Notes

Saint-Venant's Problem for Anisotropic Circular Cylinders

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1. Introduction

In [1] a method of solving Saint-Venant's problem for inhomogeneous and anisotropic elastic beams is presented. This method points out the importance of the auxiliary generalized plane strain problems in the treatment of Saint-Venant's problem.

Special cases of Saint-Venant's problem for anisotropic elastic cylinders are considered in [2].

In this paper we consider the Saint-Venant's problem in the case of anisotropic and homogeneous circular cylinders. In the first part of the paper solutions of the auxiliary generalized plane strain problems are established. Then, extension, bending and torsion problems are solved. The solution is a polynomial of second degree in the Cartesian coordinate \( x_1 \). In the last part of the paper, we consider the case when the loading acting on one of the ends is statically equivalent to a force \( R(R_i) \) and a moment \( M(M_i) \). The solution of this problem is a polynomial of degree three in \( x_1 \).

2. Statement of the Problem

Throughout this paper a rectangular coordinate system \( Ox_k \) \((k = 1, 2, 3)\) is used. We consider a circular cylindrical beam of homogeneous and anisotropic elastic material which occupies the region \( R \) of space, whose boundary is \( S \). We suppose that the circular cylinder is bounded by plane ends perpendicular to the generators. The boundary of the generic cross-section \( \Sigma \), which is a circle of radius \( a \), is denoted by \( L \). Throughout this paper the axis \( Ox_3 \) of our coordinate system will be directed parallel to the generators of the cylinder. The cylinder is assumed to be of length \( h \), and one of its bases is taken to lie in the \( x_1 Ox_2 \)-plane, while the other is in the plane \( x_3 = h \).

Unless otherwise specified, we shall employ the usual conventions: Greek subscripts are understood to range over the integers \((1, 2)\) whereas Latin subscripts are confined to the range \((1, 2, 3)\); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.
Let $u_i$ denote the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

$$2e_{ij} = u_{i,j} + u_{j,i}.$$ 

The stress-strain relations in the case of an anisotropic elastic medium are

$$t_{ij} = C_{ijrs}e_{rs},$$

where $t_{ij}$ are components of the stress tensor and $C_{ijrs}$ are the components of the elasticity tensor which obey the symmetry relations

$$C_{ijrs} = C_{jirs} = C_{reij}.$$

The equations of equilibrium, in absence of body forces, are

$$t_{iii} = 0, \text{ in } R,$$

and the equations of compatibility are

$$e_{iji} + e_{rirsij} - e_{ir,rs} - e_{rs,ir} = 0, \text{ in } R.$$ 

The circular cylinder is supposed to be free from lateral loading, so that the conditions on the lateral surface are

$$t_{i,n} = 0,$$

where $(n_1, n_2, 0)$ are the direction cosines of the outward normal to the lateral surface.

We assume that the load of the beam is distributed over its ends in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on the end located at $x_3 = 0$ be statically equivalent to a force $R(R_i)$ and a moment $M(M_i)$ such that, for $x_3 = 0$, we have the conditions

$$\int_{\Sigma} t_{3\alpha}d\sigma = -R_\alpha,$$

$$\int_{\Sigma} t_{3\alpha}d\sigma = -R_\alpha, \quad \int_{\Sigma} x_{\alpha}t_{3\beta}d\sigma = e_{3\alpha}\beta M_\beta, \quad \int_{\Sigma} e_{3\alpha}\beta x_{i}t_{3\beta}d\sigma = -M_\beta,$$

where $\epsilon_{i\beta}$ is the alternating symbol.

The relations between $t_{ij}$ and $e_{rs}$ must be reversible, hence we can write

$$e_{ij} = S_{ijrs}t_{rs},$$

where

$$C_{ijmn}S_{mnr} = \frac{1}{2} (\delta_{ir}\delta_{j\beta} + \delta_{ir}\delta_{j\beta}).$$

We will have occasion to use four special problems $P^{(m)} (m = 1, 2, 3, 4)$ of generalized plane strain [1]. In what follows we denote by $u_i^{(m)}$, $t_{ij}^{(m)}$ the components of the displacement vector and the components of the stress tensor from the
problem $P^{(m)}$. The problems $P^{(m)}$ are characterized by the equations [1]

$$
2e^{(m)}_{ij} u_{ij}^{(m)} + u_{ij}^{(m)} = 2e^{(m)}_{ij} = u_{ij}^{(m)}, \quad e^{(m)}_{33} = 0,
$$

(7)

$$
l^{(m)}_{ij} = C_{ijrs} e^{(m)}_{rs}, \quad (m = 1, 2, 3, 4),
$$

and the boundary conditions

$$
l^{(4)}_{ai} n_a = -C_{ia33} x^4 n_a, \quad l^{(4)}_{ai} n_a = -C_{ia33} n_a, \quad l^{(4)}_{ai} n_a = e_{a3} C_{ia33} x^4 n_a, \quad \text{on } L.
$$

Let us determine the solution of the problem $P^{(m)}$ ($m = 1, 2, 3, 4$). We begin with the problem $P^{(4)}$ in which we assume that

$$
l^{(4)}_{ij} = A_{ij}^4 + A_{ij}^o x^4,
$$

where $A_{ij}^4$, $A_{ij}^o$ are unknown constants. Substituting these values in the equilibrium Eqs. (8) and in the boundary conditions (9), we obtain

$$
A_{ij}^4 = 0, \quad A_{ij}^0 = e_{3a} [C_{ij33} + (\delta_{ia} \delta_{j3} + \delta_{ia} \delta_{j3}) Q + \delta_{ia} \delta_{j3} P_a],
$$

where $Q$, $P_1$, $P_2$ are unknown constants. Since

$$
0 = e_{33}^{(4)} = S_{333} x^4 = e_{3a} (2S_{333} Q + S_{333} P_a) x^4,
$$

it results

$$
P_a = -2S_{333}^{-1} S_{a33} Q.
$$

Moreover, if we use the compatibility equations

$$
e_{11,22} + e_{22,11} = 2e_{12,12}, \quad e_{23,1} = e_{31,3},
$$

we obtain

$$
Q = - \frac{1}{2} (B_{a33})^{-1},
$$

where

$$
B_{ijrs} = S_{ijrs} - S_{ia33} S_{j3rs} S_{rs33}.
$$

As in [2] we can prove that $B_{333}$, $S_{333}$ do not vanish. Therefore, the components of the stress tensor in problem $P^{(4)}$ are given by

$$
l^{(4)}_{ij} = e_{a3} x_a [C_{ij33} + (B_{a33})^{-1} \left( - \frac{1}{2} (\delta_{ia} \delta_{j3} + \delta_{ia} \delta_{j3}) + \delta_{ia} \delta_{j3} S_{333}^{-1} S_{a33} \right)].
$$

(11)

The components of the displacement vector corresponding to (11) are given by

$$
u_a^{(4)} = (B_{a33})^{-1} \left( e_{3a} B_{a33} x^4 x^4 - \frac{1}{2} e_{3a} B_{a33} x^4 x^4 \right),
$$

$$
u_3^{(4)} = (B_{a33})^{-1} B_{a33} x^4 x^4.
$$

(12)
In a similar manner we determine the solutions of the other problems. Thus, we obtain

\[ t_{ij}(i) = -C_{ij33}x_3 + \delta_{ij} \delta_{j3} S_{3333}^{-1} y_{i3} + \varepsilon_{3,\beta3} S_{i333} S^{-1}_{3333}(\varepsilon_{3,\alpha} C_{ij33} x_3 - t_{ij}^{(4)}) , \]

\[ t_{ij}(3) = -C_{ij33} + \delta_{ij} \delta_{j3} S_{3333}^{-1} , \]

\[ u_3^{(0)} = S_{3333}^{-1} \left( S_{i333} x_3 x_3 - \frac{1}{2} S_{\eta333} x_3 x_3 \delta_{i3} - \varepsilon_{3,\beta3} S_{3333} u_3^{(4)} \right) , \]

\[ u_3^{(3)} = S_{3333}^{-1} (S_{i333} x_3 + \delta_{i3} S_{\eta33} x_3) . \]

3. Extension, Bending and Torsion

Let the loading applied on \( x_3 = 0 \) be statically equivalent to a force \( R(0, 0, R_3) \) and a moment \( M(M_1, M_2, M_3) \). Thus, for \( x_3 = 0 \) we have the following conditions

\[ \int \xi \, t_{aa} \, d\sigma = 0 , \quad \int \xi \, t_{33} \, d\sigma = -R_3 , \]

\[ \int \xi \, x_3 t_{aa} \, d\sigma = \varepsilon_{3,\beta3} M_\beta , \quad \int \xi \, \varepsilon_{3,\beta3} x_3 t_{33} \, d\sigma = -M_3 . \]

The problem consists in solving Eqs. (1)–(3) with conditions (4) and (15). The solution has the form

\[ u_3 = -\frac{1}{2} a_3 x_3^2 - a_4 \varepsilon_{3,\beta3} x_3 x_3 + S_{3333}^{-1} S_{\eta333} x_3 (a_1 x_1 + a_2 x_2 + a_3) + \]

\[ -\frac{1}{2} S_{3333}^{-1} S_{\eta333} x_3 (\varepsilon_{3,\alpha} + a_4 + \varepsilon_{3,\beta3} \alpha \beta S_{3333}^{-1} S_{3333} ) x_3 \]

\[ \cdot (B_{3333})^{-1} (a_4 + \varepsilon_{3,\beta3} \alpha \beta S_{3333}^{-1} S_{3333} ) , \]

\[ u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3 + S_{3333}^{-1} S_{\eta333} x_3 (\varepsilon_{3,\alpha} + 2a_3 x_3 S_{3333}^{-1} S_{3333} ) + \]

\[ \varepsilon_{3,\beta3} x_3 x_3 B_{3333} (B_{3333})^{-1} (a_4 + \varepsilon_{3,\beta3} \alpha \beta S_{3333}^{-1} S_{3333} ) , \]

where \( a_m (m = 1, 2, 3, 4) \) are unknown constants.

The components of the stress tensor corresponding to the displacements (16) are

\[ t_{aa} = 0 , \quad t_{aa} = -\frac{1}{2} \varepsilon_{3,\beta3} x_3 (B_{3333})^{-1} (a_4 + \varepsilon_{3,\beta3} \alpha \beta S_{3333}^{-1} S_{3333} ) , \]

\[ t_{33} = S_{3333}^{-1} (a_1 x_1 + a_2 x_2 + a_3 + \varepsilon_{3,\beta3} x_3 (S_{3333} B_{3333})^{-1} S_{3333} (a_4 + \varepsilon_{3,\beta3} \alpha \beta S_{3333}^{-1} S_{3333} ) . \]

Equilibrium Eqs. (2) and boundary conditions (4) are identically satisfied. Conditions (15) are satisfied if

\[ a_3 = \frac{4}{\alpha a^4} \varepsilon_{3,\beta3} (S_{3333} M_\beta - S_{\eta333} M_3) , \]

\[ a_3 = -\frac{1}{\alpha a^2} S_{3333} M_3 , \quad a_4 = -\frac{4}{\alpha a^4} (S_{3333} M_3 - S_{\eta333} M_3) . \]
4. Flexure

The same calculation as in the treatment of the complete problem is used in order to solve the flexure problem. For this reason we investigate the problem defined by Eqs. (1)—(3) and boundary conditions (4)—(6). The solution of this problem has the form

\[ u_3 = (b_1 x_1 + b_2 x_2 + b_3) x_3 + \frac{1}{2} \left( c_1 x_1 + c_2 x_2 + c_3 \right) x_3^2 \]

\[ + \sum_{m=1}^{4} (b_m + x_3 c_m) u_3^{(m)} + v_3(x_1, x_2), \tag{17} \]

where \( b_m, c_m \) \((m = 1, 2, 3, 4)\) are unknown constants, \( u_3^{(m)} \) \((m = 1, 2, 3, 4)\) are given by (12), (14) and \( v_3(x_1, x_2) \) are unknown functions.

Taking into account (11), (13), from (1) and (17) we obtain

\[ t_{\alpha \beta} = s_{\alpha \beta} + \sum_{m=1}^{4} c_m C_{\alpha \beta \lambda \kappa} u_k^{(m)}, \]

\[ t_{\alpha \beta} = s_{\alpha \beta} + \sum_{m=1}^{4} c_m C_{\alpha \beta \lambda \kappa} u_k^{(m)} - \frac{1}{2} \epsilon_{\alpha \beta \lambda \kappa \sigma \tau} \left( B_{\sigma \tau} \right)^{-1} \]

\[ \cdot \left[ b_4 + c_4 x_3 + \varepsilon_{34 \lambda} (b_\beta + c_\beta x_3) S_{1333} S_{3333}^{-1} \right], \tag{18} \]

where \( s_{ij} = C_{ij \kappa \lambda} \epsilon_{\kappa \lambda}, \quad 2 \varepsilon_{\alpha \beta} = v_{\alpha \beta}, \quad 2 \varepsilon_{33} = v_{33}, \quad \varepsilon_{33} = 0. \tag{19} \]

By substituting the components of the stress tensor (18) in (2) and (4), we obtain the following plane strain problem for the functions \( v_3(x_1, x_2) \)

\[ s_{\alpha i, \alpha} + f_3 = 0, \quad \text{in } \Sigma, \tag{20} \]

\[ s_{\alpha i} n_\alpha = p_\alpha, \quad \text{on } L, \tag{21} \]

where

\[ f_\beta = \sum_{m=1}^{4} c_m C_{\alpha \beta \kappa \lambda} u_k^{(m)} + \frac{1}{2} \varepsilon_{\alpha \beta \lambda \kappa \sigma \tau} \left( B_{\sigma \tau} \right)^{-1} (c_4 + \varepsilon_{34 \lambda} c_\lambda S_{1333} S_{3333}^{-1}), \]

\[ f_3 = \sum_{m=1}^{4} c_m C_{1333} u_k^{(m)} + S_{1333}^{-1} (c_4 + \varepsilon_{34 \lambda} c_\lambda S_{1333} S_{3333}^{-1}), \]

\[ p_\alpha = -\sum_{m=1}^{4} c_m C_{\alpha \beta \kappa \lambda} u_k^{(m)} n_\lambda. \]
From the necessary and sufficient condition [1] for the existence of the solution of the generalized plane strain problem (19)—(21) we obtain

\[ c_3 = 0, \quad c_4 + \varepsilon_{3ij} c_i S_{3333}^{-1} S_{3333}^{-1} = 0. \] (22)

On the basis of equilibrium Eqs. (2) and boundary conditions (4), from (5), (18) and (22) it follows

\[ c_a = -\frac{4}{\pi a^4} S_{3333} R_a, \quad c_4 = \frac{4}{\pi a^4} \varepsilon_{3ij} S_{3333} R_{ij}. \]

Let us determine the solution \( v_i \) of the generalized plane strain problem (19)—(21). In this connection we take the components of the stress tensor \( s_{ij} \) in the form

\[ s_{ij} = -S_{3333}^{-1} C_{ij3} c_4 \left( S_{6333} x_i x_j - \frac{1}{2} S_{6633} x_i x_j \delta_{ij} \right) + D_{ij} x_i + D_{ij}^p x_i x_n, \]

where \( D_{ij}, D_{ij}^p, D_{ij}^p \) are unknown constants.

The substitution of these expressions in (20) and (21) gives

\[ s_{ij} = -S_{3333}^{-1} C_{ijk} c_4 \left( S_{6333} x_i x_j x_k - \frac{1}{2} S_{6633} x_i x_j \delta_{ij} \right) + \frac{1}{2} Q_4 (3x_1^2 + 3x_2^2 - a^2) \delta_{ij} - 2x_n x_i, \]

\[ s_{ij} = -S_{3333}^{-1} C_{ijkl} c_4 \left( S_{6333} x_i x_j x_k x_l - \frac{1}{2} S_{6633} x_i x_j x_k \delta_{ij} \right) + \frac{1}{2} Q_4 (3x_1^2 + 3x_2^2 - a^2) \delta_{ij} - 2x_n x_i, \]

where \( Q_m (m = 1, 2, 3, 4) \) are unknown constants. Since

\[ 0 = e_{33} = S_{33ij} \delta_{ij}, \] (24)

the constitutive Eqs. (19) can be written in the form

\[ \varepsilon_{ij} = B_{ijmn} s_{mn}. \] (25)

From (24), (25) it follows

\[ s_{33} = -S_{3333}^{-1} C_{33j} c_4 \left( S_{6333} x_i x_j x_k - \frac{1}{2} S_{6633} x_i x_j \delta_{ij} \right) + S_{3333}^{-1} C_{33j} S_{3333} x_i x_j x_k \]

\[ - S_{3333}^{-1} (S_{3333} Q_3 + S_{3333} Q_4) (3x_1^2 + 3x_2^2 - a^2) \]

\[ + 2S_{3333}^{-1} (S_{33x_3} Q_3 x_3 x_3 + S_{33x_4} Q_4 x_4 x_3) - 2S_{3333}^{-1} S_{3333} x_3 x_4 Q_4 x_4 \]

\[ - 2S_{3333}^{-1} S_{3333} x_3 x_4 x_3 x_4 Q_4 x_4 \eta. \]

The components of the strain tensor corresponding to (23), (26) are

\[ \varepsilon_{ij} = H_{ijx} x_i x_j + 2B_{ij34} Q_3 e_{33x} x_n - a^2 (B_{ij34} Q_3 + B_{ij34} Q_4), \] (27)
where

\[ H_{ij}^{\alpha \beta} = \frac{1}{2} S_{3333}^{-1} S_{33i} (S_{33\alpha \beta} + S_{33\beta \alpha}) - \frac{1}{4} S_{3333}^{-1} (\delta_{i\beta} \delta_{j\alpha} + \delta_{i\alpha} \delta_{j\beta}) (S_{33\alpha \beta} + S_{33\beta \alpha}) \]

\[ + \frac{1}{4} S_{3333}^{-1} S_{33\alpha \beta} (\delta_{i\beta} \delta_{j\alpha} + \delta_{i\alpha} \delta_{j\beta}) + 3 (B_{ij\alpha} Q_{\alpha} + B_{ij\beta} Q_{\beta}) \delta_{i\beta} \]

\[ - (2 B_{3j\beta} Q_{\beta} + B_{3j3\alpha} Q_{\alpha} + B_{ij3\beta} Q_{\beta}) + 2 S_{3333}^{-1} B_{ij3\alpha} \delta_{i\beta} \]

\[ - S_{3333}^{-1} (B_{3j3\alpha} + B_{ij3\alpha}). \]  

(28)

If we now make use of the compatibility Eqs. (10) we get \( Q_4 = 0 \). Moreover, we obtain for the determination of the unknown constants \( Q_i \), the following system

\[ L_{ij} Q_{ij} = g_i, \]

where

\[ L_{ij} = L_{ij3} = B_{3j3\alpha} \delta_{i\beta} + 2 B_{3j3\beta}, \quad L_{33} = L_{333} = B_{3\alpha33} + 2 B_{333\alpha}, \]

\[ 1 - S_{3333}^{-1} S_{33\alpha \beta} (\delta_{i\beta} \delta_{j\alpha} + \delta_{i\alpha} \delta_{j\beta}) + 3 (B_{ij\alpha} Q_{\alpha} + B_{ij\beta} Q_{\beta}) \delta_{i\beta} \]

\[- (2 B_{3j\beta} Q_{\beta} + B_{3j3\alpha} Q_{\alpha} + B_{ij3\beta} Q_{\beta}) + 2 S_{3333}^{-1} B_{ij3\alpha} \delta_{i\beta} \]

\[- S_{3333}^{-1} (B_{3j3\alpha} + B_{ij3\alpha}). \]

We now can determine the unknown functions \( v_i(x_1, x_2) \). Thus, from (27), we obtain

\[ v_1 = -a^2 (B_{13\beta \alpha} Q_{3} + B_{13\beta \alpha} Q_{\beta}) x_\beta + \frac{1}{3} H_{11}^{11} x_1^3 + H_{11}^{12} x_1^2 x_2 + H_{11}^{12} x_1 x_2^2 \]

\[ + \frac{1}{3} (2 H_{12}^{12} - H_{12}^{12}) x_3^2, \]

\[ v_2 = -a^2 (B_{23\beta \alpha} Q_{3} + B_{23\beta \alpha} Q_{\beta}) x_\beta + \frac{1}{3} (2 H_{12}^{12} - H_{11}^{11}) x_1^3 + H_{22}^{11} x_1^2 x_2 \]

\[ + H_{22}^{12} x_1 x_2^2 + \frac{1}{3} H_{22}^{22} x_2^3, \]

\[ v_3 = -2a^2 (B_{33\beta \alpha} Q_{3} + B_{33\beta \alpha} Q_{\beta}) x_\beta + \frac{2}{3} H_{31}^{11} x_1^3 + 2 H_{31}^{12} x_1 x_2 \]

\[ + 2 H_{32}^{12} x_2^2 + \frac{2}{3} H_{32}^{22} x_2^3, \]

where \( H_{ij}^{\alpha \beta} \) are given by (28).

Taking into account the relations (18), (22), (23) and (26) from (6) it results

\[ b_3 = \frac{4}{\pi a^4} S_{3333} M_{\beta} - S_{3333} M_{\alpha}, \]

\[ b_4 = \frac{4}{\pi a^4} S_{3333} M_{\beta} - S_{3333} M_{\alpha}, \]
Thus, the semi-inverse method is used to obtain a closed-form solution for the stresses and displacements in rotating circular cylinders made of anisotropic material. This paper provides a further example of the applications of auxiliary generalized plane strain problems in the deformation of cylinders.

References


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