New Finslerian Version of Schur's Lemma and its applications

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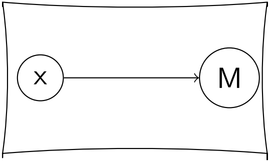
The difference between Riemann and Finsler Geometry

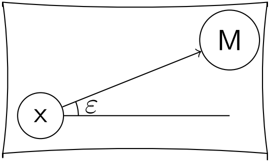
Under the influence of gravitational forces, what is the trajectory the person should walk to reach a given destination in the shortest time?

Riemannian Geometry



- $\overline{v}_x = c$ and the gravitational force is acting perpendicularly.
- *distance*(x, M)=geographic distance is what Riemannian distance is modeling.

Finsler Geometry



- $\overline{v}_x = c$ on a plane of angle ε.(Minkowski plane)
- the most efficient time minimizing paths are not the Riemannian geodesics, but the geodesics of $F = \alpha^2/(\alpha - \beta)$

Definitions

A *Riemannian metric* on M is a family of (positive-definite) inner products $g_p \colon T_pM \times T_pM \to \mathbb{R}$, $p \in M$ such that, for every pair of differentiable vector fields $X, Y \in M$, $p \mapsto g_p(X|_p, Y|_p) p \mapsto g_p(X|_p, Y|_p)$ defines a smooth function.

A *Finsler metric* on a manifold M is a function $F : TM \to [0; \infty)$ with following properties:

1. Smoothness: $F(x; y)$ is C^∞ on T_0M.
2. Homogeneity: $F(x; \lambda y) = \lambda F(x; y); \lambda > 0$
3. Regularity/Convexity: $(g_{ij}(x, y))$ is positive definite, where

$$g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$$

Geometric framework

We consider

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{1}$$

a system of n homogeneous differential equations of second order. The system (1) can be identified with a vector field named spray, given by:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i} \tag{2}$$

The curvature tensor of the nonlinear connection $\Gamma = [J, S]$ is

$$R = \frac{1}{2}[h, h] = R^i_{jk}\frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \tag{3}$$

where $h := \frac{1}{2}(I + \Gamma)$ and $v := \frac{1}{2}(I - \Gamma)$ are the projectors of Γ.

New condition for isotropic sprays

The curvature tensors $\Phi = v \circ [S, h]$ and R are related by

$$3R = [J, \Phi], \ \Phi = i_S R. \tag{4}$$

Lemma (Bucătaru I, C.G.-2019)

A spray S is isotropic if and only if there exists a semi-basic 1-form $\xi \in \Lambda^1(T_0M)$ such that its curvature tensor R is given by:

$$R = \xi \wedge J - d_J\xi \otimes \mathcal{C}, \tag{5}$$

where $\xi = \frac{1}{3}\left(\alpha + d_J\rho\right).$

The curvature 1-form of a Finsler metric

For a Finsler metric of constant flag curvature the curvature tensor is given by

$$R = \xi \wedge J \Rightarrow R^i_{jk} = \xi_j \delta^i_k - \xi_k \delta^i_j, \qquad (6)$$

where

$$\xi = \kappa F d_J F \Rightarrow \xi_i = \kappa \frac{1}{2} \frac{\partial F^2}{\partial y^i} = \kappa g_{ij} y^j.$$

Lemma (Bucătaru I, C.G.-2019)

[Differential Bianchi identies] In dimension $n \geq 3$, the curvature 1-form of an isotropic spray satisfies $d_h \xi = 0$.

New Finslerian Version of Schur Lemma

Theorem (Bucătaru I, C.G.-2019)

[Finslerian version of Schur's Lemma for $n \geq 2$] Consider S the geodesic spray of a Finsler metric F. Then F has constant curvature if and only if satisfies the following CFC-conditions:

$$S \text{ is isotropic} \quad \text{(this condition is always true for n=2);} \tag{7}$$

and the curvature 1*-form satisfies:*

$$d_J\xi = 0; \tag{8}$$

$$d_h\xi = 0 \quad \text{(this condition is always true for } n \geq 3\text{).} \tag{9}$$

The correspondent in the Riemannian case

Remark

We consider S the affine spray of a Finsler metric that is reducible to a Riemannian metric, $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$. Hence the CFC-conditions from the theorem become:

- S is isotropic.
- $d_J\xi = 0 \Leftrightarrow d_J\kappa = 0$ is always satisfied.
- $d_h\xi = 0 \Leftrightarrow d_h\kappa = 0 \Leftrightarrow R_{ij,k} - R_{ik,j} = 0$.

The invariance of the *CFC*-conditions

Definition

Two sprays S and $\widetilde{S}$ are *projectively related* if their geodesics coincide up to an orientation preserving reparameterization.

Remark

We consider S and $\widetilde{S} = S - 2P\mathcal{C}$ two projectively related sprays. Then:

- $\widetilde{S}$ is isotropic if and only if S is isotropic,
- $d_J\widetilde{\xi} = d_J\xi + \delta_S P$,
  $\delta_S P = d_J SP - 2d_h P$ is the Euler Lagrange 1-form,
- $d_{\widetilde{h}}\widetilde{\xi} = d_h\xi$.

Fislerian Version of Beltrami Theorem

Theorem (Bucătaru I, C.G.-2019)

[Finslerian version of Beltrami's Theorem for $n \geq 2$] Consider F and $\widetilde{F}$ two projectively related Finsler metrics. If $\widetilde{F}$ has constant curvature then F has also constant curvature if and only if the projective factor P is a Hamel function.

Lemma (Bucătaru I, C.G.-2018)

Consider that F and $\widetilde{F}$ are two projectively related Finsler functions that are reducible to Riemannian metrics. If F is of constant flag curvature then the projective factor satisfies

$$d_h d_J P = a_{ij} dx^i \wedge dx^j = 0, \; a_{ij} = \frac{1}{2}\left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j}\right). \tag{10}$$

Projectively flat Randers metrics

The family of projectively flat Finsler metrics that are reducible to a Riemannian metric is given by:

$$F = \frac{\sqrt{|y|^2 + \mu\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 + \mu|x|^2}. \tag{11}$$

Lemma (C.G.-2019)

The family of projectively flat Randers metrics of negative constant flag curvature whose projective factor is proportional to the metric is given by

$$F = \frac{\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 - 4c^2|x|^2} + \frac{2c\langle x, y\rangle}{1 - 4c^2|x|^2}. \tag{12}$$

In this case, the constant c represents the coefficient of proportionality between the projective factor and the metric.

A new family of projectively related Finsler metrics obtained through a Randers deformation

We consider F given by (12). We make a Randers deformation of the metric

$$F \to \widetilde{F} = F + \widetilde{b}, \tag{13}$$

where $\widetilde{b}$ is given by $\widetilde{b}(x, y) = b_i(x)y^i$. Since $\delta_S \widetilde{F} = 0 \Leftrightarrow \delta_S \widetilde{b} = 0$, it follows that F and $\widetilde{F}$ are projectively related.
The projective factor is given by:

$$P = \frac{S(\widetilde{F})}{2\widetilde{F}} = \frac{S(F + \widetilde{b})}{2(F + \widetilde{b})} = \frac{S(\widetilde{b})}{2(F + \widetilde{b})}. \tag{14}$$

New familes of Finsler metrics of negative flag curvature

We assume that $P = \nu\widetilde{b},\ \nu \in \mathbb{R}$ and we get

$$S_0\widetilde{b} - 2\nu\widetilde{b}^2 = 0. \tag{15}$$

Finally we got that the 1-form $\widetilde{b}$ is given by

$$\widetilde{b}(x,y) = \frac{\langle e, y\rangle}{4\nu^2\left(\langle e, x\rangle + f\right)}, f > 0, |e| < 1. \tag{16}$$

Therefore, the metric obtained through this deformation is

$$\widetilde{F} = \frac{\sqrt{|y|^2 - 4\nu^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1 - 4\nu^2|x|^2} - \frac{2\nu\langle x, y\rangle}{1 - 4\nu^2|x|^2} + \frac{\langle e, y\rangle}{4\nu^2\left(\langle e, x\rangle + f\right)}, \tag{17}$$

with $\widetilde{\kappa} = -c^2$.

New Finsler metrics of zero flag curvature inspired by squared metrics

We consider the following deformation of the Finsler metric (12):

$$\widetilde{F} = f(x)\frac{F^2}{a}, \tag{18}$$

where f is a positive function and a is given by (12).

Lemma (C.G.-2019)

Let $F = a + b$ a Randers metric. Then $\widetilde{F} = f(x)\frac{F^2}{a}$ is projectively related to F if and only if the following relation is satisfied

$$\frac{F^2}{a} d_h f - S(f) d_J \left(\frac{F^2}{a}\right) + fS(a) d_J \left(\frac{F^2}{a^2}\right) = 0. \tag{19}$$

A new Finsler metric of zero flag curvature

$$P = \frac{S(f)}{2f} - \frac{S_0 a - 2cFa}{2a} = \frac{Sf}{2f} - \frac{4cab - 2cFa}{2a} = \frac{S_0 f}{2f} - 2cb + cF. \quad (20)$$

We notice that

$$\delta_S P = \delta_S \left(\frac{S_0 f}{2f} - 2cb + cF \right) = \delta_S \left(\frac{S_0 f}{2f} \right) \quad (21)$$

We assume that $S_0 f = 4cfb$ and we get

$$\widetilde{F} = \frac{\eta(\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)} + 2c\langle x, y\rangle)^2}{(1 - 4c^2|x|^2)^2\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}, \ \eta \in \mathbb{R}^+, \quad (22)$$

with $\widetilde{\kappa} = 0$.

A new metric of zero flag curvature obtained through a conformal transformation

Lemma (C.G.-2019)

We consider F and $\widetilde{F}$ two projectively flat Finsler metrics and $\overline{F} = g(x)\widetilde{F} + \frac{f(y)}{F}\widetilde{F}$ a metric obtained by a multiplication of the projectively flat metric $\overline{F}$ with the 0-homogeneous function $g(x) + \frac{f(y)}{F}$, where g and f are considered such that $\widetilde{F}$ is positive. Then $\overline{F}$ is projectively flat if and only if the following relation is satisfied

$$\widetilde{F} d_{h_0} g - S_0 g d_J \widetilde{F} - \frac{S_0 f d_J \widetilde{F}}{F} + \frac{\widetilde{F} S_0 f d_J F}{F^2} - \frac{S_0 \widetilde{F} d_J f}{F} + \frac{f S_0 \widetilde{F} d_J F}{F^2} + \frac{\widetilde{F} S_0 F d_J f}{F^2} + \frac{f S_0 F d_J \widetilde{F}}{F^2} - \frac{2 f \widetilde{F} S_0 F d_J F}{F^3} = 0. \quad (23)$$

New Finsler metric of zero flag curvature

$$P = \frac{S_0\overline{F}}{2\overline{F}} = \frac{S_0 g \cdot \widetilde{F} + g S_0 \widetilde{F} + \frac{S_0 f}{F}\widetilde{F} + \frac{f S_0 \widetilde{F}}{F} - f\frac{\widetilde{F}}{F^2} S_0 F}{2\left(g\widetilde{F} + \frac{f}{F}\widetilde{F}\right)} \tag{24}$$

We recall that F and $\widetilde{F}$ are two projectively flat Finsler metrics for which $S_0 F = 2cF^2$ and $S_0\widetilde{F} = 4cF\widetilde{F}$.
Therefore, (24) becomes:

$$P = \frac{S_0 g \cdot \widetilde{F} + \frac{S_0 f}{F}\widetilde{F} - 2cf\widetilde{F}}{2\left(g\widetilde{F} + \frac{f}{F}\widetilde{F}\right)} + 2cF. \tag{25}$$

New Finsler metric of zero flag curvature

Taking into account the conditions imposed on the functions f and g it follows that we can make the following extra assumption

$$S_0 g = 2cf \text{ and } S_0 f = 0. \tag{26}$$

With the assumptions considered in (26) we get that the projective factor associated is

$$P = 2cF, \tag{27}$$

which is a Hamel function. We can write now the expression for the new metric as follows

$$\begin{aligned} \overline{F} = {} & \frac{\eta(\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)} + 2c\langle x, y\rangle)^2}{(1 - 4c^2|x|^2)^2\sqrt{|y|^2 - 4c^2\left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}} \\ & \cdot \left(2c\langle a, x\rangle + e + \frac{\langle a, y\rangle}{\frac{\sqrt{|y|^2 - 4c^2(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - 4c^2|x|^2} + \frac{2c\langle x, y\rangle}{1 - 4c^2|x|^2}} \right). \end{aligned} \tag{28}$$

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Thank you