New Finsler metrics of constant curvature

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Finsler metric

Proposition

A Finsler metric on a manifold M is a function $F : TM \to [0; \infty)$ with following properties:

1. *Smoothness: $F(x; y)$ is C^∞ on T_0M.*
2. *Homogeneity: $F(x; \lambda y) = \lambda F(x; y); \lambda > 0$*
3. *Regularity/Convexity: $(g_{ij}(x, y))$ is positive definite, where*

$$g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$$

Examples of Finsler structures

a) *Riemannian metrics:* $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$, g_{ij} is independent of y.

b) *Randers metrics are special members of Finsler metrics which have the form*

$$F(x, y) = a + b = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i.$$

It was introduced by Randers in his study of general relativity([Randers-1941]). G. Randers used this metric to describe the asymmetrical space-time. It is an important model in physics.

c) *Square metrics:* $F = \dfrac{(a+b)^2}{a}$, where a and b are the quantities from the definition of a Randers metric.

The arc length

Let $F = F(x, y)$ a Finsler metric on a manifold M^n and $\gamma : [a, b] \to \mathbb{R}$ a smooth curve on M

1. Its arc length is the integral:

$$L(\gamma) = \int_{\gamma} F(\gamma, \dot{\gamma}) dt. \tag{1}$$

2. The First variation of the arc length is:

$$\delta L(\gamma)(V) = -\int_{\gamma} g_{\dot{\gamma}} \left(V, \nabla_{\dot{\gamma}} \frac{\dot{\gamma}}{F(\dot{\gamma})} \right) dt \tag{2}$$

Remark

The first variation leads us to the geodesic equations associated to this metric.

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The variational problem

A geodesic of the manifold (M, F)(parameterized by the arc length) is a critical curve of the energy function:

$$E(\gamma) = \int_a^b F^2(\gamma, \dot{\gamma}) ds.$$

Replacing the energy function by $E = F^2$ it follows :

$$\frac{\partial E}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial E}{\partial y^i}\right) = -2g_{ij}\left(\frac{d^2x^j}{dt^2} + \frac{1}{2}g^{jl}\left(\frac{\partial^2 E}{\partial y^l \partial x^k} - \frac{\partial E}{\partial x^l}\right)\right) = 0. \tag{3}$$

If we denote $G^j(x, y) = \frac{1}{4}g^{jl}\left(\frac{\partial^2 E}{\partial y^l \partial x^k} - \frac{\partial E}{\partial x^l}\right)$, we have a system of n homogeneous differential equations of second order :

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{4}$$

Riemann curvature

The system (4) can be identified with a vector field given by:

$$S\left(\frac{\partial E}{\partial y^i}\right) + \frac{\partial E}{\partial x^i} = 0,$$

where

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i} \tag{5}$$

The vector field written above is called geodesic spray. There are two canonical structures on TM, which we will use to develop our setting. One is the tangent structure, J, and the other one is the Liouville vector field, $\mathcal{C}$, locally given by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i,\ \mathcal{C} = y^i \frac{\partial}{\partial y^i}$$

Remark

It is well known that a spray induces a nonlinear connection, $\Gamma = [J, S]$, with the corresponding projectors h and v given by

$$h := \frac{1}{2}(I + \Gamma), \quad v := \frac{1}{2}(I - \Gamma) \tag{6}$$

and the curvature tensor

$$R := -\frac{1}{2}[h, h].$$

We introduce the Jacobi endomorphism (the Riemann curvature tensor):

$$R_{(x,y)} := R^i_k(x, y)\frac{\partial}{\partial y^i} \otimes dx^k, \tag{7}$$

where

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - S\left(\frac{\partial G^i}{\partial y^k}\right) - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

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Flag curvature

Entirely analogous to the Riemannian case, we can define a quantity, that is the corespondant of the sectional curvature, named *flag curvature*.

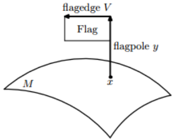
The main instruments are:

- a point $x \in M$ that will be the place of fixing the flag,
- a flagpole given by a nonzero $y \in T_x M$, and
- an edge $V \in T_x M$ transverse to the flagpole.

$$K(x, y, V) := \frac{g_{(x,y)}(R_{(x,y)}(V), V)}{g_{(x,y)}(y, y)g_{(x,y)}(V, V) - g_{(x,y)}(y, V)^2}, \qquad (8)$$

is the *flag curvature* of (y, P) with $P = span\{y, V\} \subset T_x M$.

Sectional curvature



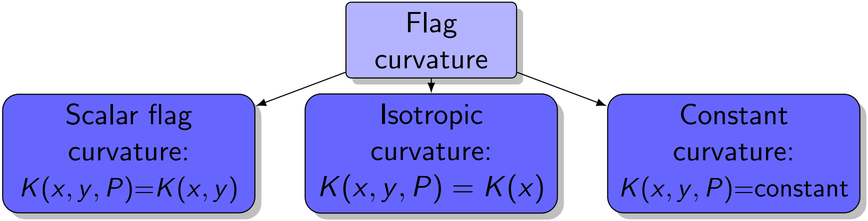
Remark

It is well known that the curvature of the flag (y, P) is independent of the choice of the flagedge V . Hence the flag curvature is usually denoted by

$$K(x, y, V) := K(x, y, P)$$

Contrast to the Riemannian case, the section P cannot completely determine the curvature in Finslerian case. One should pick a flagpole $y \in P$ to construct a flag (P, y), and then deduce the curvature

Special flag curvatures



Jacobi endomorphism

Due to the homogeneity of the spray S, information on curvature can also be obtained through the Jacobi endomorphism, which is a 1-vector valued form defined by:

$$\Phi = v \circ [S, h] = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j. \tag{9}$$

Definition

The spray S is isotropic if the Jacobi endomorphism has the following form

$$\Phi = \rho J - \alpha \otimes C \Leftrightarrow R^i_k = \rho \delta^i_k - \alpha_k y^i \Leftrightarrow R^i_k = KF^2(\delta^i_k - F^{-2} g_{kq} y^q y^i),$$

where $\rho \in C^\infty(T_0 M)$ and $\alpha = \alpha_i(x, y) dx^i \in \Lambda^1(T_0 M)$.

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Projectively related sprays

A reparameterization preserving orientation $t \to \widetilde{t}(t)$ of the system

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx^i}{dt}\right) = 0,$$

leads to a new spray $\widetilde{S} = S - 2P\mathcal{C}$.
The scalar function $P \in C^\infty(TM - \{0\})$ is 1-homogeneous and it is related to the new parameter by

$$\frac{d^2\widetilde{t}}{dt^2} = P\left(x^i(t), \frac{dx^i}{dt}\right)\frac{d\widetilde{t}}{dt}, \;\; \frac{d\widetilde{t}}{dt} > 0. \tag{10}$$

Projectively related sprays

Definition

Two sprays S and $\widetilde{S}$ are *projectively related* if their geodesics coincide up to an orientation preserving reparameterization.

A Finsler metric is *projectively flat* if and onlt if it satisfies the Hamel equation:

$$\delta_{S_0} F = d_J S_0 F - 2 d_{h_0} F = 0. \tag{11}$$

In this case the projective factor $P(x, y)$ is given by

$$P(x, y) = \frac{S_0 F}{2F}. \tag{12}$$

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Weyl-type curvature tensor for Finsler spaces

Definition

Consider S a spray with Jacobi endomorphism Φ and curvature tensor R. We define the following Weyl-type curvature tensors

$$W_0 = \Phi - \frac{1}{n-1}(\operatorname{Tr}\Phi)\,J + \frac{1}{2(n-1)}d_J(\operatorname{Tr}\Phi)\otimes\mathcal{C}. \tag{13}$$

and

$$W_1 = \frac{1}{3}[J, W_0] = R - \frac{1}{2(n-1)}d_J(\operatorname{Tr}\Phi)\wedge J \tag{14}$$

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Finsler metrics of constant flag curvature

Theorem

For a Finsler metric on a manifold of dim $M \geq 3$ the following conditions are equivalent:

1. *The Finsler metric has constant flag curvature.*
2. *The Weyl type curvature tensor W_0 vanishes.*
3. *The Weyl type curvature tensor W_1 vanishes.*

Theorem

A Finsler metric on a 2-dimensional manifold has constant flag curvature if and only if the following conditions are satisfied

1. *The Weyl-type tensor (14) vanishes.*
2. $d_h\alpha = 0$.

Projectively related Weyl-type curvature tensors

Lemma

Consider $\widetilde{S} = S - 2P\mathcal{C}$ two projectively related sprays. The corresponding Weyl-curvature tensor W_1 are related by

$$\widetilde{W_1} = W_1 + \frac{1}{2}\delta_S P \wedge J + d_J d_h P \otimes \mathcal{C}. \tag{15}$$

The Weyl-type curvature tensor W_1 is projectively invariant if and only if the projective factor associated to the deformation satisfies $\delta_S P = 0$ and hence is a Hamel function.

Lemma

Consider S and $\widetilde{S} = S - 2P\mathcal{C}$ two projectively related isotropic sprays with the property that P is a Hamel function. Then the derivatives with respect to the horizontal projector of the semi-basic 1-forms α and $\widetilde{\alpha}$ are related by

$$d_{\widetilde{h}}\widetilde{\alpha} = d_h\alpha - d_R P - P d_J\alpha + \alpha \wedge d_J P. \tag{16}$$

Proposition

We consider F and $\widetilde{F}$ two projectively related Finsler metrics. If the initial metric F is of constant flag curvature and the projective factor is a Hamel function then $\widetilde{F}$ is of constant flag curvature.

New condition for isotropic sprays

The curvature tensors Φ and R are related by

$$3R = [J, \Phi], \ \Phi = i_S R. \tag{17}$$

Lemma

A spray S is isotropic if and only if there exists a semi-basic 1-form $\xi \in \Lambda^1(T_0M)$ such that its curvature tensor R is given by:

$$R = \xi \wedge J - d_J \xi \otimes \mathcal{C}, \tag{18}$$

where $\xi = \frac{1}{3} \left(\alpha + d_J \rho \right).$

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New Finslerian Version of Schur Lemma

Theorem (Finslerian version of Schur's Lemma for $n \geq 2$)

Consider S the geodesic spray of a Finsler metric F. Then F has constant curvature if and only if:

$$S \text{ is isotropic} \quad \text{(this condition is always true for n=2);} \tag{19}$$

and the curvature 1-form satisfies:

$$d_J\xi = 0; \tag{20}$$

$$d_h\xi = 0 \quad \text{(this condition is always true for } n \geq 3\text{).} \tag{21}$$

Finslerian Version of Beltrami Theorem

Theorem (Finslerian version of Beltrami's Theorem for $n \geq 2$)
Consider F and $\widetilde{F}$ two projectively related Finsler metrics. If $\widetilde{F}$ has constant curvature then F has also constant curvature if and only if the projective factor P is a Hamel function.

Projectively flat Randers metrics

Proposition

A Randers metric $F = a + b$ is projectively flat if and only of the Riemannian metric a is projectively flat and the 1-form $b_i dx^i$ is closed.

Lemma

Let $F = \sqrt{g_{ij}(x)y^i y^j}$ be a positive definite Finsler metric on an open domain (open and convex) $\mathcal{U} \subset \mathbb{R}^n$ that is reducible to a Riemannian metric. Then F is projectively flat if and only if the following relation is satisfied

$$g_{ij,l} = 2\psi_l g_{ij} + \psi_i g_{jl} + \psi_j g_{il}, \; P(x,y) = \psi_l(x) y^l. \qquad (22)$$

In this case, P is the projective factor of F.

Projectively flat Randers metrics

The family of projectively flat Finsler metrics that are reducible to a Riemannian metric is given by:

$$F = \frac{\sqrt{|y|^2 + \mu\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 + \mu|x|^2}. \tag{23}$$

Lemma

The family of projectively flat Randers metrics of negative constant flag curvature whose projective factor is proportional to the metric is given by $F = a + b$, where:

$$a = \frac{\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 - 4c^2|x|^2} \text{ and } b = \frac{2c\langle x, y\rangle}{1 - 4c^2|x|^2}. \tag{24}$$

In this case, the constant c represents the coefficient of proportionality between the projective factor and the metric.

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   - Finsler metric
   - The variational problem
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2. **New Weyl-type tensors in Finsler geometry**
   - Weyl-type curvature tensor for Finsler spaces
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4. **New families of projectively related Finsler metrics of constant curvature**
   - New Finsler metrics of negative flag curvature
   - New Finsler metrics of zero flag curvature

Outline

A new family of projectively related Finsler metrics obtained through a Randers deformation

We consider F given by (24). We make a Randers deformation of the metric

$$F \to \widetilde{F} = F + \widetilde{b}, \tag{25}$$

where $\widetilde{b}$ is given by $\widetilde{b}(x, y) = b_i(x)y^i$. Since $\delta_S \widetilde{F} = 0 \Leftrightarrow \delta_S \widetilde{b} = 0$, it follows that F and $\widetilde{F}$ are projectively related.
The projective factor is given by:

$$P = \frac{S(\widetilde{F})}{2\widetilde{F}} = \frac{S(F + \widetilde{b})}{2(F + \widetilde{b})} = \frac{S(\widetilde{b})}{2(F + \widetilde{b})}. \tag{26}$$

New familes of Finsler metrics of negative flag curvature

We assume that $P = \nu\widetilde{b},\ \nu \in \mathbb{R}$ and we get

$$S_0\widetilde{b} - 2\nu\widetilde{b}^2 = 0. \tag{27}$$

Finally we got that the 1-form $\widetilde{b}$ is given by

$$\widetilde{b}(x,y) = \frac{\langle e, y\rangle}{4\nu^2\left(\langle e, x\rangle + f\right)}, f > 0, |e| < 1. \tag{28}$$

Therefore, the metric obtained through this deformation is

$$\widetilde{F} = \frac{\sqrt{|y|^2 - 4\nu^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 - 4\nu^2|x|^2} - \frac{2\nu\langle x, y\rangle}{1 - 4\nu^2|x|^2} + \frac{\langle e, y\rangle}{4\nu^2\left(\langle e, x\rangle + f\right)}, \tag{29}$$

with $\widetilde{\kappa} = -c^2$.

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New Finsler metrics of zero flag curvature inspired by squared metrics

We consider the following deformation of the Finsler metric (24):

$$\widetilde{F} = f(x)\frac{F^2}{a}, \text{ where f is a positive function and } a \text{ is given by (24).} \tag{30}$$

Lemma

Let $F = a + b$ a Randers metric. Then $\widetilde{F} = f(x)\frac{F^2}{a}$ is projectively related to F if and only if the following relation is satisfied

$$\frac{F^2}{a} d_h f - S(f) d_J\left(\frac{F^2}{a}\right) + fS(a) d_J\left(\frac{F^2}{a^2}\right) = 0. \tag{31}$$

A new Finsler metric of zero flag curvature

$$P = \frac{S(f)}{2f} - \frac{S_0 a - 2cFa}{2a} = \frac{Sf}{2f} - \frac{4cab - 2cFa}{2a} = \frac{S_0 f}{2f} - 2cb + cF. \quad (32)$$

We notice that

$$\delta_S P = \delta_S \left(\frac{S_0 f}{2f} - 2cb + cF \right) = \delta_S \left(\frac{S_0 f}{2f} \right) \quad (33)$$

We assume that $S_0 f = 4cfb$ and we get

$$\widetilde{F} = \frac{\eta(\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)} + 2c\langle x,y\rangle)^2}{(1-4c^2|x|^2)^2\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)}}, \ \eta \in \mathbb{R}^+, \quad (34)$$

with $\widetilde{\kappa} = 0$.

A new metric of zero flag curvature obtained through a conformal transformation

Lemma

We consider F and $\widetilde{F}$ two projectively flat Finsler metrics and $\overline{F} = g(x)\widetilde{F} + \frac{f(y)}{F}\widetilde{F}$ a metric obtained by a multiplication of the projectively flat metric $\overline{F}$ with the 0-homogeneous function $g(x) + \frac{f(y)}{F}$, where g and f are considered such that $\widetilde{F}$ is positive. Then $\overline{F}$ is projectively flat if and only if the following relation is satisfied

$$\widetilde{F} d_{h_0} g - S_0 g d_J \widetilde{F} - \frac{S_0 f d_J \widetilde{F}}{F} + \frac{\widetilde{F} S_0 f d_J F}{F^2} - \frac{S_0 \widetilde{F} d_J f}{F} + \frac{f S_0 \widetilde{F} d_J F}{F^2} + \frac{\widetilde{F} S_0 F d_J f}{F^2} + \frac{f S_0 F d_J \widetilde{F}}{F^2} - \frac{2 f \widetilde{F} S_0 F d_J F}{F^3} = 0. \tag{35}$$

New Finsler metric of zero flag curvature

$$P = \frac{S_0 \overline{F}}{2\overline{F}} = \frac{S_0 g \cdot \widetilde{F} + g S_0 \widetilde{F} + \frac{S_0 f}{F}\widetilde{F} + \frac{f S_0 \widetilde{F}}{F} - f\frac{\widetilde{F}}{F^2} S_0 F}{2\left(g\widetilde{F} + \frac{f}{F}\widetilde{F}\right)} \tag{36}$$

We recall that F and $\widetilde{F}$ are two projectively flat Finsler metrics for which $S_0 F = 2cF^2$ and $S_0 \widetilde{F} = 4cF\widetilde{F}$.
Therefore, (36) becomes:

$$P = \frac{S_0 g \cdot \widetilde{F} + \frac{S_0 f}{F}\widetilde{F} - 2cf\widetilde{F}}{2\left(g\widetilde{F} + \frac{f}{F}\widetilde{F}\right)} + 2cF. \tag{37}$$

New Finsler metric of zero flag curvature

Taking into account the conditions imposed on the functions f and g it follows that we can make the following extra assumption

$$S_0 g = 2cf \text{ and } S_0 f = 0. \tag{38}$$

With the assumptions considered in (38) we get that the projective factor associated is

$$P = 2cF, \tag{39}$$

which is a Hamel function. We can write now the expression for the new metric as follows

$$\overline{F} = \frac{\eta(\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)} + 2c\langle x, y\rangle)^2}{(1-4c^2|x|^2)^2\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}} \cdot \left(2c\langle a, x\rangle + e + \frac{\langle a, y\rangle}{\frac{\sqrt{|y|^2-4c^2(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-4c^2|x|^2} + \frac{2c\langle x,y\rangle}{1-4c^2|x|^2}}\right). \tag{40}$$

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Thank you