Saint-Venant’s problem and semi-inverse solutions in linear viscoelasticity

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Summary. This paper gives a treatment of Saint-Venant’s problem for viscoelastic cylinders by reformulating the quasi-static equilibrium equations as a formal integro-differential operator over the cross section of the cylinder, with the axial variable playing the role of a parameter. Then, the conditions are established that the solution of Saint-Venant’s problem may be treated as a generalized plane strain problem. Further, two classes of semi-inverse solutions to Saint-Venant’s problem are described. These classes are used in order to obtain a solution for the relaxed Saint-Venant’s problem.

1 Introduction

Saint-Venant’s problem consists in determining the equilibrium of an elastic cylinder loaded by surface forces distributed over its plane ends. Saint-Venant’s treatment of the foregoing problem rests on a relaxed formulation in which the detailed assignment of the terminal tractions is abandoned in favor of prescribing merely the appropriate stress resultants. An elegant analysis and important properties of the solutions of the relaxed Saint-Venant’s problem have been established recently by Ieșan [1], [2].

In this paper, we study the solutions to Saint-Venant’s problem and we describe a Saint-Venant’s procedure for the relaxed Saint-Venant’s problem rephrased for a viscoelastic cylinder. More precisely, in Section 2 we formulate Saint-Venant’s problem for a viscoelastic cylinder made of an anisotropic and inhomogeneous material. The existence and uniqueness of the solution to this three-dimensional problem may be established using the results described by Fichera [3]. However, the three-dimensional boundary value problem is less tractable for the viscoelastic cylinder. With a view towards a treatment of the problem by means of the two-dimensional problems, which are more tractable, we define in Section 3 the generalized plane strain state for a viscoelastic material. Then an analysis of Saint-Venant’s problem is given in Section 4 by means of plane cross section solutions. In fact, we treat Saint-Venant’s problem by reformulating the equilibrium equations as a formal integro-differential operator in a suitable function space over the cross section of the cylinder, with the axial variable playing the role of a parameter. Then we deduce the conditions upon the solution of Saint-Venant’s problem in order to treat it as a generalized plane strain.

Further, we point out two classes of semi-inverse solutions in the set of solutions of Saint-Venant’s problem that may be expressed in terms of a plane displacement. These classes are relevant to obtain a solution to the relaxed Saint-Venant’s problem for a viscoelastic cylinder.

The first of them, the primary solution class, corresponds to solutions of Saint-Venant’s problem for which the partial derivative with respect to the axial coordinate gives rise to a rigid displacement. The solutions of this class are characterized by the property that the cross section...
components of the resultant force about the origin for the tractions acting on the cross section are vanishing, while the axial component of the resultant force and the resultant moment field are independent of the axial coordinate. Any solution of the primary solution class is expressed in terms of four canonical displacements and depends on four arbitrary continuous functions depending only on time variable.

The second class of semi-inverse solutions consists of the solutions of Saint-Venant's problem for which the resultant force and resultant moment are independent of the axial coordinate. A solution of this class is characterized by the property that its partial derivative with respect to the axial coordinate gives rise to a primary solution. Such a solution depends on six arbitrary continuous functions depending only on the time variable.

Finally, in Section 5 we use the above classes of semi-inverse solutions in order to obtain a solution for the relaxed Saint-Venant's problem.

2 Saint-Venant's problem

We consider a prismatic cylinder $B$ with plane ends and select a rectangular system of coordinates such that one end of the cylinder lies in the $(x_1, x_2)$-coordinate plane and contains the origin $0$. We denote by $\partial B$ the boundary of $B$. We suppose that the length of the cylinder is $L$ and that $D(x_3) \subset \mathbb{R}^2$ represents the bounded cross section at distance $x_3$ from the plane end containing the origin. The boundary $\partial D$ of each cross section is assumed sufficiently smooth to admit application of the divergence theorem in the plane of cross section. The lateral boundary of the cylinder is $\pi = \partial D \times [0, L]$. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1, 2)$, whereas Latin subscripts $-$ unless otherwise specified $-$ are confined to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; superposed dots denote differentiation with respect to the time variable; where no confusion may occur, we suppress the dependence upon the spatial variables.

We assume that the body occupying $B$ is a linearly viscoelastic material that is at rest at all times $t < 0$. Let $u_i$ be the components of displacement field over $B$. Then

$$e_{ij}(u) = \frac{1}{2} (u_i,j + u_j,i)$$

are the components of the strain field associated with $u$. The stress-strain relation has the form $[4]$

$$S_{ij}(u) = G_{ijrs}(0) e_{rs} + \int_0^t G_{ijrs}(t - z) e_{rs}(z) \, dz.$$  

Here $S_{ij}(u)$ are the components of the stress field associated with $u$, while $G_{ijrs}$ stands for the components of the relaxation tensor. We assume that $G_{ijrs}$ is symmetric and that the $G_{ijrs} = G_{ijrs}(x_1, x_2; t)$

are smooth functions on $\bar{B} \times [0, \infty)$. Moreover, we assume that $G_{ijrs}(0)$ is positive-definite in $B$.

Let $s_j(u)$ be the components of the surface traction at regular points of $\partial B$ corresponding to the stress field $S_{ij}(u)$, defined by

$$s_j(u) = S_{ij}(u) n_j,$$

where $n_j$ are the components of the outward unit normal to $\partial B$. 
We call a vector field \( u \) a quasi-static equilibrium displacement field for \( B \), if for each time \( t \in [0, T) \), we have \( u \in C^3(B) \cap C^2(B) \) and is continuous with respect to \( t \) on \( [0, T) \) and

\[
S_{ij} \dot{u}(t) = 0
\]  
(5)

holds on \( B \).

Saint-Venant's problem for \( B \) consists in the determination of a quasi-static equilibrium displacement field \( u \) on \( B \), subjected to the requirements

\[
s_i(u) = 0 \quad \text{on } \pi,
\]
(6)

\[
s_i(u) = s_i^{(1)} \quad \text{on } D(0), \quad s_i(u) = s_i^{(2)} \quad \text{on } D(L),
\]
(7)

for each time \( t \in [0, T) \). Here \( s_i^{(1)} \) and \( s_i^{(2)} \) are functions preassigned on \( D(0) \) and \( D(L) \), respectively, for each time \( t \in [0, T) \).

Necessary and sufficient conditions for the existence of a solution to this problem are given by

\[
\int_{D(0)} s_i^{(1)} \, da + \int_{D(L)} s_i^{(2)} \, da = 0, \quad \int_{D(0)} \varepsilon_{ijk} x_j s_k^{(1)} \, da + \int_{D(L)} \varepsilon_{ijk} x_j s_k^{(2)} \, da = 0,
\]
(8)

where \( \varepsilon_{ijk} \) is the alternating symbol.

Under suitable smoothness hypotheses on \( \pi \) and on the given forces, a solution of Saint-Venant’s problem exists being continuous with respect to time on \( [0, T) \) (Cf. Fichera [3]).

If we introduce the relations (1) and (2) into (4) and (5), it results that the solution \( u \) of the Saint-Venant's problem satisfies the boundary value problem (5) defined by the equations

\[
\mathcal{J}(u) = \left( G_{ijr}(0) u_{r,s} + \int_0^t \dot{G}_{ijr}(t-s) u_{r,s}(z) \, dz \right)_s = 0 \quad \text{in } B = D \times (0, L),
\]
(9)

and the lateral boundary conditions

\[
\mathcal{B}(u) = \left[ G_{ijr}(0) u_{r,s} + \int_0^t \dot{G}_{ijr}(t-s) u_{r,s}(z) \, dz \right]_{n_s} = 0 \quad \text{on } \pi = \partial D \times (0, L),
\]
(10)

and the end boundary conditions

\[
S_{3i}(u) = -s_i^{(1)} \quad \text{on } D(0), \quad S_{3i}(u) = s_i^{(2)} \quad \text{on } D(L),
\]
(11)

for each \( t \in [0, T) \).

Let us denote by \( \mathcal{S} \) the set of solutions of the Saint-Venant’s problem.

3 The generalized plane strain state

The state of generalized plane strain for the plane domain \( D \subset R^2 \) is characterized by the relation

\[
v_i = v_i(x_1, x_2; t), \quad (x_1, x_2) \in D, \quad t \in [0, T). \]
(12)

Such a displacement vector, in conjunction with the stress-strain-displacement relations, implies that the components of the stress field are functions of \( x_1 \) and \( x_2 \) and \( t \), i.e. \( T_{ij} = T_{ij}(x_1, x_2; t) \). Moreover, we have

\[
T_{ij}(v) = G_{ijkp}(0) v_{k,\beta} + \int_0^t \dot{G}_{ijkp}(t-s) v_{k,\beta}(z) \, dz.
\]
(13)
A vector field \( \mathbf{v} \) is an admissible displacement field provided \( \mathbf{v} \) is continuous with respect to time variable on \([0, T)\) and, moreover, for each \( t \in [0, T) \),

(i) \( \mathbf{v} \) is independent of \( x_3 \)

(ii) \( \mathbf{v} \in C^1(\bar{D}) \cap C^2(D) \).

Given body force \( f(x_1, x_2; t) \) on \( D \) and boundary force \( p(x_1, x_2; t) \) on \( \partial D \) for each \( t \in [0, T) \), the generalized plane strain problem for \( D \cup \partial D \) consists in finding an admissible displacement field \( \mathbf{v} \) which satisfies the equations of equilibrium

\[
T_{ii}(\mathbf{v}) + f_i = 0 \quad \text{in } D, \tag{14}
\]

and the boundary conditions

\[
T_{i\alpha}(\mathbf{v}) n_\alpha = p_i \quad \text{on } \partial D \tag{15}
\]

for each \( t \in [0, T) \). If we substitute the relation (13) into (14) and (15) we obtain the displacement plane boundary value problem \( (\mathcal{P}) \) for \( D \cup \partial D \), defined by

\[
\mathcal{P}(\mathbf{v}) \equiv (G_{\alpha \beta}(0) v_{k,\beta})_\alpha + \int_0^t (\dot{G}_{\alpha \beta}(t - z) v_{k,\beta}(z))_\alpha \, dz = -f_i \quad \text{in } D, \tag{16}
\]

\[
\mathcal{F}(\mathbf{v}) \equiv \left[ G_{\alpha \beta}(0) v_{k,\beta} + \int_0^t \dot{G}_{\alpha \beta}(t - z) v_{k,\beta}(z) \, dz \right] n_\alpha = p_i \quad \text{on } \partial D \tag{17}
\]

for each \( t \in [0, T) \).

The necessary and sufficient conditions for the existence of the solution \( \mathbf{v} \) for the boundary value problem \( (\mathcal{P}) \) associated to \( D \cup \partial D \), are given by

\[
\int_D f_i \, da + \int_{\partial D} p_i \, ds = 0, \tag{18}
\]

\[
\int_D \varepsilon_{3\alpha \beta} x_3 f_\beta \, da + \int_{\partial D} \varepsilon_{3\alpha \beta} x_3 p_\beta \, ds = 0. \tag{19}
\]

Under suitable smoothness hypotheses on \( \partial D \) and on the given forces a solution of the generalized plane strain problem \( (\mathcal{P}) \) exists for each \( t \in [0, T) \) (Cf. Fichera [3]). In what follows, we denote by \( \mathcal{P} \) the set of plane displacements solutions associated with the cross section of the cylinder.

4 Analysis of Saint-Venant’s problem by plane cross section solutions

The problem \( (\mathcal{F}) \) as is defined by the relations (9)–(11) is less tractable. It is important to study the possibility to reduce the system (9) and the lateral boundary conditions (10) to a generalized plane strain problem which is more tractable.

Thus, in what follows we consider the system (9) and the boundary conditions (10) on the cross section \( D \cup \partial D \). Therefore, we consider the plane boundary value problem

\[
\mathcal{F}(\mathbf{u}) = 0 \quad \text{in } D, \tag{20}
\]

and

\[
\mathcal{B}_i(\mathbf{u}) = 0 \quad \text{on } \partial D, \tag{21}
\]
considering \( x_3 \in (0, L) \) and \( t \in [0, T) \) as parameters. In this connection we propose the following question: when does the solution \( u \in \mathcal{I} \) of Saint-Venant's problem reside in the set \( \mathcal{P} \) of the plane displacements associated with the cross section \( D \) of the cylinder?

The answer to the above question will be given in the terms of the vector-valued linear functionals \( \mathcal{R} \) and \( \mathcal{M} \), whose components are defined by

\[
\mathcal{R}(u) = \int_D S_{33}(u) \, da, \quad \mathcal{M}(u) = \int_D \varepsilon_{ijk} x_j S_{3k}(u) \, da,
\]

and which represent the resultant force and the resultant moment about 0 of the tractions acting on the cross section \( D \) of the cylinder. We remark that

\[
\mathcal{R}_d(u) = \int_D x_p S_{33}(u) \, da - x_3 \varepsilon_{3ap} \mathcal{R}_p(u),
\]

\[
\mathcal{M}_3(u) = \varepsilon_{3ap} \int_D x_a S_{33}(u) \, da.
\]

In order to answer the above question, we write the plane boundary value problem (20) and (21) in the form

\[
\left[ G_{ijk}(0) u_{k,p} + \int_0^t \dot{G}_{ijk}(t-z) u_{k,p}(z) \, dz \right]_a + \left[ G_{ik3}(0) u_{k,3} + \int_0^t \dot{G}_{ik3}(t-z) u_{k,3}(z) \, dz \right]_a + S_{31,3}(u) = 0 \quad \text{in } D,
\]

and

\[
\left[ G_{ijk}(0) u_{k,p} + \int_0^t \dot{G}_{ijk}(t-z) u_{k,p}(z) \, dz \right] n_a = -\left[ G_{ik3}(0) u_{k,3} + \int_0^t \dot{G}_{ik3}(t-z) u_{k,3}(z) \, dz \right] n_a \quad \text{on } \partial D.
\]

Therefore, we can see the boundary value problem defined by (25) and (26) as a generalized plane strain boundary value problem of the above section, with

\[
f_i = \left[ G_{ik3}(0) u_{k,3} + \int_0^t \dot{G}_{ik3}(t-z) u_{k,3}(z) \, dz \right]_a + S_{31,3}(u),
\]

\[
p_i = -\left[ G_{ik3}(0) u_{k,3} + \int_0^t \dot{G}_{ik3}(t-z) u_{k,3}(z) \, dz \right] n_a.
\]

In order to satisfy the necessary and sufficient conditions (18) and (19), i.e. \( u \in \mathcal{I} \), we deduce

\[
\int_D S_{31,3}(u) \, da = 0,
\]

\[
\int_D \varepsilon_{3ap} x_a S_{3p,3}(u) \, da = 0.
\]

It is easy to observe that, under the hypothesis (3), relation (2) gives

\[
S_{31,3}(u) = S_{31}(u,3),
\]
so that the relations (29) and (30) take the form

\[ \int_D S_3(u, 3) \, da = 0, \]  
\[ \int_D \varepsilon_{3\alpha\beta} e_3 S_3(u, 3) \, da = 0. \]  

On the other hand, relations (29) and (30) prove, by means of relations (22) and (24), that

\( R_4(u) \) and \( H_3(u) \) are independent of \( x_3 \).

We are thus led to the following result.

**Proposition 1.** Let \( u \) be a solution to Saint-Venant's problem. If relation (34) holds true then \( u \) can be expressed in terms of a plane displacement.

**Corollary 1.** Let \( u \) be a solution of Saint-Venant's problem for which relation (34) holds true. Then

\( H_3(u) \) are independent of \( x_3 \).

**Proof.** In view of the equilibrium equations (5) and the boundary conditions (6), we get

\[ \int_D x_3 S_3(u, 3) \, da = \int_D x_3 S_{3, 3}(u) \, da = - \int_D x_3 S_{\phi, \phi}(u) \, da = - \int (x_3 S_{\phi, \phi}(u)) \, da \]
\[ + \int_D S_{\phi}(u) \, da = - \int x_3 s_3(u) \, ds + R_4(u) = R_4(u), \]  

so that, by means of relation (23), we obtain

\[ (H_3(u))_3 = 0. \]  

This relation implies relation (35) and the proof is complete.

**Remark 1.** Relations (34) and (36) yield

\[ \int_D x_3 S_3(u, 3) \, da = 0. \]  

Relations (32), (33) and (38) allow us to point out two classes of semi-inverse solutions to Saint-Venant's problem that can be expressed in terms of a plane displacement. In what follows, we describe the two classes of semi-inverse solutions.

The class \( C_r \). (The primary solution class). We denote by \( C_r \) the class of solutions of Saint-Venant's problem for which

\( u, 3 \) is a rigid displacement.

Obviously, in view of relations (32)—(34) and (36) and (39), we deduce that for \( u^0 \in C_r \), we have

\[ R_4(u^0) = 0, \quad R_3(u^0) \quad \text{and} \quad H_3(u^0) \] are independent of \( x_3 \).

On the other hand, if \( u^0 \in C_r \), then, from (39) we deduce

\[ u_3^0 = - \frac{1}{2} a_2(t) x_3^2 - e_{3\alpha\beta} a_4(t) x_3 x_\alpha + w_3(x_1, x_2; t), \]
\[ u_3^0 = [a_1(t) x_1 + a_2(t) x_2 + a_3(t)] x_3 + w_3(x_1, x_2; t), \]  

(41)
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except for an additive rigid displacement field. Here \( a_s(t), s = 1, 2, 3, 4, \) are arbitrary functions of \( t, \)
continuous on \([0, T)\). The corresponding stress field is the following:

\[
S_{ij}(u^0) = G_{ijs33}(0) (a_q(t) \varepsilon_{q} + a_3(t)) - a_4(t) G_{ijs33}(t-z) \varepsilon_{3s} \varepsilon_{q} \\
+ \int_{0}^{t} \left\{ G_{ijs33}(t-z) (a_q(z) \varepsilon_{q} + a_3(z)) - a_4(z) G_{ijs33}(t-z) \varepsilon_{3s} \varepsilon_{q} \right\} dz + T_{ij}(w),
\]

(42)

where

\[
T_{ij}(w) = G_{ijs33}(0) w_{k,s} + \int_{0}^{t} G_{ijs33}(t-z) w_{k,s}(z) dz.
\]

(43)

Clearly, \( T_{ij}(w) \) and, by means of relation (42), \( S_{ij}(u^0) \) are independent of the axial coordinate.
Therefore, the boundary value problem defined by (20) and (21) and the boundary value problem
defined by (25) and (26) become

\[
S_{ij}(u^0) = \mathcal{S}_{ij}(w) + \left[ G_{ijs33}(0) a_q(t) \varepsilon_{q} + \int_{0}^{t} G_{ijs33}(t-z) a_q(z) \varepsilon_{q} dz \right]_a \\
+ \left[ G_{ijs33}(0) a_3(t) + \int_{0}^{t} G_{ijs33}(t-z) a_3(z) dz \right]_a \\
- \left[ \varepsilon_{3s} \varepsilon_{i} G_{ijs33}(0) a_4(t) \varepsilon_{q} + \int_{0}^{t} \varepsilon_{3s} \varepsilon_{i} G_{ijs33}(t-z) a_4(z) \varepsilon_{q} dz \right]_a = 0 \quad \text{in } D,
\]

(44)

\[
\mathcal{S}_{ij}(w) = \mathcal{S}_{ij}(w) + \left[ G_{ijs33}(0) a_q(t) \varepsilon_{q} + \int_{0}^{t} G_{ijs33}(t-z) a_q(z) \varepsilon_{q} dz \right]_a \\
+ \left[ G_{ijs33}(0) a_3(t) + \int_{0}^{t} G_{ijs33}(t-z) a_3(z) dz \right]_a \\
- \left[ \varepsilon_{3s} \varepsilon_{i} G_{ijs33}(0) a_4(t) \varepsilon_{q} + \int_{0}^{t} \varepsilon_{3s} \varepsilon_{i} G_{ijs33}(t-z) a_4(z) \varepsilon_{q} dz \right]_a = 0 \quad \text{on } \partial D.
\]

(45)

It follows from the divergence theorem and relations (40) that the necessary and sufficient
conditions to solve the above problem are satisfied for any functions \( a_s(t), s = 1, 2, 3, 4. \) Thus,
\( w \) represents the solution of the plane boundary value problem defined by (44) and (45).

We denote by \( w^{(0)} \) a solution of the boundary value problem (44) and (45) when \( a_1 = \delta_{ij},
\]
\( a_2 = 0, \) and by \( w^{(4)} \) a solution of the boundary value problem (44) and (45) when \( a_1 = 0 \) and
\( a_4 = 1. \) Clearly, we have

\[
w = \sum_{s=1}^{4} a_s \otimes w^{(s)},
\]

(46)

where we have used the notation

\[
(f \otimes g)(t) = f(0) g(t) + \int_{0}^{t} f(t-z) g(z) dz.
\]

(47)

Therefore, \( w^{(s)}, s = 1, 2, 3, 4, \) are characterized by the equations

\[
\mathcal{S}_{ij}(w^{(0)}) + (G_{ijs33}(t) \varepsilon_{q})_a = 0, \quad (\beta = 1, 2),
\]

\[
\mathcal{S}_{ij}(w^{(3)}) + G_{ijs33}(t) = 0,
\]

\[
\mathcal{S}_{ij}(w^{(4)}) - \varepsilon_{3s} G_{ijs33}(t) \varepsilon_{q} = 0, \quad \text{in } D,
\]

(48)
and the boundary conditions
\[
\mathcal{F}_1(w^{(\beta)}) + G_{i33}(t) x_\beta n_a = 0, \quad (\beta = 1, 2),
\]
\[
\mathcal{F}_2(w^{(3)}) + G_{i33}(t) n_a = 0,
\]
\[
\mathcal{G}_3(w^{(4)}) - \varepsilon_{3\beta\gamma} G_{i\beta\gamma}(t) x_\beta n_a = 0 \quad \text{on } \partial D.
\] (49)

In what follows we assume that the displacement fields \( w^{(s)} (s = 1, 2, 3, 4) \) are known. Then, the vector field \( u^{0} \) can be written in the form
\[
u^{0} = \sum_{s=1}^{4} a_s \otimes u^{(s)},
\] (50)
where
\[
u^{(s)}(\beta) = \frac{1}{2} x_3 x_\beta + w^{(s)}(\beta), \quad \nu^{(s)}(3) = x_3 + w^{(3)}(3),
\]
\[
u^{(s)}(4) = \varepsilon_{3\beta\gamma} x_\beta x_3 + w^{(4)}(3), \quad \nu^{(s)}(4) = w^{(4)}(3).
\] (51)

It follows from (42) and (50) that
\[
S_{ij}(u^{0}) = \sum_{s=1}^{4} a_s \otimes S_{ij}(u^{(s)}),
\] (52)
where
\[
S_{ij}(u^{(s)}) = T_{ij}(w^{(s)}) + G_{ij33}(t) x_\alpha, \quad S_{ij}(u^{(3)}) = T_{ij}(w^{(3)}) + G_{ij33}(t),
\]
\[
S_{ij}(u^{(4)}) = T_{ij}(w^{(4)}) - \varepsilon_{3\beta\gamma} G_{ij\beta\gamma}(t) x_\beta.
\] (53)

Obviously, relations (48), (49) and (53) give
\[
S_{ii,\gamma}(u^{(s)}) = 0 \quad \text{in } D,
\]
\[
S_{ii}(u^{(s)}) n_a = 0 \quad \text{on } \partial D.
\] (54)
(55)

These relations imply
\[
\int_D S_{aa}(u^{(s)}) \, da = \int_D \left\{ S_{aa}(u^{(s)}) + x_\alpha S_{33,\alpha}(u^{(s)}) \right\} \, ds = \int_D (x_\alpha S_{33}(u^{(s)})_{\alpha} \, da = 0,
\] (56)
for \( s = 1, 2, 3, 4. \)

Finally, we note that for \( u^{0} \in C_t, \) we have
\[
\mathcal{H}_3(u^{0}) = \sum_{s=1}^{4} D_{3s} \otimes a_s, \quad \mathcal{M}_3(u^{0}) = \sum_{s=1}^{4} \varepsilon_{3\beta\gamma} D_{3s} \otimes a_s, \quad \mathcal{M}_3(u^{0}) = \sum_{s=1}^{4} D_{4s} \otimes a_s,
\] (57)
where
\[
D_{3s} = \int_D x_\beta S_{33}(u^{(s)}) \, da, \quad D_{3s} = \int_D S_{33}(u^{(s)}) \, da, \quad D_{4s} = \int_D \varepsilon_{3\beta\gamma} x_\beta S_{33}(u^{(s)}) \, da, \quad s = 1, 2, 3, 4.
\] (58)

Let \( \vec{u}(t) \) be the four-dimensional vector field \( (a_1(t), a_2(t), a_3(t), a_4(t)) \). We shall write \( u^{0}(\vec{u}) \) for the displacement vector \( u^{0} \) defined by relation (50), indicating thus its dependence on the functions \( a_1(t), a_2(t), a_3(t) \) and \( a_4(t) \).
In view of relations (36) and (38) we are led to introduce the following class of semi-inverse solutions for Saint-Venant's problem.

The class \( C_{t+} \). We denote by \( C_{t+} \) the class of semi-inverse solutions of Saint-Venant's problem for which the conditions (34) hold true, and, moreover,

\[ \mathbf{u}_{33} \] is a rigid displacement. (59)

For \( \mathbf{u}^* \in C_{t+} \) it follows that \( \mathbf{u}_{3}^{*} \in C_{t} \) and, by means of the above discussion, we have

\[ \mathbf{u}_{3}^{*} = \mathbf{u}_{3}^{0}\{\hat{\mathbf{b}}\}, \] (60)

and hence

\[ \mathbf{u}^* = \int_{0}^{x_3} \mathbf{u}_{3}^{0}\{\hat{\mathbf{b}}\} d\mathbf{x}_3 + \mathbf{u}_{3}^{0}\{\hat{\mathbf{c}}\} + \mathbf{w}(x_1, 0; t), \] (61)

where \( \mathbf{w} \) is a vector field in \( \mathcal{P} \), and \( \hat{\mathbf{b}} \) and \( \hat{\mathbf{c}} \) are arbitrary four-dimensional vector fields, depending only on the time \( t \) on \( [0, T) \).

The components of the stress field corresponding to the displacement \( \mathbf{u}^* \) defined by (61) have the form

\[ S_{ij}(\mathbf{u}^*) = \sum_{s=1}^{4} (c_s + x_3 b_3) \otimes S_{ij}(\mathbf{u}_{3}^{0}) + k_{ij} + T_{ij}(\mathbf{w}^*), \] (62)

where

\[ k_{ij} = \sum_{s=1}^{4} G_{ij3}\otimes b_s \otimes w_s(\cdot). \] (63)

In view of relation (56), the relations (34), (58), (62), and (63) yield

\[ \sum_{s=1}^{4} D_{3s} \otimes b_3 = 0, \quad \sum_{s=1}^{4} D_{as} \otimes b_s = 0. \] (64)

Because the conditions (34) are satisfied, on the basis of relations (54) and (55), the plane boundary value problem defined by (25) and (26) reduces to

\[ \mathcal{P}(\mathbf{w}^*) + k_{3i, a} + \sum_{s=1}^{4} b_i \otimes S_{3j}(\mathbf{u}_{3}^{0}) = 0 \quad \text{in } D, \] (65)

\[ \mathcal{F}(\mathbf{w}^*) + k_{3i, a} n_a = 0 \quad \text{on } \partial D. \] (66)

The necessary and sufficient conditions for the existence of a solution of this problem are satisfied on the basis of relations (56) and (64).

Thus, we have

**Proposition 2.** If \( \mathbf{u}^* \in C_{t+} \), it has the form (61), where \( \hat{\mathbf{b}} \) satisfies the conditions (64). Moreover, \( \mathbf{w}^* \) can be obtained from the generalized plane strain problem defined by relations (65) and (66).

**Remark 2.** Let \( \mathbf{u}^* \in C_{t+} \). Then, from relations (36), (56), and (62), we have

\[ \mathcal{P}(\mathbf{u}^*) = \sum_{s=1}^{4} D_{3s} \otimes b_3, \quad \mathcal{P}_3(\mathbf{u}^*) = \sum_{s=1}^{4} D_{3s} \otimes c_s + \int_{D} \left[ k_{33} + T_{33}(\mathbf{w}^*) \right] d\mathbf{a}, \]

\[ \mathcal{M}_3(\mathbf{u}^*) = \epsilon_{3a\beta} \left[ \sum_{s=1}^{4} D_{3s} \otimes c_s + \int_{D} \left[ x_3 (k_{33} + T_{33}(\mathbf{w}^*)) d\mathbf{a} \right] \right], \]

\[ \mathcal{M}_3(\mathbf{u}^*) = \sum_{s=1}^{4} D_{3s} \otimes c_s + \int_{D} \epsilon_{3a\beta} x_3 (k_{33} + T_{33}(\mathbf{w}^*)) d\mathbf{a}, \]
where the components of the vector field $\dot{b}$ satisfy the conditions (64), and $\dot{e}$ is an arbitrary vector field, depending only on $t$.

**Remark 3.** Finally, we remark that the solutions of Saint-Venant’s problem residing within $C_I$ correspond to the end loads

$$s_{I}^{(1)} = -\sum_{s=1}^{4} S_{3s}(u^{(0)}) \otimes a_s, \quad s_{I}^{(2)} = \sum_{s=1}^{4} S_{3s}(u^{(0)}) \otimes a_s$$

the solutions of Saint-Venant’s problem residing within $C_{II}$ correspond to the end loads

$$s_{I}^{(1)} = -\sum_{s=1}^{4} S_{3s}(u^{(0)}) \otimes c_s - k_{3s} - T_{3s}(w^e),$$

$$s_{I}^{(2)} = \sum_{s=1}^{4} S_{3s}(u^{(0)}) \otimes (c_s + Lb_s) + k_{3s} + T_{3s}(w^e).$$

5 The relaxed Saint-Venant’s problem

The relaxed Saint-Venant’s problem for the viscoelastic cylinder $B$ consists in the determination of a quasi-static equilibrium displacement field $u$ that satisfies the condition

$$s_1(u) = 0 \quad \text{on } \pi,$$

and global conditions at each end

$$\mathcal{R}_1(u) = -R_1(t), \quad \mathcal{M}_1(u) = -M_1(t) \quad \text{on } x_3 = 0,$$

where $R_1$ and $M_1$ are continuous functions preassigned on $[0, T)$. Similar conditions are assumed on the end located at $x_3 = L$.

In what follows we proceed to determine a solution of the relaxed Saint-Venant’s problem. In view of the discussions in the previous Section, we use the decomposition of the relaxed problem into problems $(P_1)$ and $(P_2)$ characterized by

$(P_1)$ (extension-bending-torsion): $R_z = 0,$

$(P_2)$ (flexure): $F_3 = M_3 = 0.$

The solution of the problem $(P_1)$. In view of the results of the previous Section, a solution of the problem $(P_1)$ has the form

$$u_1 = u^{(0)} = \sum_{s=1}^{4} a_s \otimes u^{(0)}.$$  (72)

Relations (57) and (71) give for determination of the unknown functions $a_s(t)$, $s = 1, 2, 3, 4$, the following system:

$$\sum_{s=1}^{4} D_{3s} \otimes a_s = c_{3s} \otimes \dot{M}_0, \quad \sum_{s=1}^{4} D_{3s} \otimes a_s = -R_3, \quad \sum_{s=1}^{4} D_{4s} \otimes a_s = -M_3.$$  (73)

Let us denote by $\mathcal{D}(t)$ the $4 \times 4$-matrix whose components are $D_{rs}(t)$, $(r, s = 1, 2, 3, 4)$. Let $K(t)(R_3, M_1, M_2, M_3) = (-M_2, M_1, -R_3, -M_3)^T$ and $\dot{a} = \dot{a}^T$. Then the above system can be written in the following matrix form:

$$\mathcal{D}(0) \ddot{a}(t) + \int_{0}^{t} \mathcal{D}(t - z) \dot{a}(z) \, dz = K(t).$$  (74)
Now, we observe that \( S_{ij}(u^0) \) at \( t = 0 \) coincide with the components of the stress in the auxiliary strain problems from classical elasticity corresponding to an elastic material with the positive-definite elasticity tensor \( G_{ij}(0) \). Therefore, it follows from (58) that \( \mathcal{D}(0) \) coincides with the corresponding matrix \([2]\) for an elastic medium with the elastic coefficients \( G_{ij}(0) \). It was shown that the matrix \( \mathcal{D}(0) \) is invertible \([2]\). Therefore, we deduce from (74) that

\[
\frac{\partial}{\partial t} a(t) + \int_0^t [\mathcal{D}(0)]^{-1} \mathcal{D}(t - z) \, d\zeta \, dz = [\mathcal{D}(0)]^{-1} K(t).
\]

(75)

Since \( R_3(t) \) and \( M_1(t) \) are continuous on \([0, T)\), the Volterra integral equation (75) has one and only one solution \( a(t) \) continuous in \([0, T)\), which can be obtained by the method of successive approximations \([5]\).

Therefore, the solution of the problem \((P_1)\) is given by relation (72), where \( u^0 \) are defined by the relation (50), and the unknown functions \( a_i(t) \) are determined by means of the system (73).

The solution of the problem \((P_2)\). On the basis of the results from the previous Section, we seek a solution of the problem \((P_2)\) in the class \( C_1 \). Therefore, we seek a solution in the form (61), i.e.

\[
u_{II} = u^*,
\]

where

\[
\begin{align*}
u^*_i &= -\frac{1}{6} b_i(t) x_i^3 - \frac{1}{2} c_i(t) x_i^2 - \frac{1}{2} b_i(t) e_{3a}x_a x_i^2 - c_i(t) e_{3a}x_a x_i \\
&\quad + \sum_{s=1}^{4} (c_s + x_3 b_s) \otimes w^s + w^*_i, \\
u^*_3 &= \frac{1}{2} \left(b_1(t) x_3 + b_3(t)\right) x_3^2 + (c_1(t) x_3 + c_3(t)) x_3 + \sum_{s=1}^{4} (c_s + x_3 b_s) \otimes w^s + w^*_3,
\end{align*}
\]

(77)

and the unknown functions \( b_s(t) \) satisfy the conditions

\[
\sum_{s=1}^{4} D_{3s} \otimes b_s = 0, \quad \sum_{s=1}^{4} D_{2s} \otimes b_s = 0.
\]

(78)

From relations (67) and (71), we get

\[
\sum_{s=1}^{4} D_{3s} \otimes b_s = -R_3,
\]

(79)

so that from the integral system (78) and (79) we can determine uniquely the functions \( b_s(t) \), \( s = 1, 2, 3, 4 \). In what follows, we assume that the functions \( b_s(t) \) are known. Then the vector \( w^* \) can be determined from the generalized plane strain problem defined by relations (65) and (66).

Further, from the relations (67) and (71), we get for the determination of the unknown functions \( c_s(t) \), \( s = 1, 2, 3, 4 \), the following integral system:

\[
\begin{align*}
\sum_{s=1}^{4} D_{3s} \otimes c_s &= -\int_D x_s(k_{33} + T_{33}(w^*)) \, da, \\
\sum_{s=1}^{4} D_{2s} \otimes c_s &= -\int_D (k_{33} + T_{33}(w^*)) \, da, \\
\sum_{s=1}^{4} D_{2s} \otimes c_s &= -\int_D e_{3a}x_a(k_{33} + T_{33}(w^*)) \, da.
\end{align*}
\]

(80)
Therefore, a solution of the problem (P2) has the form (77), where the unknown functions \( b_s(t) \) and \( c_q(t) \) are determined by means of the Volterra integral equations systems defined by relations (78), (79), and (80), respectively; the vector field \( w^* \) is determined as a solution of the generalized plane strain problem defined by relations (65) and (66).

Finally, we note that the relaxed Saint-Venant’s problem has a solution of the form

\[
    u = u_l + u_{ll},
\]

where \( u_l \) and \( u_{ll} \) are defined by relations (72) and (76), respectively.

References


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