Bochner-Simons Formulas And The Rigidity Of Biharmonic Submanifolds

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Simons type formulas

(Simons - 1968) The expression of the Laplacian of the squared norm of the second fundamental form of a minimal submanifold.
For minimal hypersurfaces in $\mathbb{S}^{m+1}$:

$$\frac{1}{2}\Delta|A|^2 = -|\nabla A|^2 - |A|^2(m - |A|^2)$$

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(Cheng, Yau - 1977) A general Simons type formula for Codazzi tensors on M^m:

$$\frac{1}{2}\Delta|S|^2 = -|\nabla S|^2 - \langle S, \mathrm{Hess}(\mathrm{trace}\, S)\rangle - \frac{1}{2}\sum_{i,j=1}^{m} R_{ijij}(\lambda_i - \lambda_j)^2$$

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With $S = A$ this equation recovers Nomizu and Smyth's result as well as the Simons' one.

Harmonic and biharmonic maps

Let $\phi : (M, g) \to (N, h)$ be a smooth map.

Energy functional

$$E(\phi) = E_1(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\phi) = \tau_1(\phi) &= \operatorname{trace}_g \nabla d\phi \\ &= 0 \end{aligned}$$

Critical points of E:
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Bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau_2(\phi) &= -\Delta^\phi \tau(\phi) - \operatorname{trace}_g R^N(d\phi, \tau(\phi))d\phi \\ &= 0 \end{aligned}$$

Critical points of E_2:
biharmonic maps

The biharmonic equation

(Jiang - 1986)

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \operatorname{trace}_g R^N(d\phi, \tau(\phi))d\phi = 0$$

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$$\Delta^\phi = -\operatorname{trace}_g \left(\nabla^\phi \nabla^\phi - \nabla^\phi_\nabla\right)$$

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- it is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is proper-biharmonic
- a submanifold $\phi : M \to N$ is a biharmonic submanifolds if the immersion ϕ is biharmonic

Biharmonic submanifolds

Theorem (Balmuş, Montaldo, Oniciuc - 2012)

A hypersurface Σ^m in a Riemannian manifold N is biharmonic iff

$$\Delta f + f|A|^2 - f\,\mathrm{Ricci}^N(\eta,\eta) = 0$$

and

$$2A(\mathrm{grad} f) + mf\,\mathrm{grad} f - 2f(\mathrm{Ricci}^N(\eta))^\top = 0,$$

where $f = (1/m)\,\mathrm{trace}\,A$ is the mean curvature function, $H = f\eta$, and $(\mathrm{Ricci}^N(\eta))^\top$ is the tangent component of the Ricci curvature of N in the direction of η.

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Remark

CMC hypersurfaces M^m in $\mathbb{S}^{m+1}$ are biharmonic iff $|A|^2 = m$. Therefore, their classification is a natural goal after Chern, do Carmo, and Kobayashi's classification of minimal hypersurfaces with $|A|^2 = m$.

Conjectures on biharmonic submanifolds

Conjecture (Balmuş, Montaldo, Oniciuc - 2012)

Proper-biharmonic submanifolds of $\mathbb{S}^n$ are CMC.

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The only proper-biharmonic hypersurfaces of $\mathbb{S}^{m+1}$ are (open parts of) either hyperspheres $\mathbb{S}^m(1/\sqrt{2})$ or standard products of spheres $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Biconservative submanifolds

Let $\phi : M \to N$ be a smooth map between Riemannian manifolds.
The stress-energy tensor S_2 of the bienergy:

$$S_2(X,Y) = \frac{1}{2}|\tau(\phi)|^2\langle X,Y\rangle + \langle d\phi, \nabla\tau(\phi)\rangle\langle X,Y\rangle - \langle d\phi(X), \nabla_Y\tau(\phi)\rangle - \langle d\phi(Y), \nabla_X\tau(\phi)\rangle$$

We have

$$\operatorname{div} S_2 = -\langle \tau_2(\phi), d\phi\rangle.$$

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A submanifold $\phi : M \to N$ of a Riemannian manifold N is called biconservative if $\operatorname{div} S_2 = 0$, i.e., $\tau_2(\phi)^\top = 0$.

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For hypersurfaces M^m of space forms:

$$S_2 = -\frac{m^2 f^2}{2} I + 2mfA.$$

Moreover, $\nabla S_2 = 0$ iff $\nabla A = 0$.

A Simons type formula

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Theorem (F., Loubeau, Oniciuc - 2018)

Let $\phi : M^m \to N^{m+1}(c)$ be a hypersurface in a space form. Then

$$\begin{aligned}\frac{1}{2}\Delta|S_2|^2 =& 4cm^4f^4 - 4m^3f^3(\operatorname{trace} A^3) - 4m^2f^2|A|^2(cm - |A|^2)\\ & - 8m^4f^2|\operatorname{grad} f|^2 - 4m^2f^2|\nabla A|^2 + 4m^2f\langle \operatorname{grad} s, \operatorname{grad} f\rangle\\ & - 8m^2 \operatorname{div}(f \operatorname{Ricci}(\operatorname{grad} f)) - 2m^2 \operatorname{div}\left(|A|^2 \operatorname{grad} f^2\right)\\ & + \frac{m^5}{8}\Delta f^4 - 4cm^2(m-1)\Delta f^2 - 10m^2 f\langle \tau_2^\top(\phi), \operatorname{grad} f\rangle\\ & - 4m^2f^2 \operatorname{div}\left(\tau_2^\top(\phi)\right) - 2\left|\tau_2^\top(\phi)\right|^2 + 4mf\langle \nabla\tau_2^\top(\phi), A\rangle.\end{aligned}$$

Theorem (F., Loubeau, Oniciuc - 2018)

Let $\phi : M^m \to N^{m+1}(c)$ be a constant-scalar-curvature biconservative hypersurface in a space form. Then

$$\frac{3m^2}{2}\Delta f^4 = 4f^2\left\{cm^2f^2 - mf(\operatorname{trace} A^3) - |A|^2(cm - |A|^2) - 2m^2|\operatorname{grad} f|^2 - |\nabla A|^2\right\}.$$

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Corollary

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a biharmonic hypersurface with constant scalar curvature. Then the following equations hold

$$\frac{3m^2}{2}\Delta f^4 = 4f^2\left\{m^2f^2 - mf(\operatorname{trace} A^3) - |A|^2(m - |A|^2) - 2m^2|\operatorname{grad} f|^2 - |\nabla A|^2\right\}$$

and

$$\Delta f = f(m - |A|^2).$$

Rigidity results

Theorem (F., Loubeau, Oniciuc - 2018)

Let $\phi : M^m \to N^{m+1}(c)$ be a compact biconservative hypersurface in a space form $N^{m+1}(c)$, with $c \in \{-1, 0, 1\}$. If M is not minimal, has constant scalar curvature and $\mathrm{Riem}^M \geq 0$, then M is either

1. $\mathbb{S}^m(r)$, $r > 0$, *if* $c \in \{-1, 0\}$; *or*
2. $\mathbb{S}^m(r)$, $r \in (0, 1)$, *or the product* $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$, *where* $r_1^2 + r_2^2 = 1$, $m_1 + m_2 = m$, *and* $r_1 \neq \sqrt{m_1/m}$, *if* $c = 1$.

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Corollary

Let $\phi : M^m \to \mathbb{S}^{m+1}$ be a compact proper-biharmonic hypersurface with constant scalar curvature and $\mathrm{Riem}^M \geq 0$. Then M is either $\mathbb{S}^m(1/\sqrt{2})$ or the product $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

A Bochner formula

(Mok, Siu, Yeung - 1993) A non-linear Bochner type formula, involving the 4-tensor defined on a Riemannian manifold (M,g):

$$Q(X,Y,Z,W) = \langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle,$$

the map

$$\sigma_{24}(X,Y,Z,W) = (X,W,Z,Y),$$

and, given a symmetric $(1,1)$-tensor S, the 1-form θ defined as the contraction $C((Q\circ\sigma_{24})\otimes g^*, \nabla S\otimes S))$, where g^* is the dual of g.

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Theorem (F., Loubeau, Oniciuc - 2018)

On a Riemannian manifold M with curvature tensor R we have

$$\operatorname{div}\theta = \langle T,S\rangle + |\operatorname{div} S|^2 - |\nabla S|^2 + \frac{1}{2}|W|^2,$$

where $T(X) = -\operatorname{trace}(RS)(\cdot,X,\cdot)$ and $W(X,Y) = (\nabla_X S)Y - (\nabla_Y S)X$.

More rigidity results

Proposition (F., Loubeau, Oniciuc - 2018)

Let $\phi : M^m \to \mathbb{S}^{m+1}$ *be a compact proper-biharmonic hypersurface with* $\mathrm{Riem}^M \geq 0$*, such that*

$$f^2|\nabla A|^2 - |A|^2|\operatorname{grad} f|^2 + |A|^2(m - |A|^2)f^2 \geq 0.$$

Then M *is either* $\mathbb{S}^m(1/\sqrt{2})$ *or the product* $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$*,* $m_1 + m_2 = m$*,* $m_1 \neq m_2$*.*

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Corollary

Let $\phi : M^m \to \mathbb{S}^{m+1}$ *be a compact proper-biharmonic hypersurface with* $\mathrm{Riem}^M \geq 0$, *such that*

$$\left(\frac{m^2(m+26)}{4(m-1)}f^2 - |A|^2\right)|\operatorname{grad} f|^2 + |A|^2(m - |A|^2)f^2 \geq 0.$$

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Let $\phi : M^m \to N^n(c)$ *be a compact PNMC biconservative submanifold in a space form with* $\mathrm{Riem}^M \geq 0$ *and* $m \leq 10$*. Then* M *is a PMC submanifold and* $\nabla A_H = 0$*.*

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Corollary

Let $\phi : M^m \to N^{m+1}(c)$ be a compact biconservative hypersurface in a space form such that its mean curvature does not vanish at any point, $\mathrm{Riem}^M \geq 0$, and $m \leq 10$. Then M is one of the hypersurfaces in The Rigidity of Biconservative Hypersurfaces Theorem.

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D. Fetcu, E. Loubeau, and C. Oniciuc, *Bochner-Simons formulas and the rigidity of biharmonic submanifolds*, `arXiv:1801.07879`.