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Qualitative properties of the solution set of differential inclusions

In the frame of the present project the following activities took place: documentation and research, preparation and writing scientific articles, published in ISI and BDI journals, participation at national/international conferences. The proposed objectives were completely realized. We mention that in 2016 **4 papers** have been published (3 papers in ISI journals and 1 paper in a BDI journal).

The research objective for this year was: *To get new Filippov-Plis type results with applications in invariance, near viability and relaxation problems. Continuation.* This research objective was realized. In the following we give a presentation of the main results obtained.

Let X be a real Banach space and $A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semigrup $\{S(t) : X \to X;\, t \geq 0\}$. Let $I \subset \mathbb{R}$ be a nonempty interval and $F : I \times X \rightsquigarrow X$ a given multi-function. We consider the quasi-autonomous semilinear differential inclusion

$$x'(t) \in Ax(t) + F(t, x(t)). \tag{1}$$

By a solution of (1) on $[\tau, T] \subset I$ we mean a continuous function $x : [\tau, T] \to X$ for which there exists $f \in L^1(\tau, T; X)$ with $f(s) \in F(s, x(s))$ a.e. for $s \in [\tau, T]$ and

$$x(t) = S(t-\tau)x(\tau) + \int_\tau^t S(t-s) f(s)\, ds$$

for all $t \in [\tau, T]$.

We introduce the hypotheses that we need to get a Filippov-Plis type result.

(H1) There exists a constant $c > 0$ such that $|F(t,x)| \leq c(1+|x|)$ for any $t \in I$ and any $x \in X$.

(H2) For any $x \in X$, $F(\cdot, x)$ has an integrable selection, i.e., there exists $f \in L^1(t_0, T; X)$ such that $f(t) \in F(t,x)$ for a.a. $t \in [t_0, T]$.

(A) The semigroup $\{S(t);\ t \geq 0\}$ is compact, i.e., $S(t)$ is a compact operator for any $t > 0$.

The hypothesis (A) implies that the space X is separable.

We obtained a variant of Filippov-Plis lemma, which has many applications in optimal control, approximation of differential inclusions etc.

Theorem. *Assume (H1), (H2) and (A). Further, suppose that F is almost lower semicontinuous and $F(t,\cdot)$ is one-sided Perron with respect to the Perron function $\omega(\cdot,\cdot)$. Let $h : I \to R_+$ be a Lebesgue integrable function. If $y(\cdot)$ is a solution of*

$$\begin{cases} y' \in Ay + F(t, y(t)) + h(t)\overline{\mathbb{B}} \\ y(t_0) = y_0, \end{cases} \tag{2}$$

then, for every $\delta > 0$ there exists a solution $x(\cdot)$ of (1) such that

$$|x(t) - y(t)| < r(t),$$

where $r(\cdot)$ is the maximal solution of

$$\begin{cases} r'(t) = \omega(t, r(t)) + h(t) + \delta \\ r(t_0) = |x_0 - y_0|. \end{cases} \tag{3}$$

Definition. The continuous function $y(\cdot)$ is said to be outer ε-solution of (1) if

$$y(t) = S(t-t_0)x_0 + \int_{t_0}^{t} S(t-\tau) f_y(\tau) d\tau,$$

where $f_y(\cdot)$ is strongly measurable and $dist(f_y(t), F(t, y(t))) \leq h_y(t)$ for a.a. $t \in [t_0, T]$, with $h_y(t) \leq 2|F(t, x + \overline{\mathbb{B}})|$ for any $t \in [t_0, T]$ and $x \in X$ and

$$\int_{t_0}^{T} h_y(s) ds < \varepsilon.$$

Using the above new Fillipov-Plis type result, we obtained the following relaxation theorem, which is very important in optimal control theory.

Theorem. *Assume (H1), (H2) and (A). Moreover, suppose that F is almost continuous and $F(t, \cdot)$ is one-sided Perron with respect to the Perron function $\omega(\cdot, \cdot)$. Then, the solution set of* (1) *is dense in the outer limit solution set of* (1).

Let $K : I \rightsquigarrow X$ be a multi-function. We denote by $\mathcal{K}$ the graph of K, i.e. $\mathcal{K} = \{(t, x)\, ; t \in I, x \in K(t)\}$.

We recall that the graph $\mathcal{K}$ is said to be viable with respect to (1) if, for each initial data $(\tau, \xi) \in \mathcal{K}$ with $[\tau, \tau+\theta] \subset I$ for some $\theta > 0$, there exist $T > \tau$ with $[\tau, T] \subset I$ and at least one solution $x : [\tau, T] \to X$ of (1) such that $x(\tau) = \xi$ and $x(t) \in K(t)$ for all $t \in [\tau, T]$.

Traditionally, criteria for viability have been given in terms of tangency conditions. For example, in [M. Necula, M. Popescu, I.I. Vrabie, 2009], the authors proved that if $F : \mathcal{K} \rightsquigarrow X$ is upper semicontinuous with weakly compact and convex values and A generates a compact C_0-semigroup, then the graph $\mathcal{K}$ is viable with respect to (1) if and only if $F(\tau, \xi)$ is A-quasi-tangent to $\mathcal{K}$ at (τ, ξ) a.e. for $\tau \in I$ and for all $\xi \in K(\tau)$.

For $\varepsilon > 0$, we introduce the notion of ε-solution.

Definition 1. A function $x : [\tau, T] \to X$ is called an ε-solution of (1) if it is a solution of

$$x'(t) \in Ax(t) + F(t, x(t) + \varepsilon B)$$

on $[\tau, T]$.

The concepts of approximate/near viability (weak invariance) generalize the concept of viability and in turn rely on the notion of ε-solutions/solutions corresponding to the differential inclusion (1). Roughly speaking, approximate viability means the existence of ε-solutions which remain arbitrarily close to a given graph starting from initial states in that graph. It is interesting to note that the approximate viability is equivalent to an appropriate tangency condition under very general assumptions on F and the semigroup. Especially, convexity and compactness of the values of F are not assumed and even we do not require the semigroup to be of compact type.

Definition 2. Let $E(\cdot)$ be an integrally bounded multi-function. Let $(\tau, \xi) \in \mathcal{K}$. We say that $E(\cdot)$ is A-quasi-tangent to $\mathcal{K}$ at (τ, ξ) if

$$\liminf_{h \to 0_+} \frac{1}{h} dist\left(S(h)\xi + \int_{\tau}^{\tau+h} S(\tau + h - s) E(s) ds; K(\tau+h)\right) = 0$$

We present a Filippov-type result that we shall use to get our invariance results.

Let $f \in L^1(\tau, T; X)$ and let $y : [\tau, T] \to X$ be the solution of the Cauchy problem

$$y'(t) = Ay(t) + f(t), y(\tau) = y_0.$$

Let $F : [\tau, T] \times X \rightsquigarrow X$ be a multi-function.We list the following assumptions on F.

(F1) For each $x \in X$, the multi-function $F(\cdot, x)$ is measurable.

(F2) There exist $\beta > 0$ and $k \in L^1(\tau, T; \mathbb{R}_+)$ such that, for a.e. $t \in [\tau, T]$, the multi-function $F(t,\cdot)$ is $k(t)$-Lipshitzian on $y(t) + \beta B$.

(F3) The function $t \to dist(f(t); F(t, y(t)))$ belongs to $L^1(\tau, T; \mathbb{R}_+)$.

Theorem. (H. Frankowska) *Let X be a separable Banach space. Let F be a multifunction with non-empty and closed values satisfying (F1),(F2) and (F3). Let $\delta \geq 0$ and $M = \sup_{t \in [0, T-\tau]} ||S(t)||$. Let us set*

$$\gamma(t) = dist(f(t); F(t, y(t))),$$

$$m(t) = M exp\left(M \int_{\tau}^{t} k(s)\, ds\right),$$

$$\eta(t) = m(t)\left(\delta + \int_{\tau}^{t} \gamma(s)\, ds\right).$$

If $\eta(T) < \beta$, then, for all $x_0 \in X$ with $|x_0 - y_0| \leq \delta$ and all $\varepsilon > 0$, there exists a solution $x : [\tau, T] \to X$ of (1) *with $x(\tau) = x_0$, such that*

$$|x(t) - y(t)| \leq \eta(t) + (t - \tau)m(t)\varepsilon,$$

for all $t \in [\tau, T]$.

We obtained sufficient conditions for the graph $\mathcal{K}$ to be approximate/near viable with respect to (1) via an appropriate tangency condition expressed in terms of the tangency concept introduced in Definition 2. To this end, let us first introduce the concept of approximate/near viability for the graph $\mathcal{K}$ with respect to (1).

Definition 3. We say that $\mathcal{K}$ is approximate viable, respectively, near viable with respect to (1), if for any $(\tau, \xi) \in \mathcal{K}$ with $[\tau, \tau + \theta] \subset I$ for some $\theta > 0$, there exists $T > \tau$ such that $[\tau, T] \subset I$ and for any $\varepsilon > 0$, there exists an ε-solution, respectively, a solution $x : [\tau, T] \to X$ of (1) with $x(\tau) = \xi$ and $dist(x(t); K(t)) \leq \varepsilon$, for all $t \in [\tau, T]$.

Let us state the following standing hypotheses.

(T) For each $(\tau, \xi) \in \mathcal{K}$ with $[\tau, \tau + \theta] \subset I$ for some $\theta > 0$, we have $F(\cdot, \xi)$ is A-quasi-tangent to $\mathcal{K}$ at (τ, ξ).

(K) The multi-function $K : I \rightsquigarrow X$ is with non-empty and closed values and uniformly continuous, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $K(t) \subset K(s) + \varepsilon B$, whenever $|t - s| \leq \delta$.

Definition 4. We say that the multi-function F is integrally bounded if for each $(\tau, \xi) \in I \times X$ there exists $l \in L^1(I, \mathbb{R}_+)$ and $\rho_1 > 0$ such that $F(t, x) \subset l(t)B$, a.e. for $t \in I$ and for all $x \in B(\xi, \rho)$.

In the next theorem and under hypothesis (K), we prove the sufficiently of the tangency condition (T) for $\mathcal{K}$ to be approximate viable with respect to (1).

Theorem. *Let F be an integrally bounded multi-function with non-empty and closed values. Let $K : I \rightsquigarrow X$ satisfy (K). If (T) holds true, then K is approximate viable with respect to* (1).

The proof of this Theorem is based on a result regarding the existence of approximate solutions given below. We recall that if $\{S(t) : X \to X; t \geq 0\}$ is a C_0-semigroup, then there exist $N \geq 1$ and $\omega \in \mathbb{R}$ such that $S(t) \leq Ne^{\omega t}$ for every $t \geq 0$. Furthermore, the graph $\mathcal{K}$ is said to be closed from the left if for each $(\tau_n, \xi_n) \in \mathcal{K}$ with $(\tau_n)_n$ non-decreasing, $\lim \tau_n = \tau$ and $\lim \xi_n = \xi$, we have $(\tau, \xi) \in \mathcal{K}$. It is easy to see that under hypothesis (K) the graph $\mathcal{K}$ is closed from the left.

Lemma. *Let F be an integrally bounded multi-function with non-empty and closed values. Let K be a locally closed from the left graph. We take $(\tau,\xi)\in K$ with $[\tau,\tau+\theta]\subset I$ for some $\theta>0$. Let $T>\tau$ and $\rho_2>0$ be such that $([\tau,T]\times B(\xi,\rho_2))\cap K$ is closed from the left. Let l and ρ_1 be as in Definition 4. Let $\rho=min\{\rho_1,\rho_2\}$. If (T) holds true, then we may diminish T, if necessary, such that for each $\varepsilon\in(0,1)$ there exist $\sigma:[\tau,T]\to[\tau,T]$ non-decreasing, $\theta:\{(t,s);\tau\le s\le t\le T\}\to[0,T-\tau]$ measurable, $f,g\in L^1(\tau,T;X)$ and $u:[\tau,T]\to X$ continuous, such that:*

(i) $t-\varepsilon\le\sigma(t)\le t$, for all $t\in[\tau,T]$ and $\sigma(T)=T$;

(ii) $u(\sigma(t))\in K(\sigma(t))\cap B(\xi,\rho)$, for all $t\in[\tau,T]$;

(iii) $f(t)\in F(t,u(\sigma(t)))$, a.e. for $t\in[\tau,T]$, and $f(t)\le l(t)$, a.e. for $t\in[\tau,T]$;

(iv) $g(t)\le\varepsilon$, a.e. for $t\in[\tau,T]$;

(v) $\theta(t,s)\le t-s$ for each $\tau\le s\le t\le T$ and $t\to\theta(t,s)$ is non-expansive on $(s,T]$ and for each $t\in(\tau,T]$, $s\to\theta(t,s)$ is measurable on $[\tau,t)$;

(vi) $u(t)=S(t-\tau)\xi+\int_\tau^t S(t-s)f(s)ds+\int_\tau^t S(\theta(t,s))g(s)ds$, for all $t\in[\tau,T]$;

(vii) $u(t)-u(\sigma(t))\le\varepsilon$, for all $t\in[\tau,T]$.

It is important to note that under the same hypotheses on F, if we assume instead of (T) that $F(\tau,\xi)$ is A-quasi-tangent to $\mathcal{K}$ at (τ,ξ) for each $(\tau,\xi)\in\mathcal{K}$ with $[\tau,\tau+\theta]\subset I$ for some $\theta>0$, then the main difference between our lemma and the results mentioned above resides in the first part of (iii), which becomes $f(t)\in F(\sigma(t),u(\sigma(t)))$, for each $t\in[\tau,T]$. The use of the new tangency condition (T) leads to eliminate this variation.

Now, we establish some sufficient conditions that $\mathcal{K}$ be near viable with respect to (1). To this end, let us list the basic assumptions that we shall refer to in the sequel.

(F5) For each $(\tau,\xi)\in\mathcal{K}$ with $[\tau,\tau+\theta]\subset I$ for some $\theta>0$, there exist $T>\tau$ with $[\tau,T]\subset I$, an open set $\Omega\subset X$ containing ξ and $k\in L^1(\tau,T;\mathbb{R}_+)$ such that $F(t,x)\subset F(t,y)+k(t)|x-y|B$, for each $x,y\in\Omega$ and a.e. for $t\in[\tau,T]$.

(F6) There exists $k\in L^1(I;\mathbb{R}_+)$ such that $F(t,x)\subset F(t,y)+k(t)|x-y|B$, for each $x,y\in X$ and a.e. for $t\in I$.

Notice that (F6) is a particular case getting from (F5).

Theorem. *Let X be a separable Banach space. Let F be an integrally bounded multi-function with non-empty and closed values satisfying (F1) and (F5) and let $K:I\rightsquigarrow X$ satisfy (K). If (T) holds true, then K is near viable with respect to* (1).

We give a result concerning (exact) viability for the graph $\mathcal{K}$ with respect to (1), in the case when F is single valued and locally Lipshitz. More exactly, let $f:I\times X\to X$ be a given function and let us consider the semilinear differential equation

$$x'(t)=Ax(t)+f(t,x(t)) \tag{4}$$

The following theorem gives a criterion for the graph $\mathcal{K}$ to be viable with respect to (4).

Theorem. *Let X be a separable Banach space. Let f be integrally bounded, $f(\cdot,x)$ measurable for each $x\in X$ and $f(t,\cdot)$ locally Lipshitz. Let K satisfy (K). If $f(\cdot,\xi)$ is A-quasi-tangent to K at (τ,ξ) for each $(\tau,\xi)\in K$ with $[\tau,\tau+\theta]\subset I$ for some $\theta>0$, then K is viable with respect to* (4).

Under hypothesis (F7) below, we give some necessary conditions for approximate viability for a graph $\mathcal{K}$ with respect to (1).

(F7) Let $(\tau,\xi)\in\mathcal{K}$ be such that $[\tau,\tau+\theta]\subset I$ for some$\theta>0$. Assume that there exist $l\in L^1_{loc}(I,\mathbb{R}_+)$, $W:\mathbb{R}_+\to\mathbb{R}_+$ non-decreasing and continuous at 0 with $W(0)=0$ and an open set $\Omega\subset X$ containing ξ such that $F(t,x)\subset F(t,\xi)+l(t)W(|x-\xi|)B$, for each $x\in\Omega$ and, a.e. for $t\in[\tau,\tau+\mu]$, for some $\mu>0$. Furthermore, we assume that

$$\frac{1}{t}\int_\tau^{\tau+t}|l(s)|ds\le M$$

for some $M>0$ and for all $t\in]0,\mu]$.

We suppose that F has a sublinear growth, i.e., there exists $c \in L^1(I; \mathbb{R}_+)$ such that

(F4) $||F(t,x)|| \le c(t)(1+|x|)$,

a.e. for $t \in I$ and for every $x \in X$. Here $||F(t,x)|| = \underset{v \in F(t,x)}{sup} |v|$.

Theorem. *Let X be a separable Banach space. Let F be a non-empty and closed valued multi-function satisfying (F1) and (F4). If K is approximate viable with respect to* (1)*, then, at each point $(\tau, \xi) \in K$ at which (F7) is satisfied, $F(\cdot, \xi)$ is A-quasi-tangent to K at (τ, ξ).*

We show here that, under additional compactness hypotheses on F and $S(t)$, the tangency condition (T) is sufficient for the graph $\mathcal{K}$ to be (exact) viable with respect to (1).

Theorem. *Let A generate a compact C_0-semigroup. Let I be a non-empty and bounded interval and let $F : I \times X \rightsquigarrow X$ be an integrally bounded multifunction, strongly weakly upper semi-continuous with non-empty, convex and weakly compact values. A sufficient condition that a locally closed from the left graph K to be (exact) viable with respect to* (1) *is the tangency condition (T).*

Next, we prove a viability result for the graph $\mathcal{K}$ with respect to (1) in the case when F is of Carathedory type. To this end, we recall first that a multi-function $F(t,\cdot)$ is said to be globally Lipshitz uniformly, a.e. for $t \in I$, if there exists a constant $L > 0$ such that

$$F(t,x) \subset F(t,y) + L|x-y|B,$$

for all $x, y \in X$ and, a.e. for $t \in I$.

Theorem. *Let X be a reflexive and separable Banach space. Let A generate a compact C_0-semigroup and let K satisfy (K). Let I be a non-empty and bounded interval and let $F : I \times X \rightsquigarrow X$ be a multi-function with non-empty, closed, bounded and convex values. Assume that $F(\cdot, x)$ is measurable for each $x \in X$, $F(t,\cdot)$ is globally Lipshitz uniformly, a.e. for $t \in I$ and there exists $x_0 \in X$ such that $F(t, x_0) \le M$, a.e. for $t \in I$ for some $M > 0$. If the tangency condition (T) holds true, then K is globally viable with respect to* (1)*, that is, for each $(\tau, \xi) \in K$ with $[\tau, \tau+\theta] \subset I$ for some $\theta > 0$, for every $T > \tau$ with $[\tau, T] \subset I$, there exists a solution $x : [\tau, T] \to X$ of* (1) *with $x(\tau) = \xi$ and $x(t) \in K(t)$ for all $t \in [\tau, T]$.*

An application considered concerns a relation between solutions of the relaxed (convexified) differential inclusion

$$x'(t) \in Ax(t) + \overline{co} F(t, x(t)) \tag{5}$$

and ε-solutions of the differential inclusion (1). We recall that in [Frankowska, H, 1990], the author proved, in the frame of separable Banach spaces and for an integrally bounded multi-function F satisfying (F1) and (F6), that the set of solutions of (1) is dense in the set of solutions of the relaxed inclusion (5) in the metric of uniform convergence. Here, under weaker assumptions on F, we give a variant of the result from [Frankowska, H, 1990]. More exactly, we have the following result.

Theorem. *Let X be a separable Banach space. Let F be a multi-function with non-empty and closed values satisfying the sublinear growth condition (F4) and let $x : [\tau, T] \to X$ be the solution of the differential inclusion*

$$x'(t) \in Ax(t) + \overline{co} F(t, x(t))$$

on $[\tau, T] \subset I$ with $x(\tau) = \xi$. Assume that $\overline{co} F$ satisfies (F7) at each $(t_0, x(t_0))$ with $t_0 \in [\tau, T[$. Then, for any $\varepsilon > 0$, there exists an ε-solution $\overline{x} : [\tau, T] \to X$ of (1) *with $\overline{x}(\tau) = \xi$, satisfying*

$$|\overline{x}(t) - x(t)| \le \varepsilon$$

for each $t \in [\tau, T]$.

It was proved in [O. Cârjă, 2010] that if A generates a compact C_0-semigroup and $F : X \rightsquigarrow X$ is a globally Lipshitz multi-function with weakly compact and convex values then the set of solutions of the differential inclusion $x'(t) \in Ax(t) + F(x(t))$ depends in a Lipschitz manner on the initial states. Here, we give an analogous of this result in the quasi-autonomous semilinear case.

Theorem. *Let X, A, and F be as in Theorem 4.2. Let $x_0 \in X$ and let $x : [t_0, T] \to X$ be the solution of* (1) *on $[t_0, T] \subset I$ with $x(t_0) = x_0$. Then, for each $y_0 \in X$, there exists a solution $y : [t_0, T] \to X$ of* (1) *with $y(t_0) = y_0$ satisfying*

$$|x(t) - y(t)| \leq \mu e^{\mu L(t-t_0)} |x_0 - y_0|$$

for all $t \in [t_0, T]$, where $\mu = e^{\omega(T-t_0)}$.

The research results have been presented to the following conference: *7th European Congress of Mathematics*, 18-22 iulie 2016, Berlin (A. I. Lazu, Generalized solutions for semilinear differential inclusions in Banach spaces).

In **2016** there have been published **4 articles (3** in **ISI journals)**, articles financially supported by the project, confirmed by the corresponding text from the Acknowledgement section.

1. O. Benniche, O. Cârjă, *Approximate and near weak invariance for nonautonomous differential inclusions*, J. Dyn Control Syst., 2016, DOI 10.1007/s10883-016-9312-0.
2. O. Cârjă, T. Donchev, A. I. Lazu, *Generalized solutions of semilinear evolution inclusions*, SIAM J. Optim. 26 (2), 1365–1378, 2016.
3. O. Benniche, O. Cârjă, *Viability for quasi-autonomous semilinear evolution inclusions*, Mediterranean Journal of Mathematics, 2016, DOI 10.1007/s00009-016-0739-z.
4. R. Ahmed, T. Donchev, A. I. Lazu, A. Nosheen, M. Rafaqat, *On the solution set of multivalued fuzzy fractional systems*, International Journal of Pure and Applied Mathematics 109 (1), 49-66, 2016.

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