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Qualitative properties of the solution set of differential inclusions

In the frame of the present project the following activities took place: documentation and research, preparation and writing scientific articles, published in ISI and BDI journals, participation at national/international conferences. The proposed objectives were completely realized. We mention that in 2015 **3 papers** have been published and other 2 papers are submitted to ISI journals.

The research objective for this year was: *The study of invariance problems for impulsive and fractional differential inclusions in Banach spaces.* This research objective was realized. In the following we give a presentation of the main results obtained.

Let X be a real Banach space and $I = [a, b)$, where $a < b \leq +\infty$. Let $F : [a, b) \times X \rightsquigarrow X$ be a given multifunction. As usual, an exact solution of

$$x'(t) \in F(t, x(t)) \tag{1}$$

on $[\tau, T] \subset I$ is a function $x : [\tau, T] \to X$ a.e. differentiable on $[\tau, T]$ with $x' \in L^1(\tau, T; X)$, such that for each $t \in [\tau, T]$ we have

$$x(t) = x(\tau) + \int_\tau^t x'(s)\, ds,$$

which satisfies (1) a.e. on $[\tau, T]$. Let $K : I \rightsquigarrow X$ be a multifunction. We denote by $\mathcal{K}$ the graph of K, i.e.,

$$\mathcal{K} = \{(t, x); t \in I, x \in K(t)\}.$$

Roughly speaking, the classical concept of weak invariance for $\mathcal{K}$ with respect to (1) requires that for each initial data (τ, ξ) in $\mathcal{K}$ there exists a solution of (1) starting from ξ at $t = \tau$ and such that its graph remains in $\mathcal{K}$ at least for a short time. A more general concept, called approximate weak invariance, was introduced by Clarke, Ledyaev, Radulescu (J. Dyn. Control Syst. 1997) in Hilbert spaces for the autonomous case $x'(t) \in F(x(t))$ with K independent of t. It involves ε-solutions, which are in fact solutions of $x'(t) \in F(x(t) + \varepsilon B)$, where B is the closed unit ball. Approximate weak invariance requires the existence of ε-solution for which

$$dist\,(x(t); K) \leq \varepsilon,$$

for t in some interval. It is interesting to note that the approximate weak invariance is equivalent to a tangency condition under very general assumptions on F. Adding natural convexity and compactness assumptions on F, passing to limit when $\varepsilon \to 0$, we get exact weak invariance.

Definition 1. A function $x : [\tau, T] \to X$ is called an ε-solution of (1) if it is a solution of the differential inclusion

$$x'(t) \in F(t, x(t) + \varepsilon B)$$

on $[\tau, T]$.

Definition 2. We say that $\mathcal{K}$ is approximate weakly invariant with respect to (1) if for any $(\tau,\xi)\in\mathcal{K}$ there exists $T>\tau$ such that $[\tau,T]\subset I$ and for any $\varepsilon>0$ there exist a function $\sigma:[\tau,T]\to[\tau,T]$ satisfying $t-\varepsilon\le\sigma(t)\le t$ for each $t\in[\tau,T]$ and an ε-solution $x:[\tau,T]\to X$ of (1) with $x(\tau)=\xi$ and $\mathrm{dist}(x(t),K(\sigma(t)))\le\varepsilon$, for each $t\in[\tau,T]$.

The basic tangential hypothesis we shall refer to in the sequel is the following. Let $(\tau,\xi)\in\mathcal{K}$.
(H1) $F(\cdot,\xi)$ is quasi-tangent to $\mathcal{K}$ at (τ,ξ), i.e.,

$$\lim_{h\to0^+}\inf\frac{1}{h}dist\left(\xi+\int_\tau^{\tau+h}F(s,\xi)\,ds;K(\tau+h)\right)=0.$$

The next result gives sufficient conditions for a locally closed from the left graph $\mathcal{K}$ to be approximate weakly invariant with respect to (1).

Theorem 1. *Let K be locally closed from the left and F an integrally bounded mulifunction. If (H1) is satisfied for each $(\tau,\xi)\in K$, then K is approximate weakly invariant with respect to* (1).

The next theorem is a variant of the previous one when the graph $\mathcal{K}$ is X-closed and F satisfies a sublinear growth condition.

Theorem 2. *Let K be X-closed and F a multifunction satisfying the sublinear growth condition: there exists $c\in L^1(I,\mathbb{R}_+)$ such that*

$$F(t,x)\subset c(t)(1+\|x\|)B,$$

for each $x\in X$ and a.e. $t\in I$. Assume that (H1) is satisfied for each $(\tau,\xi)\in K$. Then, for any $(\tau,\xi)\in K$, $T>\tau$ with $[\tau,T]\subset I$ and any $\varepsilon>0$ there exist a nondecreasing function $\sigma:[\tau,T]\to[\tau,T]$ such that $t-\varepsilon\le\sigma(t)\le t$ for each $t\in[\tau,T]$ and an ε-solution $x:[\tau,T]\to X$ of (1) *with $x(\tau)=\xi$ satisfying*

$$dist(x(t),K(\sigma(t)))\le\varepsilon,$$

for each $t\in[\tau,T]$.

Theorem 3. *Let X be a separable Banach space. Let K be locally closed from the left and F an integrally bounded multifunction with nonempty and closed values. Assume that for each $(\tau,\xi)\in K$ $F(\cdot,\xi)$ is $\varepsilon-\delta$ l.s.c. at τ. If $F(\tau,\xi)$ is quasi-tangent to K at (τ,ξ) for each $(\tau,\xi)\in K$, then K is approximate weakly invariant with respect to* (1).

For certain class of Caratheodory multifunctions we showed that a necessary condition for approximate weak invariance for $\mathcal{K}$ is that (H1) be satisfied a.e. for $\tau\in I$ and for all $\xi\in K(\tau)$. To this end we considered the following assumptions.

(H2) For each $x\in X$ the multifunction $F(\cdot,x)$ is measurable on I.

(H3) There exists $l\in L^1(I,\mathbb{R}_+)$ such that for each $(\tau,\xi)\in\mathcal{K}$ there exist a nondecreasing function $W:\mathbb{R}_+\to\mathbb{R}_+$ continuous in 0 with $W(0)=0$ and a bounded open set $\Omega\subset X$ containing ξ, such that

$$F(t,x)\subset F(t,\xi)+l(t)W(\|x-\xi\|)B,$$

for each $x\in\Omega$ and a.e. $t\in I$.

(H4) The multifunction $K:I\rightsquigarrow X$ is Lipschitz.

Theorem 4. *Let X be a separable Banach space and F be a multifunction with nonempty and closed values satisfying (H2), (H3) and the sublinear growth condition. Let K satisfying (H4). If K is approximate weakly invariant with respect to* (1) *then (H1) is satisfied a.e. for $\tau\in I$ and for all $\xi\in K(\tau)$.*

Corollary. *Let X be a separable Banach space. Let K be a locally closed from the left graph and F a multifunction with nonempty and closed values satisfying (H2), (H3) (l is supposed continuous) and the sublinear growth condition. Let K satisfying (H4). A necessary and sufficient condition in order that K be approximate weakly invariant with respect to* (1) *is that (H1) be satisfied for each* $(\tau, \xi) \in K$.

Many real problems arising in population dynamics, economy and nanoelectronics are described by impulsive differential equations (IDEs). Let us consider the IDE of the form:

$$x'(t) = f(t, x(t)), \ t \in I = [0,1], \ \Phi_i(t,x) \neq 0, \ x(0) = x_0, \tag{2}$$

$$\Delta x|_{\Phi_i(t,x)=0} = S_i(t,x)\,|_{\Phi_i(t,x)=0}, \ i = 1, ..., r. \tag{3}$$

Here $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function, $\Phi_i : I \times \mathbb{R}^n \to \mathbb{R}$ and $S_i : I \times \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous switching surfaces and jump functions respectively.

Recall that the piecewise continuously differentiable function $x(\cdot)$ is said to be a solution of (2) if:
A) $x(\cdot)$ is right continuously differentiable and satisfies (2) for all $t \in I$ for which $\Phi_i(t, x(t)) \neq 0$, $i = 1, ..., r$.
B) $x(\bar{t}) = x(\bar{t} - 0) + S_i(\bar{t}, x(\bar{t} - 0))$ for every $\bar{t} \in I$ for which $\Phi_i(\bar{t}, x(\bar{t} - 0)) = 0$.

The points $\bar{t}$ will be called further events. One of the most important problem regarding $(2) - (3)$ is to determine the events of the solutions.

We showed that under weak condition the Runge-Kutta methods can be successfully adapted to be applied in case of $(2) - (3)$. We will call the solutions obtained via Runge-Kutta method "approximate solutions".

We suppose that $f(\cdot, \cdot)$ is sufficiently smooth (hence it is locally Lipschitz) and it satisfies growth condition, i.e., there exists a continuous function $v : I \times \mathbb{R}^+ \to \mathbb{R}^+$ with $|f(t,x)| \leq v(t, |x|)$ such that the maximal solution of $\dot{r} = v(t, r)$ exists on I for any initial condition $r(0) \geq 0$. Suppose that:

A1. $\Phi_i(\cdot, \cdot)$ and $S_i(\cdot, \cdot)$ are Lipschitz with constants M and μ respectively.

A2. $\Phi_i(t, x + S_i(t,x)) \neq \Phi_j(t,x), \forall j \neq i, \forall x \in \mathbb{R}^n$ and $\forall t \in I$.

We assume further that either A3 or A4 holds.

A3. $\Phi_i(t,x) < \Phi_{i+1}(t,x)$ for every $(t,x) \in I \times \mathbb{R}^n$ and the following two conditions are satisfied:
1) there exists a constant $\alpha < 0$ such that

$$\frac{\partial \Phi_i}{\partial t} + \left\langle \frac{\partial \Phi_i}{\partial x}, f(t,x) \right\rangle \leq \alpha$$

for $i = 1, ..., r$ and for every $(t,x) \in I \times \mathbb{R}^n$, where the derivatives exists. Here $\frac{\partial \Phi_i}{\partial x}$ is the gradient on $\Phi_i(t, \cdot)$;
2) $\Phi_i(t,x) \geq \Phi_i(t, x + S_i(t,x)), \forall (t,x) \in I \times \mathbb{R}^n$.

A4. $\Phi_i(t,x) > \Phi_{i+1}(t,x)$ for every $(t,x) \in I \times \mathbb{R}^n$ and the following two conditions are satisfied:
1) there exists a constant $\beta > 0$ such that

$$\frac{\partial \Phi_i}{\partial t} + \left\langle \frac{\partial \Phi_i}{\partial x}, f(t,x) \right\rangle \geq \beta$$

for $i = 1, ..., r$ and for every $(t,x) \in I \times \mathbb{R}^n$, where the derivatives exists.
2) $\Phi_i(t,x) \leq \Phi_i(t, x + S_i(t,x)), \forall (t,x) \in I \times \mathbb{R}^n$.

One can study also hybrid systems, where the right-hand side changes after events, i.e.,

$$x'(t) = f_i(t, x(t)), \ t \in [\tau_i, \tau_{i+1}), \text{ after}$$

$$\Phi_i(t_i, x(t_i - 0)) = 0, \ x(0) = x_0.$$

$f_0(t, x) = f(t, x)$ in (2). Such problems arise in the control of real life processes. Our method is also useful for such systems.

The beating phenomena occurs if a solution of $(2)-(3)$ hits some switching surface (finite or countably) times. In order to prevent beating phenomena, many authors assume that $\Phi_i(\cdot,\cdot)$ is continuously differentiable and, moreover, $\frac{\partial \Phi_i}{\partial t} + \left\langle \frac{\partial \Phi_i}{\partial x}, f(t, x) \right\rangle \neq 0$. If these conditions hold, then in any compact connected set A3 or A4 holds. We show that if the standing hypotheses hold true, then this phenomena is impossible.

Lemma 1. *Let A1, A2 and A3 or A4 hold and let $x(\cdot)$ be a solution of $(2)-(3)$. Then every equation $\Phi_i(t, x(t)) = 0$ w.r.t. the time t admits no more than one solution, where $i = 1, ..., r$.*

Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $r(\cdot)$ be a function such that $g(\tau) = r(p) = 0$ $(\tau, p \in \mathbb{R})$ and $|g(t) - r(t)| \leq \varepsilon$ for some $\varepsilon > 0$.

Proposition 1. *Let $\gamma > 0$. If either $(g(t) - g(s))(t - s) \leq -\gamma (t - s)^2$, or $(t - s)(g(t) - g(s)) \geq \gamma (t - s)^2$ $\forall t, s \in \mathbb{R}$, then $|\tau - p| \leq \frac{\varepsilon}{\gamma}$.*

Consider the system (2). Let $x(\cdot)$ be a solution and $y(\cdot)$ be a continuous function such that $|x(t) - y(t)| \leq \varepsilon$. Since $\Phi_i(\cdot,\cdot)$ is $M-$Lipschitz, one has that $|\Phi_i(t, x(t)) - \Phi_i(t, y(t))| \leq M\varepsilon$. For $\tau, s \in (a, b) \subset I$ if $\Phi_i(\tau, x(\tau)) = \Phi_i(s, y(s)) = 0$, then $|\tau - s| \leq \frac{M\varepsilon}{\gamma}$, where $\gamma = \min\{|\alpha|, \beta\}$. Taking into account Proposition 1, one can conclude that the error after events may increase rapidly.

Notice that due to the growth condition we may assume that $|f(t, x)| \leq K$, i.e., every exact solution is piecewise $K-$ Lipschitz.

It follows that A3 or A4 needs to be satisfied only in some $\varepsilon-$neighborhood of the set $\mathcal{A} := \{(t, x); \Phi_i(t, x) = 0 \text{ for some } i = 1, ..., r\}$. However, in this case beating phenomena is possible.

Further, we will denote the events of function $x(\cdot)$ by $\tau_i(x)$.

Proposition 2. *Under the conditions of Lemma 1, there exists a constant $\lambda > 0$ such that the solution $x(\cdot)$ of (2) satisfies $\tau_{i+1}(x(t)) - \tau_i(x(t)) \geq \lambda$ for every $t \in [0, 1]$ and $i = 1, 2, ..., r - 1$.*

We study now discrete approximations of (2) with Runge-Kutta scheme. Given a natural number N, set $h = \frac{1}{N}$ and let $t_j = jh$ be a uniform grid on $[0, 1]$, where $j = 0, 1, ..., N$. An s-stage RK method computes iteratively the solution of (2) without impulses using the following relation

$$\eta_h(t_{j+1}) = \eta_h(t_j) + h \sum_{\nu=1}^{s} b_\nu k_\nu, \tag{3}$$

$$k_\nu = f\left(t_j + c_\nu h, \eta_h(t_j) + h \sum_{l=1}^{\nu} a_{\nu l} k_l\right), \ \nu = 1, ..., s. \tag{4}$$

Set

$$\varphi_i(t) = \Phi_i(t, x(t)), \ t \in [0, 1],$$
$$\varphi_{i,h}(t_j) = \Phi_i(t_j, \eta_h(t_j)), \ j = 0, 1, ..., N.$$

We calculate approximations with the RK method for the differential system (2) for subsequent grid points $t_j,\ j = 1, ..., N$. Due to our assumptions, every approximate solution $y(\cdot)$ fetches $i+1$-th switching surface after i-th. For convenience, we denote $\tau_0 = t_0$ (the zero event). We study the system $(2)-(3)$ on $[\tau_{i-1}, 1]$ and associate to it the equation

$$\dot{\varphi}_i(t) = \frac{\partial \Phi_i}{\partial t}(t, x) + \langle \nabla_x \Phi_i, f(t, x) \rangle,\ \varphi_i(\tau_{i-1}) = \overline{\varphi}.$$

Clearly, $\varphi_i(t) = \Phi_i(t, y(t))$. In the case when A3 (A4) holds, we start with minimal i for which $\Phi_i(\tau_0, x(\tau_0)) > 0\ (< 0)$.

If $\varphi_i(t_j)\varphi_i(t_{j+1}) < 0$, then $\tau_i(y) \in (t_j, t_{j+1})$. We can use several polynomial extensions of $\varphi_i(\cdot)$ and solve $\varphi_i(t) = 0$ to find approximation $\overline{\tau}$ of $\tau_i(y)$. Afterward, we can calculate the value of $\varphi_i(\overline{\tau})$ with the RK method starting from $\overline{\varphi}$. If it is large, then use the same or other piecewise polynomial extension. These extensions also depend on the smoothness of $\Phi_i(\cdot, \cdot)$.

After calculating the approximate event with one strategy τ_i^1, we evaluate $\varphi_{i,h}(\tau_i^1)$ and verify in which subinterval $\varphi_{i,h}(\cdot)$ changes its sign. Then continue. The approximate events should be included in the grid points.

Let the error of the RK method be $O(h^p)$. This motivates the stopping criterion resulting in the common estimate $O(h^q)$ for some $q > p$.

If we find the approximate $\tau_i(y)$, then we study $(2)-(3)$ on $[\tau_i(y), 1]$ and analogously start with the minimal $k > i$ for which $\Phi_k(\tau_i, y(\tau_i)) > 0\ (< 0)$ in case A3 (A4).

We say that two solutions $x(\cdot)$ and $y(\cdot)$ of $(2)-(3)$ are at distance $\rho(x(\cdot), y(\cdot)) \leq \varepsilon$ if they intersect successively the impulsive surfaces, i.e., $\tau_i(x) < \tau_{i+1}(y)$ or vice versa. Moreover,

$$\sum_{i=1}^{r} \left(\tau_i^+ - \tau_i^-\right) < \varepsilon$$

and $|x(t) - y(t)| < \varepsilon$ for every $t \in I \backslash \left(\cup_{i=1}^r \left[\tau_i^-, \tau_i^+\right]\right)$. Here $\tau_i^- = \min\{\tau_i(x), \tau_i(y)\}$ and $\tau_i^+ = \max\{\tau_i(x), \tau_i(y)\}$. We say that the RK admits $O(h^p)$ order of approximation if the exact solution $x(\cdot)$ and the appropriate solution $y(\cdot)$ obtained via the RK method are at distance $O(h^p)$.

Theorem 1. *Under the standing hypotheses, the measure of distance between the exact solution $x(\cdot)$ ans the approximate solution $\eta_h(\cdot)$ is $O(h^p)$ for very small step size.*

Moreover, we considered the fractional differential inclusion

$$D_C^q x(t) \in F(t, x(t)),\ x(t_0) = x_0,\ t \in I = [t_0, T], \tag{4}$$

where F is a multifunction from $I \times E$ to E, where E is the space of fuzzy sets. We denoted by D_C^q the fractional Caputo derivative of order $0 < q < 1$, i.e.

$$D_C^q x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \dot{x}(s)\, ds,\ t_0 < t < T.$$

The solutions of (4) can be defined as the solutions of the following fractional integral inclusion

$$x(t) \in x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s))\, ds,\ t \in I. \tag{5}$$

More exactly, $x(\cdot)$ is a solution of (5) if there is a strongly measurable selection $f(t) \in F(t, x(t))$ such that for any $t \in I$ we have

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s)\, ds.$$

We studied the properties of the solution set of (5). To this end we first established the following result regarding the fractional integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, w(s))\, ds, \ t \in I. \tag{6}$$

Theorem. *Let* $\eta > 0$ *be such that* $D\left(f(t,x), \widehat{0}\right) \le \eta \left(1 + D\left(x, \widehat{0}\right)\right)$. *Assume that* $f(\cdot, x)$ *is strongly measurable, while* $f(t, \cdot)$ *is locally Lipschitz, i.e. for every* $x \in E$ *there exists a neighborhood* U_x *and a constant* L_x *such that* $D(f(t,y), f(t,z)) \le L_x D(y,z)$ *for every* $y, z \in U_x$. *Then the differential equation* (6) *admits an unique solution on the interval* I, *which depends continuously on* x_0.

Proposition. *Suppose that there exists a positive constant* $\lambda > 0$ *such that*

$$\max_{v \in F(t,x)} D\left(v, \widehat{0}\right) \le \lambda \left(1 + D\left(x, \widehat{0}\right)\right).$$

Then, the solution set of

$$D_C^q x(t) \in \overline{co} F(t, x(t) + B), \ x(t_0) = x_0$$

is Holderian of degree q (if nonempty).

We considered the following standing hypotheses on F :

(H1.) $F(\cdot, \cdot)$ is almost upper semicontinuous with nonempty convex compact values and there exists a constant $\lambda > 0$ such that

$$\max_{v \in F(t,x)} D\left(v, \widehat{0}\right) \le \lambda \left(1 + D\left(x, \widehat{0}\right)\right).$$

(H2.) $F(\cdot, \cdot)$ satisfies the compactness type condition, i.e., there exists a Perron function $w(\cdot, \cdot)$ such that $\beta(F(t, A)) \le \frac{1}{2} w(t, \beta(A))$ for any bounded set $A \subset E$.

We denoted by β the Hausdorff measure of noncompactness.

Definition.

a) The set $Y \subset E$ is said to be connected if for any two open disjoint sets O_1, O_2 such that $Y \subset O_1 \cup O_2$ either $Y \bigcap O_1$ or $Y \bigcap O_2$ is empty.

b) The set Y is contractible if there exists a point $a \in Y$ and a homotopy $H : E \times [0,1] \to E$ such that $H(\cdot, \cdot)$ is continuous, $H(x, 0) = x$ and $H(x, 1) = a$ for all $x \in E$.

Theorem. *Under the standing hypotheses, the solution set of* (5) *is nonempty compact and connected.*

This theorem can be reformulated as follows.

Theorem. *Under the standing hypotheses, the solution set of* (4) *is nonempty compact and connected.*

In **2015** there have been published **3 articles (one in ISI journal)**, articles financially supported by the project, confirmed by the corresponding text from the Acknowledgement section. Also, there have been submitted two papers in ISI journals.

1. Q. Din, T. Donchev, A. Nosheen, M. Rafaqat, Runge-Kutta methods for differential equations with variable time of impulses, Numerical Functional Analysis and Optimization, 36 (2015), 777-791. (ISI, IF=0.591, SRI= 0.651).

2. T. Donchev, A. Nosheen, R. Ahmed, On the solution set of fuzzy systems, Contemporary Methods in Mathematical Physics and Gravitation, vol. 1, No. 2 (2015), 55-70.

3. T. Donchev, D. Kolev, A. Nosheen, M. Rafaqat, A. Zeinev, Numerical methods for delayed differential equations with discontinuities, Pliska Stud. Math. Bulgar., vol. 23 (2014) , 55-65.

4. O. Benniche, O. Cârjă, Approximate and near weak invariance for nonautonomous differential inclusions, submitted to J. Dyn Control Syst.

5. O. Cârjă, T. Donchev, A. I. Lazu, Generalized solutions of semilinear evolution inclusions, submitted to SIAM J. Optim.

IF=Impact factorul (2014) according to Web of Knowledge.

SRI=Influence score according to
http://uefiscdi.gov.ro/userfiles/file/CENAPOSS/RIS2015.pdf

The research results have been presented to the following conferences: *International Conference on Applied and Pure Mathematics,* November 6-8, 2015, Iaşi (A. I. Lazu, Generalized solutions of semilinear evolution inclusions), *Iaşi Academic Days, Scientific Session of Comunications, "O. Mayer" Mathematical Institute*, October 10, 2015 (O. Cârjă, Viability for quasi-autonomous semilinear evolution inclusions).

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