A PHRAGMÈN-LINDELÖF PRINCIPLE IN DYNAMIC LINEAR THERMOELASTICITY

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A PHRAGMÈN–LINDELOF PRINCIPLE IN
DYNAMIC LINEAR THERMOELASTICITY

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A dynamic analogue of the Phragmèn–Lindelöf principle is established for an unbounded body composed of an inhomogeneous and anisotropic linear thermoelastic material. The asymptotic behavior of either a volume measure of the total energy or alternative cross-sectional measure is studied. It is shown that the cross-sectional measure of a thermoelastic process either asymptotically grows faster than an increasing exponential function or asymptotically decays faster than a decreasing exponential function. A uniqueness theorem valid for infinite domains is established, and some explicit bounds for the decay rate are given.

A recent paper [1] has explored aspects of a dynamical Saint Venant's principle for a bounded thermoelastic solid subjected to the action of nonzero boundary loads only on a plane end face. The method of proof is based on a first-order differential inequality governing an appropriate volume measure of the total energy. Spatial decay estimates are established that provide exponential spatial decay of end effects away from the loaded end. The decay rate is controlled by the factor \( \exp\left(\frac{-z}{v(t)}\right) \), where \( z \) is the distance from the loaded end, \( t \) is the time variable, and the positive function \( v(t) \) depends also on the thermoelastic coefficients.

This article continues the aforementioned study by embarking upon an examination of an anisotropic and inhomogeneous thermoelastic body that occupies a regular unbounded region \( B \) of the three-dimensional space. We suppose that for a finite time \( T > 0 \), the body is subjected to the external given data having a bounded support \( \bar{D}_T, \bar{D}_T \subset B \). Then we are able to establish that the total energy on \( [0, T] \) of the body contained in a part lying at a distance not less than \( r \) from the region \( \bar{D}_T \) is either unbounded or is bounded above by some exponentially decaying function for large values of \( r \). The conclusions provide a relationship with the Phragmèn–Lindelöf theorem, while the spatial decay constituent becomes immediately recognizable as Saint Venant's principle established in [1]. As an immediate consequence of these results, a uniqueness theorem is proved for linear thermoelastodynamics that is valid for infinite domains. Finally, we give a complete

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discussion concerning explicit bounds for the decay rate in linear thermoelastodynamics.


**PRELIMINARIES**

Let $E$ denote a real euclidean three-dimensional point space, and let $V$ be the (real) three-dimensional vector space associated with $E$. Let $\{e_i\} (i = 1, 2, 3)$ be an orthonormal basis of $V$ and $O$ any fixed point of $E$; then, the reference frame $\{O, e_i\}$ is introduced in $E$, and, for any $x \in E$, $x - O = x_i e_i$, with $x_i \in \mathbb{R}$ (the sum over repeated indices is implied and $i$ ranges from 1 to 3). Also, for any $v \in V$, $v = v_i e_i$, with $v_i \in \mathbb{R}$. For the sake of simplicity we put

$$|v| = (v_i v_i)^{1/2} \quad \forall v \in V$$  

(1)

We denote by $Lin$ the nine-dimensional vector space of all linear mappings from $V$ to $V$ (second-order tensors on $V$). For any $v \in V$ and for any $T \in Lin$, the symbol $Tv$ stands for the value of $T$ at $v$. A fourth-order tensor $C$ on $V$ is a linear mapping from $Lin$ to $Lin$. For any $T \in Lin$, $C[T] \in Lin$ is the value of $C$ at $T$. Let $Sym$ be the set of symmetric tensors.

A thermoelastic body $B$ is identified with a regular unbounded (open connected) region of $E$ it occupies in an assigned reference configuration. The fundamental system of field equations consists [4] of the strain-displacement relation

$$E = \frac{1}{2} (\nabla u + \nabla u^T) \quad \text{on } Q \equiv \overline{B} \times [0, \infty)$$  

(2)

the thermal gradient-temperature relation

$$g = \nabla \theta \quad \text{on } Q$$  

(3)

the equation of motion

$$\rho \ddot{u} = \text{div} S + b \quad \text{on } Q$$  

(4)

the energy equation

$$c \dot{\theta} = \theta_0 M \cdot \dot{E} - \text{div} q + h \quad \text{on } Q$$  

(5)

the stress-strain-temperature relation

$$S = C[E] + \theta M \quad \text{on } Q$$  

(6)
and the heat conduction equation
\[ q = -K \nabla g \quad \text{on } Q \tag{7} \]

Here \( u, E, S, b, \theta, g, q, \) and \( h \) are the displacement, strain, stress, body force, temperature difference, thermal gradient, heat flux, and heat supply fields, respectively; while \( \rho, C, M, K, \) and \( c \) are the density, elasticity, stress-temperature, conductivity, and the specific heat, respectively; finally, \( \theta_0 \) is the fixed uniform reference temperature and the point \( (\cdot) \) denotes the inner product of two vectors or second-order tensors. Moreover, as usual, the operator \( \nabla \) is defined by the relation
\[ \nabla f = \partial_i f e_i \quad \left( \frac{\partial}{\partial x_i} = \partial_i \right) \]
for any scalar field \( f \), so that, for any vector function \( v \), \( \nabla v \) turns out to be the second-order tensor defined by setting \( (\nabla v)e_j = \partial_j v \)
while the second-order tensor \( (\nabla v)^T \) is defined by the relation
\[ e_i : (\nabla v)e_j = e_j : (\nabla v)^T e_i \]
for any vector field \( v \) and for any second-order tensor field \( T \),
\[ \text{div } v = \partial_i v_i \quad \text{div } T = \partial_i (Te_j) \]
Finally, the superimposed dot denotes the partial derivative with respect to time \( t \).

The density \( \rho \) is assumed to be strictly positive and a continuous and bounded field on \( \bar{B} \). We introduce the notation
\[ \rho_0 = \inf \{ \rho(x) : x \in \bar{B} \} \tag{8} \]

The elasticity tensor \( C \) is assumed to be symmetric, that is,
\[ A \cdot C[B] = B \cdot C[A] \quad \forall A, B \in Sym \tag{9} \]
and positive definite, that is,
\[ A \cdot C[A] > 0 \quad \forall A \in Sym, A \neq 0 \tag{10} \]
These relations imply that there exists \( \Lambda_1(x) > 0 \), so we have
\[ |C[A]|^2 \leq \Lambda_1 A \cdot C[A] \quad \forall A \in Sym \tag{11} \]
A full discussion on upper and lower bounds for the magnitude of the strain energy density in linear anisotropic elastic materials associated with $C$ is given in a recent paper by Mehrabadi, Cowin, and Horgan [5]. The explicit values for the maximum eigenvalue $\Lambda_1$ are given there for different elastic symmetries. We assume further that $C$ is continuous and bounded on $\overline{B}$. Then the maximum elastic modulus $\mu$ as defined by

$$\mu = \sup\{\Lambda_1(x): x \in \overline{B}\}$$

(12)

is finite.

Further, we assume that the stress-temperature tensor $M$ is symmetric, that is,

$$v \cdot Mw = w \cdot Mv \quad \forall v, w \in \mathcal{V}$$

(13)

and that $c$ is a strictly positive field on $\overline{B}$. Moreover, we assume that $M$ and $c^{-1}$ are continuous and bounded fields on $\overline{B}$. Then

$$m = \sup\left\{\frac{\theta_0 |M(x)|^2}{c(x)}: x \in \overline{B}\right\}$$

(14)

is finite.

Then the relations (6), (11), (12), (14) and the inequality

$$|A \otimes B|^2 \leq (1 + \epsilon)|A|^2 + \left(1 + \frac{1}{\epsilon}\right)|B|^2 \quad \forall A, B \in \text{Sym}, \forall \epsilon > 0$$

(15)

$$|S|^2 = |C[E] + \theta M|^2 \leq (1 + \epsilon)|C[E]|^2 + \left(1 + \frac{1}{\epsilon}\right)|\theta^2|M|^2$$

$$\leq (1 + \epsilon)\mu E \cdot C[E] + \left(1 + \frac{1}{\epsilon}\right)m \frac{1}{\theta_0} c \theta^2$$

(16)

Finally, we assume that the conductivity tensor $K$ is symmetric, that is,

$$v \cdot Kw = w \cdot Kv \quad \forall v, w \in \mathcal{V}$$

(17)

and positive definite, that is,

$$v \cdot Kv > 0 \quad \forall v \in \mathcal{V}, v \neq 0$$

(18)

We also assume that $K$ is continuous and bounded on $\overline{B}$. Let $\lambda(x)$ be the largest characteristic value of the tensor $K(x)/c(x)$. Then

$$k = \sup\{\lambda(x): x \in \overline{B}\}$$

(19)
is finite and

$$w \cdot K(x)w \leq k c(x)|w|^2 \quad \forall w \in V, \forall x \in \bar{B}$$  \hspace{1cm} (20)$$

Let $$p = [u, E, S, \theta, g, q]$$ be a thermoelastic process corresponding to the body force $$b$$ and the heat supply $$h$$ in the sense defined in [4]. The corresponding surface traction $$s$$ and heat flux $$q$$ are defined at every regular point of $$\partial B \times [0, \infty)$$ by

$$s(x, t) = S(x, t)n(x)$$ \hspace{1cm} (21)$$

and

$$q(x, t) = q(x, t) \cdot n(x)$$ \hspace{1cm} (22)$$

where $$n(x)$$ is the unit outward normal to $$\partial B$$ at $$x$$.

Throughout this article we shall assume that $$m > 0$$. Obviously, if $$m = 0$$, then the basic equations become uncoupled and, therefore, can be treated separately using the methods described in [6, 7].

THE PHRAGMÈN–LINDELÖF PRINCIPLE

Given a dynamic thermoelastic process $$p = [u, E, S, \theta, g, q]$$ on $$\bar{B}$$, corresponding to the external supply $$[b, h]$$ and a given time $$T > 0$$, let $$\bar{D}_T$$ denote the set of all $$x \in \bar{B}$$ such that (i) if $$x \in \bar{B}$$, then

$$u(x, 0) \neq 0 \text{ or } \dot{u}(x, 0) \neq 0 \text{ or } \theta(x, 0) \neq 0$$ \hspace{1cm} (23)$$

or

$$b(x, \tau) \neq 0 \text{ or } h(x, \tau) \neq 0 \text{ for some } \tau \in [0, T]$$ \hspace{1cm} (24)$$

or (ii) if $$x \in \partial B$$, then

$$s(x, \tau) \cdot \dot{u}(x, \tau) \neq 0 \text{ or } q(x, \tau) \theta(x, \tau) \neq 0 \text{ for some } \tau \in [0, T]$$ \hspace{1cm} (25)$$

Roughly speaking, $$\bar{D}_T$$ represents the support of the initial and boundary data and the body supplies. In what follows, we shall assume that $$\bar{D}_T$$ is a bounded region.

Further, we consider the set $$D_r$$ of all points of $$\bar{B}$$ defined by

$$D_r = \{ x \in \bar{B} : \bar{D}_T \cap \Sigma(x, r) \neq \emptyset \}$$ \hspace{1cm} (26)$$

where $$\Sigma(x, r)$$ is the open ball with radius $$r$$ and center at $$x$$. We denote by $$S_r$$ a part of the surface $$\partial D_r$$ contained in $$B$$ and external to $$D_r$$. Moreover, we denote by $$B_r$$ a part of the body $$B$$ contained in $$B \setminus D_r$$. Figure 1 shows an illustration of these symbols and others for the simplest case when a thermoelastic process is produced by the initial disturbances only, i.e. when $$\bar{D}_T$$ is time independent and $$\bar{B} = E^3$$. 
Let us consider \( r > 0 \) and \( t \in [0, T] \). We associate with the dynamic thermoe­lastic process \( p = [u, E, S, \theta, g, q] \) the functional

\[
I(r, t) = -\int_0^t \int_{S(r)} \left[ s(\tau) \cdot \dot{\mathbf{u}}(\tau) - \frac{1}{\theta_0} q(\tau) \theta(\tau) \right] dS d\tau ds
\]  

(27)

where \( s(\tau) \) and \( q(\tau) \) are defined by the relations (21) and (22), respectively.

Let \( r_1 \) be fixed in \([0, r]\). Then the relations (25) and (27) give

\[
I(r, t) - I(r_1, t) = -\int_0^t \int_{\partial B(r_1, r)} \left[ s(\tau) \cdot \dot{\mathbf{u}}(\tau) - \frac{1}{\theta_0} q(\tau) \theta(\tau) \right] dS d\tau ds
\]  

(28)

where

\[
B(r_1, r) = D_r \setminus D_{r_1}
\]  

(29)

Figure 1. Illustration of symbols in Eqs. (26)-(30) when \( B = E^3 \) and data reduce to the initial conditions only (\( \partial D_r = \partial \bar{D} \)).
On the basis of the divergence theorem and by means of relations (2)-(7), (9), (13), (17), and (21)-(24), we get

\[
I(r, t) - I(r_1, t) = -\frac{1}{2} \int_0^t \int_{S_r} \left\{ \rho \dot{u}(s)^2 + E(s) \cdot C[E(s)] \right\}
+ \frac{1}{\theta_0} c \theta(s)^2 + \frac{2}{\theta_0} \int_0^s g(\tau) \cdot Kg(\tau) \, d\tau \right\} \, dS \tag{30}
\]

Thus, by a direct differentiation of the relation (30), we deduce

\[
\frac{dI}{dr}(r, t) = -\frac{1}{2} \int_0^t \int_{S_r} \left\{ \rho \dot{u}(s)^2 + E(s) \cdot C[E(s)] + \frac{1}{\theta_0} c \theta(s)^2 \right\}
+ \frac{2}{\theta_0} \int_0^s g(\tau) \cdot Kg(\tau) \, d\tau \right\} \, dS d\tau \tag{31}
\]

On the other hand, by means of the relations (6), (7), (21), and (22) from the relation (27) we obtain

\[
I(r, t) = -\int_0^t \int_{S_r} \left\{ S(\tau) \cdot [n \otimes \dot{u}(\tau)] + \frac{1}{\theta_0} \theta(\tau) n \cdot Kg(\tau) \right\} \, dS d\tau ds \tag{32}
\]

where \( \otimes \) represents the tensor product of two vectors.

On the basis of the Schwarz's inequality and the arithmetic-geometric mean inequality and using relations (16) and (20), we obtain

\[
|I(r, t)| \leq \left( \int_0^t \int_{S_r} \dot{u}(\tau)^2 \, dS d\tau ds \right)^{1/2}
\cdot \left( \int_0^t \int_{S_r} \left\{ (1 + \epsilon) \mu E(\tau) \cdot C[E(\tau)] + \left( 1 + \frac{1}{\epsilon} \right) m \frac{1}{\theta_0} c \theta(\tau)^2 \right\} \, dS d\tau ds \right)^{1/2}
+ \left( \int_0^t \int_{S_r} \frac{1}{\theta_0} \theta(\tau)^2 n \cdot Kn \, dS d\tau ds \right)^{1/2}
\cdot \left( \int_0^t \int_{S_r} \frac{1}{\theta_0} g(\tau) \cdot Kg(\tau) \, dS d\tau ds \right)^{1/2}
\leq t \int_0^t \int_{S_r} \left\{ \frac{\alpha}{\rho_0} \frac{1}{2} \rho_0 \dot{u}(\tau)^2 + \frac{1}{\alpha} (1 + \epsilon) \mu \frac{1}{2} E(\tau) \cdot C[E(\tau)] \right\} \, dS d\tau ds \tag{33}
\]
for all positive parameters \( \epsilon \) and \( \alpha \). Therefore, by means of the relation (8), from Eq. (33) we deduce that

\[
\left| I(r, t) \right| \leq \sqrt{\int_0^1 \int_S \left( \frac{\alpha \sqrt{\rho}}{\rho_0} \frac{\sqrt{I}}{2} \rho \tau (r) \right)^2 + \frac{\sqrt{I}}{2} \left( 1 + \epsilon \right) \mu \frac{1}{2} E(\tau) \cdot C[E(\tau)] + \frac{\sqrt{I}}{2} \left( 1 + \epsilon \right) m + \frac{k}{\beta} \frac{1}{2} \theta_0 c \theta (r)^2 + \frac{\beta}{2} \int_0^T \frac{1}{2} g(\tau) \cdot K_g(\tau) d\tau \right) dS \]  

(34)

for all arbitrary positive parameters \( \epsilon \), \( \alpha \), and \( \beta \).

We now equate the coefficients of the various energetic terms in the integral in Eq. (34), that is, we set

\[
\frac{\alpha \sqrt{I}}{\rho_0} = \frac{\sqrt{I}}{\alpha} \left( 1 + \epsilon \right) \mu = \frac{\sqrt{I}}{\alpha} \left( 1 + \epsilon \right) \frac{m}{2} = \frac{\beta}{2} = \chi
\]

(35)

Therefore, we choose

\[
\alpha = \sqrt{\rho_0 \mu [1 + e_0(t)]} \quad \beta = 2 \sqrt{\frac{I \mu [1 + e_0(t)]}{\rho_0}} \quad \epsilon = e_0(t)
\]

(36)

where \( e_0(t) \) is the positive root of the algebraic equation

\[
g(\epsilon) = \epsilon^2 + \epsilon \left( 1 - \frac{m}{\mu} - \frac{k \rho_0}{2 \mu t} \right) - \frac{m}{\mu} = 0
\]

(37)

With these choices, the inequality (34) and the relation (31) imply

\[
\sqrt{\chi(t)} \frac{dI}{dr}(r, t) + |I(r, t)| \leq 0
\]

(38)

for all \( r > 0 \) and \( t \in [0, T] \). It can be seen from the relation (31) that for all \( t \in [0, T] \), \( I(r, t) \) is a nonincreasing function with respect to \( r \) on \( (0, \infty) \).
Let $t$ be fixed in $[0,T]$. Then, it results that we have only the following two possibilities: (a) $I(r,t) \geq 0$ for all $r \geq 0$ and (b) there exists $r_{0} \geq 0$ such that $I(r_{0},t) < 0$ (the subscript $t$ for $r$ means that the value of $r$ for which $I$ can become strictly negative should be dependent on the value of $t$ fixed in the above). It will be shown in what follows that the second possibility leads to the conclusion that in this case we have a thermoelastic process having an infinite total energy on $[0,t]$.

Let us consider the first possibility. Then the first-order differential inequality (38) implies

$$\sqrt{t} \chi(t) \frac{dI}{dr}(r,t) + I(r,t) \leq 0 \quad \text{for all } r > 0$$

for fixed $t \in [0,T]$. By an integration, from Eq. (39) we deduce that

$$I(r,t) \leq I(0,t) \exp \left\{ - \frac{r}{\sqrt{t} \chi(t)} \right\}$$

for fixed $t \in [0,T]$.

Let us now consider the second case. Then we have $I(r,t) \leq I(r_{0},t) \leq 0$ for all $r \geq r_{0}$, so that the differential inequality (38) gives

$$\sqrt{t} \chi(t) \frac{dI}{dr}(r,t) - I(r,t) \leq 0 \quad \text{for all } r \geq r_{0}$$

and for fixed $t \in [0,T]$. Thus, by an integration, from (41), for fixed $t \in [0,T]$, we obtain

$$-I(r,t) \geq -I(r_{0},t) \exp \left\{ \frac{r - r_{0}}{\sqrt{t} \chi(t)} \right\} \quad \text{for all } r \geq r_{0}$$

We observe at once that the relation (42) implies that $-I(r,t)$ becomes exponentially unbounded for sufficiently large $r$ and for all fixed $t \in [0,T]$.

Thus, we have established the following result.

**Theorem 1: (Phragmèn–Lindelöf Principle).** The deformation of a thermoelastic body as measured by $|I(r,t)|$ either decays faster than the exponential $\exp(-r/\sqrt{t} \chi(t))$ as $r \to \infty$ or grows faster than the exponential $\exp((r - r_{0})/\sqrt{t} \chi(t))$ as $r \to \infty$.

Note that the total energy $\mathcal{E}(r,t)$ contained in the portion $B_{r}$ of the body on $[0,t]$ is given by

$$\mathcal{E}(r,t) = \frac{1}{2} \int_{0}^{t} \int_{B_{r}} \left\{ \rho \dot{u}(s)^{2} + E(s) \cdot C(E(s)) + \frac{1}{\theta_{0}} \varepsilon \theta(s)^{2} \right. + \left. \frac{2}{\theta_{0}} \int_{0}^{s} g(\tau) \cdot K g(\tau) d\tau \right\} dV ds$$

(43)
In view of relation (30) it follows that when Eq. (40) holds true, we have that $\mathcal{E}(r,t)$ exists for all fixed $t \in [0,T]$ and

$$I(r,t) = \mathcal{E}(r,t)$$  \hspace{1cm} (44)

while, when Eq. (42) holds true, it follows that $\mathcal{E}(r,t)$ is unbounded.

Thus, the Theorem 1 can be formulated in terms of the total energy $\mathcal{E}(r,t)$ as follows

**Theorem 2:** Let $p = [u, E, S, \theta, g, q]$ be a thermoelastic process corresponding to the initial and boundary data and the body supplies having a bounded support $\tilde{D}_T$. Then the corresponding total energy on $[0,t]$, $\mathcal{E}(r,t)$, does not exist, vanishes identically, or decays at least exponentially with $r$ for all $t \in [0,T]$.

An immediate consequence of the Theorem 2 is the following uniqueness result.

**Theorem 3:** (Uniqueness). In a class of thermoelastic processes for which the total energy on $[0,t]$, $\mathcal{E}(r,t)$, is bounded, there is at most one solution of the initial-boundary value problem associated with dynamic linear thermoelasticity.

**Proof:** Let $p = [u, E, S, \theta, g, q]$ be the difference between two solutions. Then $p$ corresponds to vanishing body supplies and satisfies

$$u(\cdot,0) = 0 \quad \dot{u}(\cdot,0) = 0 \quad \theta(\cdot,0) = 0 \quad \text{on } B$$  \hspace{1cm} (45)

and

$$s \cdot \dot{u} = 0 \quad q \theta = 0 \quad \text{on } \partial B \times (0,\infty)$$  \hspace{1cm} (46)

Choose $T$ arbitrary in $(0,\infty)$. Clearly, $\tilde{D}_T$ is empty and $I(0,t) = 0$ for all $t \in [0,T]$. Then in the class of thermoelastic processes for which the total energy $\mathcal{E}(r,t)$ is bounded, we have, by means of relations (40) and (44), that

$$\mathcal{E}(r,t) = 0 \quad \text{for all } r \geq 0 \text{ and } t \in [0,T]$$  \hspace{1cm} (47)

and the relations (10), (18), and (47) imply that $p$ vanishes on $\overline{B} \times [0,T]$. Since $T$ was arbitrarily chosen, the proof is complete.

**DECAY RATE**

It can be useful to have explicit bounds for the decay rate in relation (40). It is easy to see that for all $t \in [0,T]$ we have

$$\chi(t) = \sqrt{\frac{\mu}{t \rho_0 [1 + \epsilon_0(t)]}}$$  \hspace{1cm} (48)
Further, we note that

\[ g\left( \frac{m}{\mu} \right) = - \frac{k \rho_0 m}{2 \mu^2 t} < 0 \quad g\left( \frac{m}{\mu} + \frac{k \rho_0}{2 \mu t} \right) = \frac{k \rho_0}{2 \mu t} > 0 \quad (49) \]

so we have

\[ \frac{m}{\mu} < \epsilon(t) < \frac{m}{\mu} + \frac{k \rho_0}{2 \mu t} \quad (50) \]

On this basis, from relation (48) we deduce that

\[ t \sqrt{\frac{\mu + m}{\rho_0}} < \sqrt{\epsilon(t)} < t \sqrt{\frac{\mu + m}{\rho_0} + \frac{k}{2t}} \quad (51) \]

Thus, we have

\[ \exp\left( - \frac{r}{t \sqrt{(\mu + m)/\rho_0}} \right) < \exp\left( - \frac{r}{\sqrt{\epsilon(t)}} \right) < \exp\left( - \frac{r}{t \sqrt{(\mu + m)/\rho_0 + k/2t}} \right) \quad (52) \]

We now use Eq. (52) to study the influence of the interdependence between the mechanical and thermal effects upon the decay rate established in the above section. First of all we observe that for all \( t \in [0, T] \), from Eq. (52) we have

\[ \exp\left( - \frac{r}{t \sqrt{\mu/\rho_0}} \right) < \exp\left( - \frac{r}{\sqrt{\epsilon(t)}} \right) \quad (53) \]

and this proves that the decay rate in linear thermoelasticodynamics is slower than the analogous described in [8] for linear elastodynamics associated with the elasticity tensor \( C \), where the equations are hyperbolic rather than hyperbolic-parabolic.

Let us now consider suitable short times, namely,

\[ t < \frac{k \rho_0}{2(\mu + m)} \quad (54) \]

It is a simple matter to verify that the inequality (52) implies that

\[ \exp\left( - \frac{r}{\sqrt{\epsilon(t)}} \right) < \exp\left( - \frac{r}{\sqrt{k \epsilon(t)}} \right) \quad (55) \]
On this basis we conclude that for short times satisfying relation (54), the decay rate can be controlled by the factor $\exp\left[ -r/\sqrt{k_0} \right]$ as described in [6] for parabolic heat equation.

On the other hand, for large values of time variable, namely,

$$\frac{k\rho_0}{2(\mu + m)} \leq t \leq T$$

(56)

from the relation (52) we get

$$\exp\left( -\frac{r}{\sqrt{t}\chi(t)} \right) \leq \exp\left( -\frac{r}{t\sqrt{2}/(\mu + m)/\rho_0} \right)$$

(57)

This proves that for values of time satisfying inequality (56) the decay rate can be controlled by a factor similar to that of linear elastodynamics described in [8].

Therefore, we conclude that a decay rate for the linear dynamic thermoelasticity can be controlled by a decay factor associated with thermal effects for small times satisfying relation (54) or by a decay factor associated with mechanical effects for large values of time satisfying relation (56).

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