leq \omega(t, \|x-y\|).$$

*Then, the solution set of the initial problem* (1) *is dense in the solution set of the problem* (3).

Then, we considered the following parabolic partial derivatives equation:

$$\begin{aligned} \frac{\partial x(t,z)}{\partial t} - \Delta x(t,z)\,|x(t,z)|^{r-1} &\in F(t,z,x(t,z)) \\ x(t,z) &= 0 \text{ pe } [0,T]\times\Gamma \\ x(0,z) &= x_0(z) \text{ în } \Omega, \end{aligned} \tag{4}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma$ and $r > (n-2)/n$. The multifunction $F : [0,T]\times\Omega\times\mathbb{R} \rightsquigarrow \mathbb{R}$ has closed values and is lower semicontinuous in the third variable. Moreover, we supposed that the multifunction $G : [0,T]\times L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by

$$G(t,y) = \left\{ f \in L^1(\Omega);\ f(z) \in F(t,z,y(z)) \text{ a.e. on } \Omega \right\}$$

is measurable on $[0,T]\times L^1(\Omega)$. We showed that the problem (4) can be written in the abstract form (1). Moreover, we supposed that there exists a Perron function $\omega(\cdot,\cdot)$ such that, for any $\varphi,\psi \in L^1(\Omega)$, a.e. $t\in[0,T]$ and any $f \in \overline{co}G(t,\varphi)$, there exists $g \in G(t,\psi)$ such that

$$\begin{aligned} &\int_{\Omega^+_{\varphi-\psi}} (f(z)-g(z))\,dz - \int_{\Omega^-_{\varphi-\psi}} (f(z)-g(z))\,dz \pm \int_{\Omega^0_{\varphi-\psi}} (f(z)-g(z))\,dz \\ &\leq \omega\left(t, \int_\Omega |\varphi(z)-\psi(z)|\,dz\right), \end{aligned}$$

where $\Omega^{+(-,0)}_{\varphi-\psi} = \{z\in\Omega; \varphi(z)-\psi(z) > (<,=)\, 0\}$. Then, the hypotheses of the previous theorem are satisfied. Let $x(\cdot,\cdot) \in C\left([0,T], L^1(\Omega)\right)$ a generalized solution on $[0,T]\times\Omega$ of the relaxed problem

$$\begin{aligned} \frac{\partial x(t,z)}{\partial t} - \Delta x(t,z)\,|x(t,z)|^{r-1} &\in \overline{co}F(t,z,x(t,z)) \\ x(t,z) &= 0 \text{ on } [0,T]\times\Gamma \\ x(0,z) &= x_0(z) \text{ in } \Omega. \end{aligned}$$

Then, for $\varepsilon > 0$ we find $x_\varepsilon(\cdot,\cdot)$ a generalized solution of the problem (4) such that

$$\sup_{t\in[0,T]}\int_\Omega |x(t,z) - x_\varepsilon(t,z)|\,dz < \varepsilon.$$

The hypothesis on the multifunction $F$ from the previous theorem is stronger than the one-sided Perron condition. In the following result we supposed that $F$ is one-sided Perron, but we imposed a stronger condition on the space $X$.

**Theorem**. *Let $X$ be a Banach space with univoc duality map $J(\cdot)$. Suppose that $(H_1)$ and $(H_2)$ hold. Moreover, suppose that $F : [0,T] \times X \rightsquigarrow X$ is one-sided Perron (with the suplimentary hypothesis that, a.e. $t \in [0,T]$, $\omega(t,\cdot)$ is nondecreasing). Then, the solution set of the initial problem* (1) *is dense in the solution set of the problem* (3).

Moreover, we considered the differential inclusion

$$D_C^q y(t) \in F(t, y(t)),\; y(t_0) = y_0,\; t \in I = [t_0, T], \tag{5}$$

where $F$ is a multifunction from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$. We denoted by $D_C^q y$ the fractional Caputo derivative of order $0 < q < 1$, i.e.

$$D_C^q y(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t} (t-s)^{-q}\,\dot{u}(s)\,ds,\; t_0 < t < T.$$

The absolut continuous function $y(\cdot)$ is called solution of the problem (5) if there is a measurable selection $f_y(t) \in F(t, y(t))$ such that for any $t \in I$ we have

$$y(t) = y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} f_y(s)\,ds.$$

Consider the following hypotheses on $F$ :

(H) $F : I \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is upper semicontinuous with nonempty convex compact values. Moreover, there exists a constant $\alpha$ such that

$$|F(t,x)| \le \alpha(1 + |x|),\; \forall (t,x) \in I \times \mathbb{R}^n.$$

Let $K \subset \mathbb{R}^n$ be a closed set and let $y_0 \in K$. We studied the viability problem, i.e. the problem of existence of a solution $y(\cdot)$ for (5) such that $y(t) \in K$ for any $t \in I$.

**Definition.** Let $\varepsilon > 0$. An absolutely continuous function $y(\cdot)$ is called $\varepsilon-$*solution* for (5) if there exists a measurable function $g : I \to \mathbb{R}^n$ and a nondecreasing function $\sigma : I \to I$ such that

a) $|g(t)| \le \varepsilon$ for a.a. $t \in I$;

b) $t - \varepsilon \le \sigma(t) \le t$, $\forall t \in I$;

c) $y(\sigma(t)) \in K$, $\forall t \in I$;

d) there exists a measurable selection $f_y(t) \in F(\sigma(t), y(t)) + g(t)B$ such that for any $t \in I$,

$$y(t) = y_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t} (t-s)^{q-1} f_y(s)\,ds.$$

**Definition.** We say that the pair $(g, E)$ is *tangent to $I \times K$ in* $(\bar{t}, \overline{y}) \in I \times K$ if $y(\bar{t}) = \overline{y}$ and

$$\lim\inf_{h\to 0^+} h^{-q} d\left(\overline{y} + r(\bar{t}, g(\cdot))(h) + \frac{h^q}{\Gamma(q+1)}E, K\right) = 0.$$

We obtained the following rezult:

**Theorem**. *Let* $t_0 \leq \bar{t} < T$ *and let* $y(\cdot)$ *an absolutely continuous function on* $[t_0, \bar{t}]$ *with* $y(t_0) = y_0$. *If the pair* $\left(D_C^q y(\cdot), F(\bar{t}, y(\bar{t}))\right)$ *is tangent to* $I \times K$ *in* $(\bar{t}, \bar{y}) \in I \times K$ *then, for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *and an extension* $z(\cdot)$ *of* $y(\cdot)$ *which is* $\varepsilon$*-solution of* (5) *on* $[\bar{t}, \bar{t} + \delta]$ *and such that* $z(\bar{t} + \delta) \in K$.

**Definition.** We say that the system (5) satisfies the tangency condition in $(\bar{t}, \bar{y}) \in I \times K$ if for any $\varepsilon > 0$ there exista an $\varepsilon$-solution $y(\cdot)$ on $[t_0, \bar{t}]$ such that the pair $\left(D_C^q y(\cdot), F(\bar{t}, \bar{y})\right)$ to be tangent to $I \times K$ in $(\bar{t}, \bar{y})$.

**Theorem**. *Suppose that (H) holds. If the system* (5) *satisfies the tangency condition in any* $(\bar{t}, \bar{y}) \in I \times K$, *then it has a viable solution.*

In **2014** there have been published or accepted for publications **6 articles in ISIjournals**, articles financially supported by the project, confirmed by the corresponding text from the Acknowledgement section.

1. T. Donchev, A. Nosheen, Fuzzy differential equations under dissipative and compactness type conditions, Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 47, pp. 1-9. (ISI, IF=0.419).
2. E. Farkhi, T. Donchev, R. Baier, Existence of solutions for nonconvex differential inclusions of monotone type, C.R. Acad. Bulg. Sci., Vol. 67 (2014), No. 3, pp. 323-330. (IS, IF=0.198, SRI=0.051).
3. T. Donchev, A. Nosheen, V. Lupulescu, Fuzzy integro-differential equations with compactness type conditions, Hacettepe Journal of Mathematics and Statistics, Vol. 43 (2014), No. 2, pp. 249-257. (ISI, IF=0.433, SRI=0.222).
4. O. Cârjă, T. Donchev, V. Postolache, Relaxation results for nonlinear evolution inclusions with one-sided Perron right-hand side, Set-Valued Var. Anal., Vol. 22 (2014), No. 4, pp. 657-671. (ISI, IF=0.918, SRI=1.368).
5. O. Cârjă, T. Donchev, M. Rafaqat, R. Ahmed, Viability of fractional differential inclusions, Applied Mathematics Letters, 2014, 38, pp. 48-51. (ISI, IF=1.48, SRI=0.853).
6. T. Donchev, A. Nosheen, Value function and optimal control of differential inclusions, Annals of the Alexandru Ioan Cuza University - Mathematics, DOI: 10.2478/aicu-2014-0031, 2014. (ISI, IF=0.108, SRI=0.054).

IF=Impact factorul (2013) according to Web of Knowledge.

SRI=Influence score according to
http://uefiscdi.gov.ro/userfiles/file/CENAPOSS/Scor_Relativ_Influenta_2014.pdf

In order to achieve the objectives of the project, **reserach stages** have been performed to University of Caliornia Berkeley, USA, period May 18 - June 2, 2014, to Univesity of Architecture, Civil Engineering and Geodesy, Sofia, Bulgaria, period July 7-18, 2014, to IMAR Bucuresti, Romanian Academy, period August 7-10, 2014, to University of Crete, Greece, period August 30- September 13, 2014, to Technische Universitat Wien, Austria, period: September 22 - October 9, 2014. Also, V. Postolache participated to *NetCO 2014-Conference on "New Trends in Optimal Control"*, Tours, period June 23-27, 2014.

The research results have been presented to the following conferences: *The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, Madrid, July 7-11, 2014 (A. I. Lazu, Estimates for large time controls), *Scientific Session of Comunications, Faculty of de Mathematics,* "Al. I. Cuza" University, Iaşi, October 24, 2014 (A. I. Lazu, O. Cârjă, Estimates for slow controls), *Scientific Session of Comunications, "O. Mayer" Mathematical Institute,* Iaşi, October 2014 (O. Cârjă, On the equivalence between the minimum time problem and the minimum norm problem).

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