Complete biconservative surfaces in the hyperbolic space $\mathbb{H}^3$

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Iaşi, April 30, 2020

Article

- S. Nistor, C. Oniciuc, *Complete biconservative surfaces in the hyperbolic space* $\mathbb{H}^3$, Nonlinear Anal. 198 (2020) 111860.

Outline

1. Introducing the biconservative submanifolds

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2. Biconservative surfaces in 3-dimensional space forms
   - Local and global intrinsic results in $\mathbb{H}^3$
   - Local and global extrinsic results in $\mathbb{H}^3$

Biharmonic maps

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. Assume that M is compact and consider

- Bienergy functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \qquad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

- Euler-Lagrange equation

$$\begin{aligned} \tau_2(\varphi) &= -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0. \end{aligned}$$

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The biharmonic equation (G.Y. Jiang, 1986)

$$\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0,$$

where

$$\Delta^{\varphi} = -\operatorname{trace}_g\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

is the rough Laplacian on sections of $\varphi^{-1}TN$ and

$$R^N(X,Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z.$$

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The stress-bienergy tensor

- G.Y. Jiang, 1987 defined the stress-energy tensor S_2 for the bienergy functional, and called it the stress-bienergy tensor:

$$\begin{aligned}\langle S_2(X),Y\rangle =& \frac{1}{2}|\tau(\varphi)|^2\langle X,Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X,Y\rangle \\ & - \langle d\varphi(X), \nabla_Y\tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X\tau(\varphi)\rangle.\end{aligned}$$

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If φ is a submersion, $\operatorname{div} S_2 = 0$ if and only if φ is biharmonic.

If $\varphi : M \to N$ is an isometric immersion then $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$. In general, for an isometric immersion, $\operatorname{div} S_2 \neq 0$.

Biharmonic and biconservative submanifolds

Definition 3.1

A submanifold $\varphi : M^m \to N^n$ is called biharmonic if φ is a biharmonic map, i.e., $\tau_2(\varphi) = 0$.

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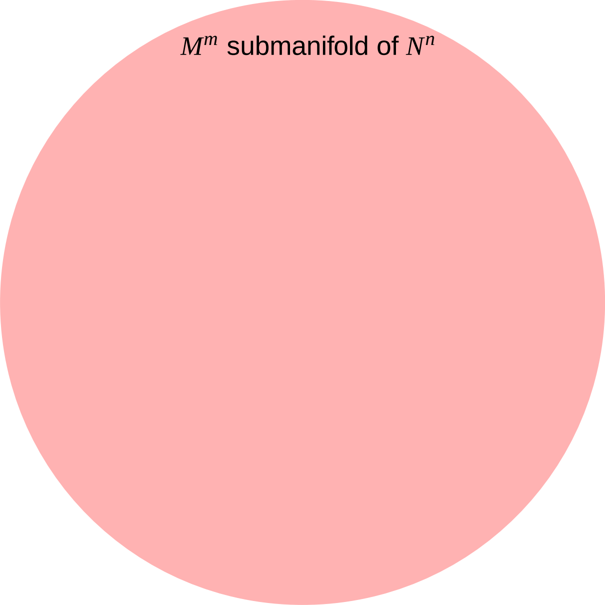
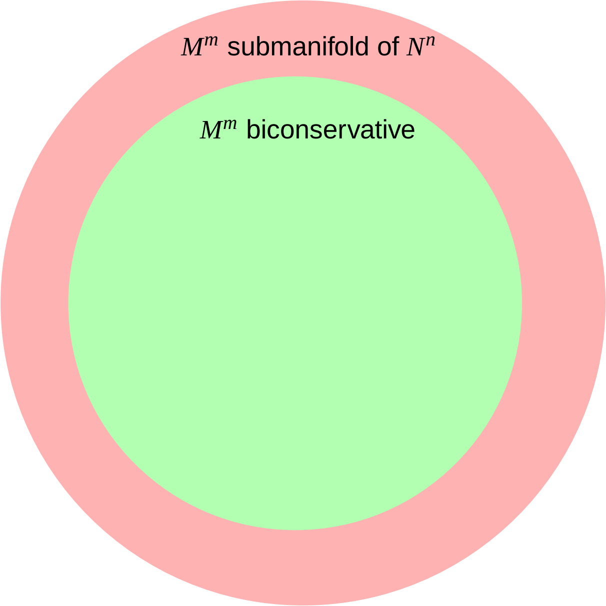
Definition 3.2

A submanifold $\varphi : M^m \to N^n$ is called biconservative if $\operatorname{div} S_2 = 0$, i.e., $\tau_2(\varphi)^\top = 0$.


M^m submanifold of N^n


M^m submanifold of N^n
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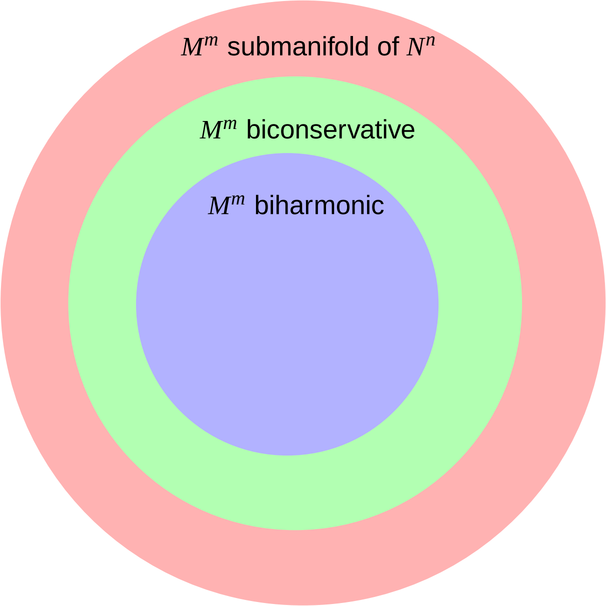
M^m submanifold of N^n
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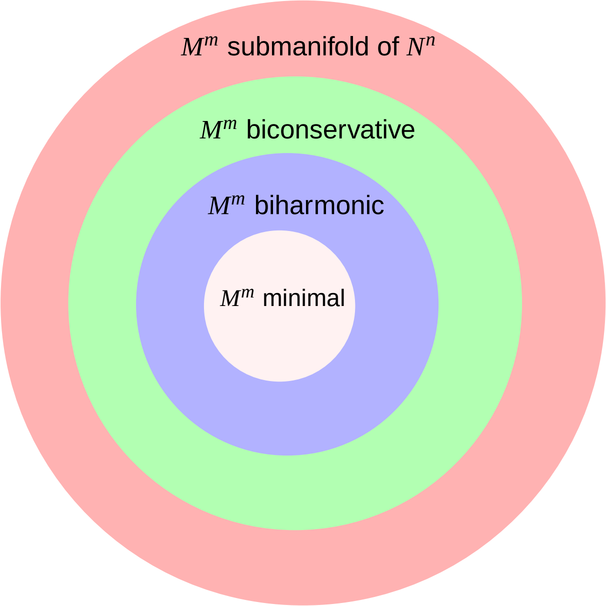
M^m submanifold of N^n
M^m biconservative
M^m biharmonic
M^m minimal

Characterization results

Proposition 3.3

A hypersurface $\varphi : M^m \to N^{m+1}(c)$ is biconservative if and only if

$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f.$$

where A is the shape operator of M and $f = \operatorname{trace} A$ is its mean curvature function.

- Every CMC hypersurface in $N^{m+1}(c)$ is biconservative.

Outline

Biconservative surfaces in $N^3(c)$

Let $\varphi : M^2 \rightarrow N^3(c)$ be a non-CMC biconservative surface.

Local results

Global results

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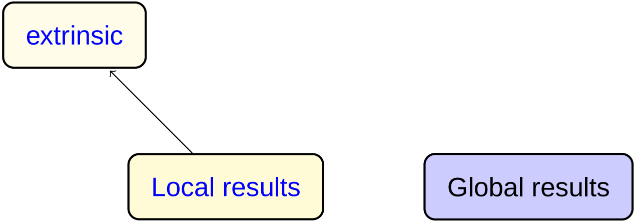

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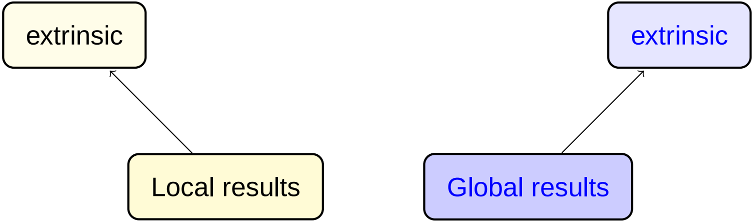

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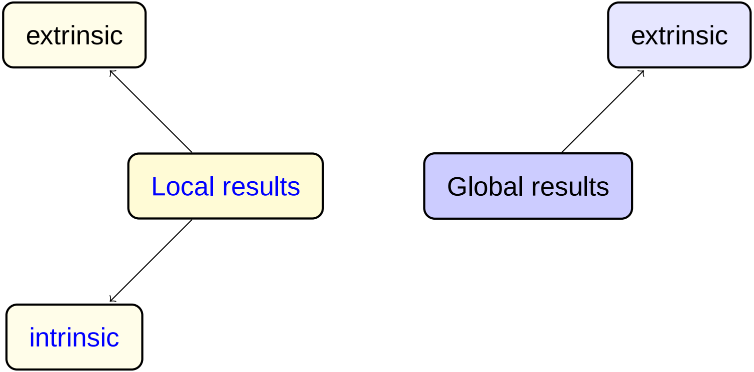

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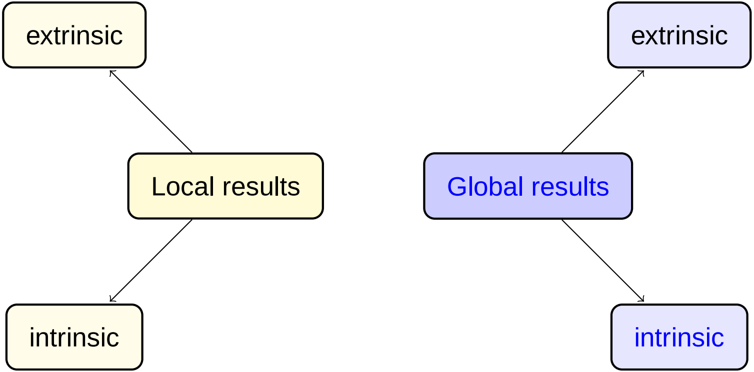

Biconservative surfaces in $N^3(c)$

Local conditions

- extrinsic $\longleftrightarrow$ $\operatorname{grad} f \neq 0$ on M
- intrinsic $\longleftrightarrow$ $c - K > 0$ on M, $\operatorname{grad} K \neq 0$ on M, and the level curves of K are certain circles

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Local conditions → intrinsic ⟷ $c - K > 0$ on M, $\operatorname{grad} K \neq 0$ on M, and the level curves of K are certain circles

Global conditions ⟷ (M, g) complete and the above properties hold on an open and dense subset of M

Local results

Theorem 4.1 ([Caddeo, Montaldo, Oniciuc, Piu – 2014])

Let $\varphi : M^2 \to N^3(c)$ be a biconservative surface with $\operatorname{grad} f \neq 0$ at any point of M. Then the Gaussian curvature K satisfies

(i) *the extrinsic condition*

$$K = \det A + c = -\frac{3f^2}{4} + c; \tag{1}$$

(ii) *the intrinsic conditions $c - K > 0$, $\operatorname{grad} K \neq 0$ on M, and its level curves are circles in M with constant curvature*

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)};$$

(iii)

$$(c-K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3}K(c-K)^2 = 0, \tag{2}$$

where Δ is the Laplace-Beltrami operator on M.

Local intrinsic characterization

Theorem 4.2 ([Fetcu, N., Oniciuc – 2016])

Let (M^2,g) *be an abstract surface and* $c\in\mathbb{R}$ *a constant. Then,* M *can be locally isometrically embedded in* $N^3(c)$ *as a biconservative surface with* $\operatorname{grad} f\neq 0$ *everywhere if and only if it satisfies the local intrinsic conditions* $c-K>0$, $\operatorname{grad} K\neq 0$, *at any point, and its level curves are circles in* M *with constant curvature*

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- We note that unlike in the minimal immersions case, if M satisfies the hypotheses from above theorem, then there exists a unique biconservative immersion in $N^3(c)$ (up to an isometry of $N^3(c)$), and not a one-parameter family.

Local intrinsic results in $\mathbb{N}^3(c)$

Theorem 4.3 ([N., Oniciuc – 2017, N. – 2017])

Let (M^2, g) be an abstract surface with Gaussian curvature K satisfying $c - K(p) > 0$ and $(\operatorname{grad} K)(p) \neq 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Then, the level curves of K are circles in M with constant curvature $\kappa = 3|\operatorname{grad} K|/(8(c-K))$ if and only if one of the following equivalent conditions holds

(i) *locally, $g = e^{2\sigma}\left(du^2 + dv^2\right)$, $\sigma = \sigma(u)$ satisfies $\sigma'' = e^{-2\sigma/3} - ce^{2\sigma}$ and $\sigma' > 0$;*

$$u(\sigma) = \int_{\sigma_0}^{\sigma} \frac{d\tau}{\sqrt{-3e^{-2\tau/3} - ce^{2\tau} + a}} + u_0, \quad \sigma, \sigma_0 \in I, a, u_0 \in \mathbb{R};$$

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(ii) $X_2(X_1 K) = 0$ *and* $\nabla_{X_2} X_2 = \frac{-3X_1 K}{8(c-K)} X_1$, *where $X_1 = \operatorname{grad} K / |\operatorname{grad} K|$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$;*

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where a *and* u_0 *are real constants and* I *is an open interval. Then* $\left(M^2, g\right)$ *is isometric to*

$$\left(D_{C_{-1}}, g_{C_{-1}}\right) = \left((0,\xi_{01}) \times \mathbb{R}, g_{C_{-1}}(\xi,\theta) = \frac{1}{\xi^2}\left(\frac{3}{-\xi^{8/3} + C_{-1}\xi^2 + 3} d\xi^2 + d\theta^2\right)\right),$$

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$\left(D_{C_{-1}}, g_{C_{-1}}\right)$ is called abstract standard biconservative surface.

Local intrinsic results in $\mathbb{H}^3$

Remarks:

- We note that
$$\lim_{\xi \searrow 0} \left| \frac{\partial}{\partial \xi} \right|^2 = \lim_{\xi \nearrow \xi_{01}} \left| \frac{\partial}{\partial \xi} \right|^2 = \infty,$$
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- The surface $\left(D_{C_{-1}},g_{C_{-1}}\right)$ is not complete since the geodesic $\theta=\theta_0$ cannot be defined on the whole $\mathbb{R}$ but only on a half line (when its arc-length parameter goes to $-\infty$ it approaches the boundary given by $\xi=0$).

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Changes of coordinates

$$g_{C_{-1}}(\xi,\theta)=\frac{1}{\xi^2}\left(\frac{3}{-\xi^{8/3}+C_{-1}\xi^2+3}d\xi^2+d\theta^2\right),\quad (\xi,\theta)\in(0,\xi_{01})\times\mathbb{R}$$

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$$\rho(\xi)=-\int_{\xi_{00}}^{\xi}\sqrt{\frac{3}{\tau^2\left(-\tau^{8/3}+C_{-1}\tau^2+3\right)}}\,d\tau$$

$$g_{C_{-1}}(\rho,\theta)=\tilde{h}^2(\rho)d\theta^2+d\rho^2,\quad (\rho,\theta)\in(\rho_1,\infty)\times\mathbb{R},$$

$$\tilde{h}(\rho)=\frac{1}{\xi(\rho)},\rho_1\in\mathbb{R}_-$$

Changes of coordinates

$$g_{C_{-1}}(\xi,\theta)=\frac{1}{\xi^2}\left(\frac{3}{-\xi^{8/3}+C_{-1}\xi^2+3}d\xi^2+d\theta^2\right),\quad (\xi,\theta)\in(0,\xi_{01})\times\mathbb{R}$$

$$\rho(\xi)=-\int_{\xi_{00}}^{\xi}\sqrt{\frac{3}{\tau^2\left(-\tau^{8/3}+C_{-1}\tau^2+3\right)}}\,d\tau$$

$$g_{C_{-1}}(\rho,\theta)=\tilde{h}^2(\rho)d\theta^2+d\rho^2,\quad (\rho,\theta)\in(\rho_1,\infty)\times\mathbb{R},$$

$$\tilde{h}(\rho)=\frac{1}{\xi(\rho)},\rho_1\in\mathbb{R}_-$$

$$\omega(\rho)=\rho-\rho_1$$

$$g_{C_{-1}}(\omega,\theta)=h^2(\omega)d\theta^2+d\omega^2,\quad (\omega,\theta)\in(0,\infty)\times\mathbb{R},$$

$$h(\omega)=\tilde{h}(\rho(\omega)),$$

Local intrinsic result

$$g_{C_{-1}}(\omega,\theta) = h^2(\omega)d\theta^2 + d\omega^2, \quad (\omega,\theta) \in (0,\infty)\times\mathbb{R}$$

Remark

We note that

$$\lim_{\omega \searrow 0} \left| \frac{\partial}{\partial \theta} \right|^2 = \frac{1}{\xi_{01}^2} \in \mathbb{R}_+^*,$$

and thus, the metric $g_{C_{-1}}$ can be smoothly extended to the boundary $\omega = 0$.

Global intrinsic result

Theorem 4.5

The surface $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}(\omega,\theta) = \Gamma^2(\omega)d\theta^2 + d\omega^2\right)$ *is complete, where the function* $\Gamma : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\Gamma(\omega) = \begin{cases} h(\omega), & \omega > 0 \\ \frac{1}{\xi_{01}}, & \omega = 0 \\ h(-\omega), & \omega < 0 \end{cases} . \tag{5}$$

Global intrinsic result

Theorem 4.5

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- $\left(\operatorname{grad}\tilde{K}_{C_{-1}}\right)(0,\theta) = 0$, for any $\theta \in \mathbb{R}$.

Global intrinsic result

Theorem 4.5

The surface $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}(\omega,\theta) = \Gamma^2(\omega)d\theta^2 + d\omega^2\right)$ *is complete, where the function* $\Gamma : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\Gamma(\omega) = \begin{cases} h(\omega), & \omega > 0 \\ \frac{1}{\xi_{01}}, & \omega = 0 \\ h(-\omega), & \omega < 0 \end{cases} . \qquad (5)$$

- $\left(\operatorname{grad} \tilde{K}_{C_{-1}}\right)(0,\theta) = 0$, for any $\theta \in \mathbb{R}$.
- As $\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = 0$ along the boundary of $\left((0,\infty) \times \mathbb{R}, g_{C_{-1}}\right)$ it follows that its boundary $\theta \to (0,\theta)$ becomes a geodesic in $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$.

Global intrinsic result

As the complete surface $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ satisfies $(\operatorname{grad} \tilde{K}_{C_{-1}})(0, \theta) = 0$, for any $\theta \in \mathbb{R}$, the existence of a (non-*CMC*) biconservative immersion from $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ in $\mathbb{H}^3$ is not guaranteed.

Global intrinsic result

As the complete surface $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ satisfies $(\operatorname{grad} \tilde{K}_{C_{-1}})(0,\theta) = 0$, for any $\theta \in \mathbb{R}$, the existence of a (non-CMC) biconservative immersion from $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ in $\mathbb{H}^3$ is not guaranteed.

Our aim is to construct such an immersion!!!!

Global intrinsic result

As the complete surface $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ satisfies $(\operatorname{grad} \tilde{K}_{C_{-1}})(0,\theta) = 0$, for any $\theta \in \mathbb{R}$, the existence of a (non-CMC) biconservative immersion from $\left(\mathbb{R}^2, \tilde{g}_{C_{-1}}\right)$ in $\mathbb{H}^3$ is not guaranteed.

Our aim is to construct such an immersion!!!!

- We will omit writing the index C_{-1} in the following construction.

Global intrinsic result

Let us consider two surfaces

$$^{1}g(\omega,\theta)=h^2(\omega)d\theta^2+d\omega^2,\qquad (\omega,\theta)\in(0,\infty)\times\mathbb{R}$$

and

$$^{2}g(\omega,\theta)=h^2(-\omega)d\theta^2+d\omega^2,\qquad (\omega,\theta)\in(-\infty,0)\times\mathbb{R}.$$

Let

$$^{1}X_1=\frac{\operatorname{grad}\, {}^{1}K}{|\operatorname{grad}\, {}^{1}K|},\qquad {}^{2}X_1=\frac{\operatorname{grad}\, {}^{2}K}{|\operatorname{grad}\, {}^{2}K|},$$

be two vector fields defined on $(0,\infty)\times\mathbb{R}$, respectively on $(-\infty,0)\times\mathbb{R}$. One obtains:

$$^{1}X_1=\frac{\partial}{\partial\omega}\ \text{and}\ {}^{2}X_1=-\frac{\partial}{\partial\omega}$$

on $(0,\infty)\times\mathbb{R}$, respectively on $(-\infty,0)\times\mathbb{R}$.

Global intrinsic result

Let us consider two surfaces

$$^1g(\omega,\theta) = h^2(\omega)d\theta^2 + d\omega^2, \qquad (\omega,\theta) \in (0,\infty) \times \mathbb{R}$$

and

$$^2g(\omega,\theta) = h^2(-\omega)d\theta^2 + d\omega^2, \qquad (\omega,\theta) \in (-\infty,0) \times \mathbb{R}.$$

Let

$$^1X_1 = \frac{\operatorname{grad} {}^1K}{|\operatorname{grad} {}^1K|}, \qquad {}^2X_1 = \frac{\operatorname{grad} {}^2K}{|\operatorname{grad} {}^2K|},$$

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on $(0,\infty) \times \mathbb{R}$, respectively on $(-\infty,0) \times \mathbb{R}$.

$X_1 = \frac{\partial}{\partial\omega}$ on $\mathbb{R}^2$

Global intrinsic result

Let us consider two surfaces

$$^{1}g(\omega,\theta)=h^2(\omega)d\theta^2+d\omega^2,\qquad (\omega,\theta)\in(0,\infty)\times\mathbb{R}$$

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$X_1=\frac{\partial}{\partial\omega}$ on $\mathbb{R}^2$ ⟶ $\{X_1,X_2\}$ positive orthonormal frame field on $\mathbb{R}^2$ ⟶ $X_2=\frac{1}{\Gamma(\omega)}\frac{\partial}{\partial\theta}$ on $\mathbb{R}^2$

Global intrinsic result

Proposition 4.6

Let $\left(\mathbb{R}^2, \tilde{g}\right)$ the above complete surface. Then, the Gaussian curvature $\tilde{K}$ of $\left(\mathbb{R}^2, \tilde{g}\right)$ satisfies $-1-\tilde{K}>0$ at any point, and the vector fields X_1 and X_2 defined above, satisfy on $\mathbb{R}^2$

$$\nabla_{X_1}X_1=\nabla_{X_1}X_2=0, \quad \nabla_{X_2}X_2=-\frac{3X_1\tilde{K}}{8\left(-1-\tilde{K}\right)}X_1, \quad \nabla_{X_2}X_1=\frac{3X_1\tilde{K}}{8\left(-1-\tilde{K}\right)}X_2.$$

Global intrinsic result

Proposition 4.6

Let $\left(\mathbb{R}^2,\tilde{g}\right)$ *the above complete surface. Then, the Gaussian curvature* $\tilde{K}$ *of* $\left(\mathbb{R}^2,\tilde{g}\right)$ *satisfies* $-1-\tilde{K}>0$ *at any point, and the vector fields* X_1 *and* X_2 *defined above, satisfy on* $\mathbb{R}^2$

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Theorem 4.7

Let $\left(\mathbb{R}^2,\tilde{g}\right)$ *the above complete surface. Then, there exists a unique biconservative immersion* $\Phi:\left(\mathbb{R}^2,\tilde{g}\right)\to\mathbb{H}^3$*. Moreover,* $\operatorname{grad} f\neq 0$ *at any point of* $\mathbb{R}^*\times\mathbb{R}$*, where* f *is the mean curvature function of the immersion* Φ*.*

Proof:

$A : C\left(T\mathbb{R}^2\right) \to C\left(T\mathbb{R}^2\right)$ defined by

$$A\left(X_1\right) = -\frac{\sqrt{-1-\tilde{K}}}{\sqrt{3}}X_1, \qquad A\left(X_2\right) = \sqrt{3\left(-1-\tilde{K}\right)}X_2.$$

Proof:

$A : C\left(T\mathbb{R}^2\right) \to C\left(T\mathbb{R}^2\right)$ defined by

$$A\left(X_1\right) = -\frac{\sqrt{-1-\tilde{K}}}{\sqrt{3}}X_1, \qquad A\left(X_2\right) = \sqrt{3\left(-1-\tilde{K}\right)}X_2.$$

A satisfies:

- The Gauss equation $\det A = 1 + \tilde{K}$;
- The Codazzi equation $\left(\nabla_{X_1} A\right)\left(X_2\right) = \left(\nabla_{X_2} A\right)\left(X_1\right)$;

Proof:

$A : C\left(T\mathbb{R}^2\right) \to C\left(T\mathbb{R}^2\right)$ defined by

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From the fundamental theorem of surfaces in $\mathbb{H}^3$, it follows that there exists an unique isometric immersion $\Phi : \left(\mathbb{R}^2, \tilde{g}\right) \to \mathbb{H}^3$ such that A is its shape operator.

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$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f,$$

which shows that Φ is biconservative.

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$$A(X_1) = -\frac{\sqrt{-1-\tilde{K}}}{\sqrt{3}}X_1, \qquad A(X_2) = \sqrt{3\left(-1-\tilde{K}\right)}X_2.$$

A satisfies:

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- The Codazzi equation $(\nabla_{X_1}A)(X_2) = (\nabla_{X_2}A)(X_1)$;

From the fundamental theorem of surfaces in $\mathbb{H}^3$, it follows that there exists an unique isometric immersion $\Phi : \left(\mathbb{R}^2, \tilde{g}\right) \to \mathbb{H}^3$ such that A is its shape operator. Moreover, the operator A satisfies

$$A(\operatorname{grad} f) = -\frac{f}{2}\operatorname{grad} f,$$

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Uniqueness: we suppose that there exist two biconservative immersions Φ_1 and Φ_2 from $\left(\mathbb{R}^2, \tilde{g}\right)$ in $\mathbb{H}^3$ and using the fact that ${\Phi_1}_{|(0,\infty)\times\mathbb{R}}$ and ${\Phi_1}_{|(-\infty,0)\times\mathbb{R}}$ are biconservative and unique (up to isometries of $\mathbb{H}^3$) we can prove that Φ_1 and Φ_2 coincide.

Outline

1. Introducing the biconservative submanifolds

2. Biconservative surfaces in 3-dimensional space forms
   - Local and global intrinsic results in $\mathbb{H}^3$
   - Local and global extrinsic results in $\mathbb{H}^3$

As there exist several models for the hyperbolic space, we will consider, in each particular situation, the most appropriate model in order to obtain a complete biconservative surface.
Let us recall that the Minkowski space $\mathbb{R}^4_1$ is given by $\mathbb{R}^4_1 = \left(\mathbb{R}^4, \langle\cdot,\cdot\rangle\right)$, where $\langle\cdot,\cdot\rangle$ is the bilinear form

$$\langle x, y\rangle = \sum_{i=1}^{3} x^i y^i - x^4 y^4, \qquad x = \left(x^1, x^2, x^3, x^4\right), \quad y = \left(y^1, y^2, y^3, y^4\right).$$

The hyperboloid model is

$$\mathbb{H}^3 = \left\{ x \in \mathbb{R}^4_1 \quad : \quad \langle x, x\rangle = -1 \text{ and } x^4 > 0 \right\},$$

that is the upper part of the hyperboloid of two sheets.

Local extrinsic result; $C_{-1} > 0$

Theorem 4.8 ([Caddeo, Montaldo, Oniciuc, Piu – 2014])

Let M^2 be a biconservative surface in $\mathbb{H}^3$ with $\operatorname{grad} f \neq 0$ everywhere. If $C_{-1} > 0$, then, locally, $M^2 \subset \mathbb{R}^4_1$ can be parametrized by

$$X_{\tilde{C}_{-1}}(\kappa, v) = \left(\frac{4\cos v}{3\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}, \frac{4\sin v}{3\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}, R(\kappa)\sinh\mu(\kappa), R(\kappa)\cosh\mu(\kappa) \right), \qquad (6)$$

for any $(\kappa, v) \in (0, \kappa_{01}) \times \mathbb{R}$, where $R(\kappa) = \frac{\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}}{3\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}$ and

$$\mu(\kappa) = \pm \int_{\kappa_{00}}^{\kappa} \frac{36\sqrt{\tilde{C}_{-1}}\tau^{7/4}}{\left(9\tilde{C}_{-1}\tau^{3/2}+16\right)\sqrt{\frac{16}{9}\tau^2 - 16\tau^4 + \tilde{C}_{-1}\tau^{7/2}}}\, d\tau + c_0, \qquad c_0 \in \mathbb{R}, \quad (7)$$

for any $\kappa \in (0, \kappa_{01})$, where $\tilde{C}_{-1} > 0$ and κ_{01} is the positive vanishing point of $16\kappa^2/9 - 16\kappa^4 + \tilde{C}_{-1}\kappa^{7/2}$, $16\kappa^2/9 - 16\kappa^4 + \tilde{C}_{-1}\kappa^{7/2} > 0$, for any $\kappa \in (0, \kappa_{01})$, $\kappa_{01} > \left(3\tilde{C}_{-1}\right)^2/2^{12}$, and κ_{00} is arbitrarily fixed in $(0, \kappa_{01})$.

Local extrinsic result; $C_{-1} > 0$

- $\left(M^2, X^*_{\tilde{C}_{-1}}\langle,\rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface $\left(D_{C_{-1}}, g_{C_{-1}}\right)$, and the link between the constants C_{-1} and $\tilde{C}_{-1}$ is

$$C_{-1} = \frac{3^{3/4}}{16}\tilde{C}_{-1} > 0.$$

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$$C_{-1} = \frac{3^{3/4}}{16}\tilde{C}_{-1} > 0.$$

- The "profile curve" σ:

$$\sigma(\kappa) = \left(\frac{4}{3\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}, 0, R(\kappa)\sinh\mu(\kappa), R(\kappa)\cosh\mu(\kappa)\right),$$

for any $\kappa \in (0, \kappa_{01})$, does not have self-intersections.

Local extrinsic result; $C_{-1} > 0$

- $\left(M^2, X^*_{\tilde{C}_{-1}}\langle,\rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface $\left(D_{C_{-1}}, g_{C_{-1}}\right)$, and the link between the constants C_{-1} and $\tilde{C}_{-1}$ is

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- The immersion $X_{\tilde{C}_{-1}}$ is, in fact, an embedding, thus the image of $X_{\tilde{C}_{-1}}$ is a regular surface in $\mathbb{H}^3$. Therefore, in order to glue two standard biconservative surfaces in $\mathbb{R}^4_1$, it is enough to glue two profile curves defining them, in this way obtaining a complete biconservative regular surface in $\mathbb{H}^3$.

Global extrinsic result; $C_{-1} > 0$

Our strategy is as follows:

Since the gluing process of the curves σ implies all its components it is more convenient to chose another model for $\mathbb{H}^3$ (the upper half space) such that, after that transformation, the curve σ would have two components.
After the gluing process is performed, we will obtain a regular curve, which is a closed subset of the upper half plane and therefore, we will get a biconservative regular surface, closed in $\mathbb{H}^3$, which has to be complete.

Global extrinsic result; $C_{-1} > 0$

Using the standard diffeomorphism from hyperboloid model to upper half space model.

$$\delta\left(x^1, x^2, x^3, x^4\right) = \left(1, \frac{2x^2}{x^1+x^4}, \frac{2x^3}{x^1+x^4}, \frac{2}{x^1+x^4}\right) \tag{8}$$

the profile curve σ becomes

$$\begin{aligned}\sigma(\kappa) &= \left(1, 0, \frac{2\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\sinh\mu(\kappa)}{4+\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\cosh\mu(\kappa)}, \frac{6\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}{4+\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\cosh\mu(\kappa)}\right)\\ &\equiv \left(\frac{2\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\sinh\mu(\kappa)}{4+\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\cosh\mu(\kappa)}, \frac{6\sqrt{\tilde{C}_{-1}}\kappa^{3/4}}{4+\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\cosh\mu(\kappa)}\right).\end{aligned}$$

Choosing appropriate values of the constant c_0 and of the sign in the expression of $\mu(k)$, we can find two profile curves σ_1 and σ_2 such that we can glue them smoothly (at least of C^3 class).

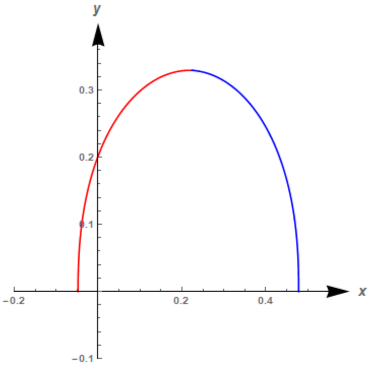

Figure 1. The profile curves σ_1 and σ_2

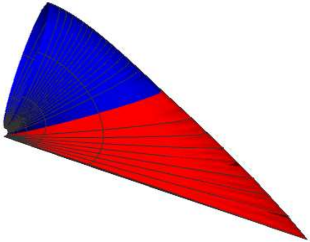

Figure 2. The corresponding surface to σ_1 and σ_2

Local extrinsic result; $C_{-1} < 0$

Theorem 4.9 ([Caddeo, Montaldo, Oniciuc, Piu – 2014])

Let M^2 *be a biconservative surface in* $\mathbb{H}^3$ *with* $\operatorname{grad} f \neq 0$ *everywhere. If* $C_{-1} < 0$*, then, locally,* $M^2 \subset \mathbb{R}^4_1$ *can be parametrized by*

$$X_{\tilde{C}_{-1}}(\kappa, v) = \left(\sqrt{2}R(\kappa)\sin\mu(\kappa) + \frac{4\cosh v}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}, \frac{4\sinh v}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}, R(\kappa)\cos\mu(\kappa), R(\kappa)\sin\mu(\kappa) + \frac{4\sqrt{2}\cosh v}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}\right), \tag{9}$$

for any $(\kappa, v) \in (0, \kappa_{01}) \times \mathbb{R}$*, where* $R(\kappa) = \frac{\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}$ *and*

$$\mu(\kappa) = \pm\int_{\kappa_{00}}^{\kappa} \frac{36\sqrt{-\tilde{C}_{-1}}\tau^{7/4}}{(9\tilde{C}_{-1}\tau^{3/2}+16)\sqrt{\frac{16}{9}\tau^2 - 16\tau^4 + \tilde{C}_{-1}\tau^{7/2}}}\, d\tau + c_0, \qquad c_0 \in \mathbb{R}, \tag{10}$$

for any $\kappa \in (0, \kappa_{01})$, $\tilde{C}_{-1} < 0$ *and* κ_{00} *arbitrarily fixed in* $(0, \kappa_{01})$*, where* κ_{01} *is the vanishing point of* $16\kappa^2/9 - 16\kappa^4 + \tilde{C}_{-1}\kappa^{7/2}$.

Local extrinsic result; $C_{-1} < 0$

- $\left(M^2, X^*_{\tilde{C}_{-1}}\langle,\rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface $\left(D_{C_{-1}}, g_{C_{-1}}\right)$, and the link between the constants C_{-1} and $\tilde{C}_{-1}$ is

$$C_{-1} = \frac{3^{3/4}}{16}\tilde{C}_{-1} < 0.$$

Local extrinsic result; $C_{-1} < 0$

- $\left(M^2, X^*_{\tilde{C}_{-1}}\langle , \rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface $\left(D_{C_{-1}}, g_{C_{-1}}\right)$, and the link between the constants C_{-1} and $\tilde{C}_{-1}$ is

$$C_{-1} = \frac{3^{3/4}}{16}\tilde{C}_{-1} < 0.$$

- The "profile curve" σ:

$$\sigma(\kappa) = \left(\sqrt{2}R(\kappa)\sin\mu(\kappa) + \frac{4}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}, 0, R(\kappa)\cos\mu(\kappa), R(\kappa)\sin\mu(\kappa) + \frac{4\sqrt{2}}{3\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}}\right),$$

for any $\kappa \in (0, \kappa_{01})$, does not have self-intersections.

Local extrinsic result; $C_{-1} < 0$

- $\left(M^2, X^*_{\tilde{C}_{-1}}\langle,\rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface $\left(D_{C_{-1}}, g_{C_{-1}}\right)$, and the link between the constants C_{-1} and $\tilde{C}_{-1}$ is

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- The immersion $X_{\tilde{C}_{-1}}$ is, in fact, an embedding, thus the image of $X_{\tilde{C}_{-1}}$ is a regular surface in $\mathbb{H}^3$. Therefore, in order to glue two standard biconservative surfaces in $\mathbb{R}^4_1$, it is enough to glue two profile curves defining them, in this way obtaining a complete biconservative regular surface in $\mathbb{H}^3$.

Global extrinsic result; $C_{-1} < 0$

Using the same diffeomorphism (8) the profile curve σ becomes

$$\sigma(\kappa) = \frac{\left(2\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\cos\mu(\kappa), 6\sqrt{-\tilde{C}_{-1}}\kappa^{3/4}\right)}{\left(1+\sqrt{2}\right)\left(4+\sqrt{9\tilde{C}_{-1}\kappa^{3/2}+16}\sin\mu(\kappa)\right)}.$$

Choosing appropriate values of the constant c_0 and of the sign in the expression of $\mu(k)$, we can find two profile curves σ_1 and σ_2 such that we can glue them smoothly (at least of C^3 class).

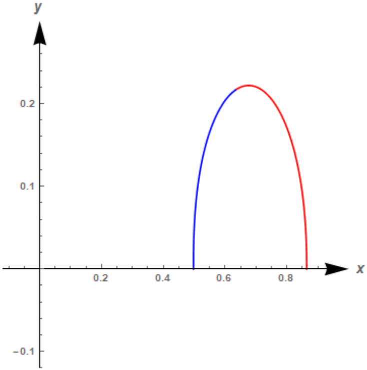

Figure 3. The profile curves σ_1 and σ_2

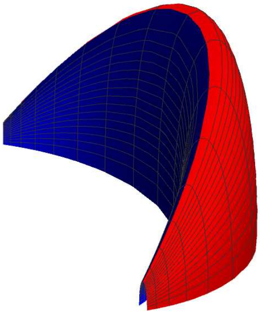

Figure 4. The corresponding surfaces to σ_1 and σ_2

Local extrinsic result; $C_{-1} = 0$

Theorem 4.10 ([Fu – 2015])

Let M^2 be a biconservative surface in $\mathbb{H}^3$ with $\operatorname{grad} f \neq 0$ everywhere. If $C_{-1} = 0$, then, locally, $M^2 \subset \mathbb{R}^4_1$ can be parametrized by

$$\begin{aligned} X(\kappa, v) \quad = \quad & \left(2^{3/4}\kappa^{3/4}\left(1 + x^2(\kappa) + v^2\right) - \frac{1}{2^{11/4}\kappa^{3/4}}, v, x(\kappa), \right. \\ & \left. 2^{3/4}\kappa^{3/4}\left(1 + x^2(\kappa) + v^2\right) + \frac{1}{2^{11/4}\kappa^{3/4}} \right), \end{aligned} \tag{11}$$

for any $(\kappa, v) \in (0, 1/3) \times \mathbb{R}$, where $x(\kappa) = \frac{1}{\kappa^{3/4}}\mu(\kappa)$ and

$$\mu(\kappa) = \pm\frac{9}{4}\int_{\kappa_{00}}^{\kappa} \frac{\tau^{3/4}}{\sqrt{1 - 9\tau^2}}\, d\tau + c_0, \tag{12}$$

with $c_0 \in \mathbb{R}$.

Local extrinsic result; $C_{-1} = 0$

- $\left(M^2, X^*\langle,\rangle\right)$ is called standard biconservative surface and is isometric to the abstract standard biconservative surface (D_0, g_0).

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- The "profile curve" σ:

$$\sigma(\kappa) = \left(y(\kappa) - \frac{1}{2^{7/4}\kappa^{3/4}}, 0, x(\kappa), y(\kappa)\right), \qquad \kappa \in \left(0, \frac{1}{3}\right)$$

where $y(\kappa) = 2^{3/4}\kappa^{3/4}\left(x^2(\kappa)+1\right) + \frac{1}{2^{11/4}\kappa^{3/4}}$, for any $\kappa \in (0, 1/3)$, does not have self-intersections.

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- The immersion X is, in fact, an embedding, thus the image of X is a regular surface in $\mathbb{H}^3$. Therefore, in order to glue two standard biconservative surfaces in $\mathbb{R}^4_1$, it is enough to glue two profile curves defining them, in this way obtaining a complete biconservative regular surface in $\mathbb{H}^3$.

Global extrinsic result; $C_{-1} = 0$

Using the same diffeomorphism (8) the profile curve σ becomes

$$\sigma(\kappa) = \frac{\left(\mu(\kappa), \kappa^{3/4}\right)}{2^{3/4}\left(\kappa^{3/2} + \mu^2(\kappa)\right)}, \qquad \kappa \in \left(0, \frac{1}{3}\right).$$

Choosing appropriate values of the constant c_0 and of the sign in the expression of $\mu(k)$, we can find two profile curves σ_1 and σ_2 such that we can glue them smoothly (at least of C^3 class).

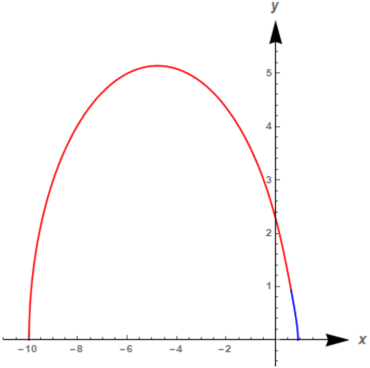

Figure 5. The profile curves σ_1 and σ_2

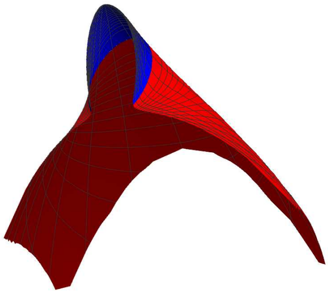

Figure 6. The corresponding surfaces to σ_1 and σ_2

Theorem 4.11

By gluing two standard biconservative surfaces along their common boundary we get a complete biconservative regular surface in $\mathbb{H}^3$. Moreover, the gradient of its mean curvature vanishes along the initial boundary which now is a geodesic of the surface.

References I

[Caddeo, Montaldo, Oniciuc, Piu – 2014] R. Caddeo, S. Montaldo, C. Oniciuc, P. Piu,
Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor, Ann. Mat. Pura Appl. (4) 193 (2014), 529–550.

[Fetcu, N., Oniciuc – 2016] D. Fetcu, S. Nistor, C. Oniciuc,
On biconservative surfaces in 3-dimensional space forms, Comm. Anal. Geom. (5) 24 (2016), 1027–1045.

[Fu – 2015] Y. Fu,
Explicit classification of biconservative surfaces in Lorentz 3-space forms, Ann. Mat. Pura Appl. (4) 194 (2015), 805–822.

[N. – 2017] S. Nistor,
Biharmonicity and biconservativity topics in the theory of submanifolds, PhD Thesis, 2017.

References II

[N., Oniciuc – 2017] S. Nistor, C. Oniciuc,
Global properties of biconservative surfaces in $\mathbb{R}^3$ *and* $\mathbb{S}^3$, Proceedings of The International Workshop on Theory of Submanifolds, Istanbul Technical University, Turkey, vol. 1 (2016), 30–56.

Thank you for your attention!