On the uniqueness and continuous data dependence of solutions in the theory of swelling porous thermoelastic soils

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Received 7 April 2003; accepted 30 April 2003

Abstract

This paper studies the uniqueness and continuous data dependence of solutions of the initial-boundary value problems associated with the linear theory of swelling porous thermoelastic soils. The formulation belongs to the theory of mixtures for porous elastic solids filled with fluid and gas with thermal conduction and by considering the time derivative of temperature as a variable in the set of constitutive equations. Some uniqueness and continuous data dependence results are established under mild assumptions on the constitutive constants. Thus, it is shown that the general approach of swelling porous thermoelastic soils is well posed. The method of proof is based on some integro-differential inequalities and some Lagrange–Brun identities.

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1. Introduction

In [1] Eringen has developed a continuum theory for a mixture consisting of three components: an elastic solid, a viscous fluid and a gas. Field equations is obtained for a heat conducting mixture. Such theory is relevant to problems in the oil exploration industry, since oil is viscous and usually it is accompanied by thermal effects and gas in underground rocks, porous solid in slurries and muddy river beds. Consolidation problems in the building industry, earthquake problems, swelling of plants and living tissues and a plethora of other problems fall into domain of mixture theory considered here. We outline here that for the isothermal approach some results concerning uniqueness, continuous dependence and spatial behaviour of the solutions have been studied in several recent papers (see, for example [2–6]). Some results have been established in [7]
concerning the existence and uniqueness of solutions in the case when the thermal effects are present, but the time derivative of temperature is not present into the set of independent variables. Other results for this particular approach can be found in [8,9]. We outline that the general properties of solutions in dynamical theory of mixtures are established by Ieșan and Nappa [10].

The purpose of this paper is to study the uniqueness and continuous data dependence of solutions for the initial-boundary value problems within the general context of the linear theory of swelling porous thermoelastic soils developed by Eringen [1]. We outline that this general approach assumes the time derivative of temperature into the constitutive equations. To this end we use a method based on some integro-differential inequalities and a method based on some Lagrange–Brun identities (see, for example, Brun [11]). Thus, we individuate some classes of solutions where it is possible to establish various estimates describing the continuous dependence of the solutions with respect to the body forces and heat supplies, provided some mild assumptions are made upon the constitutive constants.

2. Basic formulation

Throughout this paper Latin subscripts take the values 1, 2, 3, summation is carried out over repeated indices, \( x = (x_1, x_2, x_3) \) is a generic point referred to orthogonal Cartesian coordinates in \( \mathbb{R}^3 \). The suffix “, k” denotes \( \partial / \partial x_k \), that is the derivative with respect to \( x_k \) and a superposed dot denotes the time derivative. We consider a body that at time \( t = 0 \) occupies the regular region \( B \) of Euclidean three-dimensional space whose boundary is the regular surface \( \partial B \).

We assume that \( B \) is occupied by a mixture consisting of three components: an elastic solid, a compressible fluid and a gas. We shall use the superscript \( s, f \) and \( g \) to denote the elastic solid, the fluid and the gas, respectively. According to the general linear theory of swelling porous thermoelastic soils (see Eqs. (4.1)–(4.4) of [1]), the fundamental equations are

\[
\begin{align*}
\varrho_0^s \ddot{u}^s_i &= \mu \dot{u}^s_{i,j} + (\lambda + \mu) u^s_{j,ji} - \sigma^f u^f_{j,ji} - \sigma^g u^g_{j,ji} + (\gamma^f + \gamma^g - \alpha_0) T_i \\
&\quad + (\bar{\zeta}^g + \bar{\zeta}^gg^g)(\dot{u}^g_i - \dot{u}^g) + (\bar{\zeta}^f + \bar{\zeta}^ff^f)(\dot{u}^f_i - \dot{u}^f) + \bar{\zeta}_0^sf^s, \\
\varrho_0^f \ddot{u}^f_i &= \mu \dot{u}^f_{i,j} + (\lambda + \mu) u^f_{j,ji} - \sigma^f u^f_{j,ji} - \sigma^g u^g_{j,ji} - \sigma^s u^s_{j,ji} - (\lambda^f + \gamma^f) T_i, \\
&\quad - \bar{\zeta}^f (\dot{u}^f_i - \dot{u}^f) - \bar{\zeta}^g (\dot{u}^g_i - \dot{u}^g) + \bar{\zeta}_0^sf^f, \\
\varrho_0^g \ddot{u}^g_i &= -\sigma^g u^g_{j,ji} - \sigma^f u^f_{j,ji} - \sigma^s u^s_{j,ji} - (\lambda^g + \gamma^g) T_i - \bar{\zeta}^g (\dot{u}^g_i - \dot{u}^g) - \bar{\zeta}^f (\dot{u}^f_i - \dot{u}^f) + \bar{\zeta}_0^sf^g, \\
0 &= \alpha_1 \dot{T} - \sigma \dot{T} + \left( \frac{\varrho^s}{T_0} + \varrho^g \right) \dddot{u}^s_{i,l} + \left( \frac{\varrho^f}{T_0} + \varrho^g \right) \dddot{u}^f_{i,l} + \left( \frac{\varrho^g}{T_0} - \frac{\varrho}{T_0} \right) \dddot{u}^g_{l,i} - \frac{k}{T_0} T_{ji} - \frac{q_0}{T_0},
\end{align*}
\]

where \( \varrho_0^s, \varrho_0^f \) and \( \varrho_0^g \) denote the densities of the three constituents at the time \( t = 0 \), \( f^s_i, f^f_i \) and \( f^g_i \) are the body forces, \( h \) is the heat supply, \( u^s_i, u^f_i \) and \( u^g_i \) are the displacement vector fields, \( T \) is the variation of temperature from the reference absolute temperature \( T_0 > 0 \) and \( \lambda, \mu, \sigma^a, \lambda^a, \mu^a, \tau^a, \zeta^a, \gamma^a \) \((a = f, g)\), \( \lambda, \mu, \sigma^ab, \tau^ab \) \((a, b = f, g)\), \( \alpha_0, \alpha_1, \sigma \) and \( k \) are constitutive constants. The coefficients \( \zeta^ab \) \((a, b = f, g)\) have the following symmetry

\[
\zeta^ab = \zeta^ba, \quad a, b = f, g.
\]
It should be remarked that the following relations from [1]: (3.9), (3.26) (in this relation the coefficient of $r a b$ is reported as $1/C a b$ instead of $-\sigma a b$ as there seems to be according the subsequent developments) and (3.28) prove that the coefficients $r a b (a, b = f, g)$ have the following symmetry

$$\sigma a b = \sigma b a, \quad a, b = f, g. \tag{2.3}$$

Moreover, the relations (3.24) and (3.24) of [1] imply that:

$$\gamma a = \frac{1}{A_0} \gamma a, \quad a, b = f, g. \tag{2.4}$$

Further, it was shown in [1] that the Clausius–Duhem inequality implies that

$$\mu_v \geq 0, \quad 3\lambda_v + 2\mu_v \geq 0, \quad \omega \geq 0, \tag{2.5}$$

and the following symmetric matrix

$$A = \begin{pmatrix} \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma \end{pmatrix} \tag{2.6}$$

is positive semi-definite.

To the Eq. (2.1) we have to adjoin the following boundary conditions

$$u_t^s = 0, \quad u_t^f = 0, \quad u_t^g = 0, \quad T = 0, \quad \text{on } \partial B \times [0, \infty), \tag{2.7}$$

and the following initial conditions

$$u_t^s(x, 0) = \tilde{u}_t^s(x), \quad u_t^f(x, 0) = \tilde{u}_t^f(x), \quad u_t^g(x, 0) = \tilde{u}_t^g(x), \quad u_t^s(x, 0) = \tilde{u}_t^s(x),$$

$$u_t^f(x, 0) = \tilde{v}_t^f(x), \quad u_t^g(x, 0) = \tilde{v}_t^g(x), \quad u_t^s(x, 0) = \tilde{v}_t^s(x),$$

$$T(x, 0) = T^0(x), \quad T(x, 0) = \tilde{T}^0(x), \quad \text{x } \in \overline{B}, \tag{2.8}$$

where $\tilde{u}_t^s, \tilde{u}_t^f, \tilde{u}_t^g, \tilde{v}_t^f, \tilde{v}_t^g, T^0$ and $\tilde{T}^0$ are prescribed continuous functions. We denote by $\mathcal{P}$ the initial-boundary value problem defined by Eq. (2.1), the boundary conditions (2.7) and the initial conditions (2.8). In what follows we discuss the uniqueness and continuous data dependence of the solutions of the initial-boundary value problem $\mathcal{P}$. For convenience we restricted our analysis to the temperature–displacement boundary conditions, although other boundary conditions can be treated by a technique similar with that in the following.

3. Uniqueness results

In this section we study the uniqueness of solutions to the basic initial-boundary value problems of the theory of swelling porous thermoelastic soils. With a view to treat the uniqueness
question we consider two solutions of the above initial-boundary value problem \( \mathcal{P} \) corresponding to the same given data and denote by \( \{u_i^e, u_i^f, u_i^g, T\} \) their difference. Then \( \{u_i^e, u_i^f, u_i^g, T\} \) satisfies the following differential equations

\[
q_0^e \ddot{u}_i^e = \mu \dddot{u}_{i,jj} + (\lambda + \mu) \dddot{u}_{i,jj} - \sigma^f \dddot{u}_{i,jj} - \sigma^g \dddot{u}_{i,jj} + (\gamma_f + \gamma_g - \chi_0) T_j \\
+ (\zeta^g + \chi^g)(\dot{u}_i^e - \dot{u}_i^f) + (\zeta^f + \chi^f)(\dot{u}_i^f - \dot{u}_i^g),
\]

\[
q_0^f \ddot{u}_i^f = \mu \dddot{u}_{i,jj} + (\lambda + \mu) \dddot{u}_{i,jj} - \sigma^f \dddot{u}_{i,jj} - \sigma^f \dddot{u}_{i,jj} - \sigma^g \dddot{u}_{i,jj} - (\gamma_f + \gamma_f) T_j \\
- \xi^f (\ddot{u}_i^f - \ddot{u}_i^f) - \xi^f (\ddot{u}_i^f - \ddot{u}_i^f),
\]

\[
q_0^g \ddot{u}_i^g = -\sigma^g \dddot{u}_{i,jj} - \sigma^g \dddot{u}_{i,jj} - (\gamma_f + \gamma_f) T_j - \xi^g (\ddot{u}_i^g - \ddot{u}_i^g) - \xi^f (\ddot{u}_i^f - \ddot{u}_i^g),
\]

\[
0 = \chi_1 \dddot{T} - \sigma_0 \dddot{\mathcal{T}} + (\gamma_g + \gamma_f) \dddot{u}_i^g + (\gamma_f + \gamma_f) \dddot{u}_i^f + (\chi_0 - \gamma_g - \gamma_f) \dddot{u}_i^g - \frac{k}{T_0} T_{ji},
\]

and the boundary conditions

\[
u_i^e = 0, \quad u_i^f = 0, \quad u_i^g = 0, \quad T = 0, \quad \text{on } \partial \Omega \times [0, \infty),
\]

and the initial conditions

\[
u_i^e(x, 0) = 0, \quad u_i^f(x, 0) = 0, \quad u_i^g(x, 0) = 0, \quad \dot{u}_i^e(x, 0) = 0, \quad \dot{u}_i^f(x, 0) = 0,
\]

\[
u_i^g(x, 0) = 0, \quad T(x, 0) = 0, \quad \dddot{T}(x, 0) = 0, \quad x \in \partial \Omega.
\]

In order to study the uniqueness question it is sufficiently to prove that only solution of the initial boundary value problem \( \mathcal{P}_0 \) defined by the relations (3.1)–(3.3) is vanishing.

3.1. First uniqueness result

In this subsection we shall assume the following hypotheses upon the mass densities and thermoelastic coefficients of mixture:

\[
\rho_0^e > 0, \quad \rho_0^f > 0, \quad \rho_0^g > 0, \quad \chi_1 > 0,
\]

and the following four quadratic forms are positive semi-definite

\[
\Pi(U) = \mu \nu_{i,j} u_{i,j} + (\lambda + \mu) \nu_{i,j} u_{i,j} - \sigma^f \nu_{i,j} u_{i,j} - \sigma^g \nu_{i,j} u_{i,j} - 2\sigma^f \nu_{i,j} u_{i,j} - 2\sigma^f \nu_{i,j} u_{i,j} - 2\sigma^f \nu_{i,j} u_{i,j},
\]

\[
U = \{u_{i,j}^e, u_{i,j}^f, u_{i,j}^g\},
\]

\[
A_1(U_1) = \mu \dddot{u}_{i,j} + (\lambda + \mu) \dddot{u}_{i,j}, \quad U_1 = \{\dot{u}_{i,j}^f\},
\]

\[
A_2(U_2) = \chi^f (\dddot{u}_i^f - \dddot{u}_i^f) + \chi^g (\dddot{u}_i^g - \dddot{u}_i^g) + 2\chi^f (\dddot{u}_i^f - \dddot{u}_i^f) (\dddot{u}_i^g - \dddot{u}_i^g) + \chi^f (\dddot{u}_i^f - \dddot{u}_i^f),
\]

\[
U_2 = \{\dddot{u}_i^f - \dddot{u}_i^f, \dddot{u}_i^g - \dddot{u}_i^g\},
\]
\[
A_3(T) = \frac{k}{T_0} T_s T_j + \frac{1}{2} \sigma T^2.
\]  

(3.8)

The quadratic form \( A_1(U_1) \) will be non-negative, for all values of \( \tilde{u}^c_{i,j} \), if and only if
\[
\mu_v \geq 0, \quad \lambda_v + \frac{4}{3} \mu_v \geq 0,
\]  

(3.9)

while the quadratic form \( A_2(U_2) \) will be non-negative, for all values of \( \tilde{u}^c_j - \tilde{u}^c_i \) and \( \tilde{u}^b_j - \tilde{u}^b_i \), if and only if
\[
\xi^\text{ff} \geq 0, \quad \xi^\text{ff} \xi^\text{gg} - (\xi^\text{gg})^2 \geq 0.
\]  

(3.10)

Finally, we note that the quadratic form \( A_3(T) \) will be non-negative, for all values of \( \tilde{T} \) and \( T_j \), if and only if
\[
k \geq 0, \quad \sigma \geq 0.
\]  

(3.11)

We have to outline here that our hypotheses (3.9)–(3.11) are concerned with the constitutive moduli characterising the dissipation potential. As it can be seen these hypotheses are weaker than the restrictions (2.5) and (2.6) characterising the positive semi-definiteness of the dissipation potential. On the other hand, the quadratic form \( \Pi(U) \) can be posed in connection with the internal energy density \( \epsilon \) (see [1], relations (2.18), (2.22), (3.26) and (3.27)). Our hypothesis that \( \Pi(U) \) to be a positive semi-definite quadratic form is a weaker form of the hypothesis that the internal energy density to be positive semi-definite. This last hypothesis has been used by Galeş [2, p. 4154] and Quintanilla [8, p. 64] in their studies on the uniqueness and continuous dependence problems.

Let us consider \( \{u^c_i, u^c_j, u^b_i, T\} \) a solution of the initial-boundary value problem \( \mathcal{P}_0 \) and introduce the following notations
\[
\mathcal{E}(t) = \frac{1}{2} \int_B \left[ \rho_0^c \dot{u}^c_i(t) \ddot{u}^c_i(t) + \rho_0^c \dot{u}^c_j(t) \ddot{u}^c_j(t) + \rho_0^b \dot{u}^b_i(t) \ddot{u}^b_i(t) + \rho_0^b \dot{u}^b_j(t) \ddot{u}^b_j(t) + z_1 T^2(t) + \Pi(U) \right] \mathrm{d}v,
\]  

(3.12)

\[
\mathcal{D}(t) = \int_B \left[ A_1(U_1) + A_2(U_2) + A_3(T) \right] \mathrm{d}v.
\]  

(3.13)

Further, we start with the following identity
\[
\frac{\partial}{\partial t} \left\{ \frac{1}{2} \left[ \rho_0^c \dot{u}^c_i(t) \ddot{u}^c_i(t) + \rho_0^c \dot{u}^c_j(t) \ddot{u}^c_j(t) + \rho_0^b \dot{u}^b_i(t) \ddot{u}^b_i(t) + \rho_0^b \dot{u}^b_j(t) \ddot{u}^b_j(t) + z_1 T^2(t) \right] \right\}
= \rho_0^c \dot{u}^c_i(t) \ddot{u}^c_i(t) + \rho_0^c \dot{u}^c_j(t) \ddot{u}^c_j(t) + \rho_0^b \dot{u}^b_i(t) \ddot{u}^b_i(t) + \rho_0^b \dot{u}^b_j(t) \ddot{u}^b_j(t) + z_1 T(t) \dot{T}(t).
\]  

(3.14)

Then, by using the basic equations (3.1) and the boundary conditions (3.2) and the initial conditions (3.3) and an integration by parts and the divergence theorem and the notations (3.5)–(3.8), (3.12) and (3.13), we deduce that
\[
\int_0^t \sigma(s) \, ds + \int_0^t \int_B \mathcal{D}(z) \, dz \, ds + \frac{\sigma}{2} \int_0^t \int_B \mathcal{T}^2(z) \, dz \, ds - \frac{\sigma}{2} \int_B T^2(t) \, dv = 0,
\forall t \in [0, \infty).
\]

The relation (3.15) constitutes the basis for our analysis in this subsection. We first note that for \( \sigma = 0 \) the relation (3.15), when combined with the assumptions (3.4)–(3.8) and the initial conditions (3.3), furnishes

\[
u^2(t, x) = 0, \quad \nu^1(t, x) = 0, \quad \nu^0(x, t) = 0, \quad T(t, x) = 0, \quad \forall (x, t) \in \overline{B} \times [0, \infty),
\]

that is the uniqueness of solutions of the initial boundary value problem \( \mathcal{P} \).

Let us now consider the case \( \sigma > 0 \). Then we write the relation (3.15) in the following form

\[
\int_0^t \left[ \sigma(s) - \frac{1}{2} x_1 \int_B T^2(s) \, dv \right] \, ds + \int_0^t \int_B \mathcal{D}(z) \, dz \, ds + \frac{\sigma}{2} \int_0^t \int_B \mathcal{T}^2(z) \, dz \, ds + \frac{x_1}{2} \int_0^t \int_B T^2(s) \, dv \, ds - \frac{\sigma}{2} \int_B T^2(t) \, dv = 0, \quad \forall t \in [0, \infty).
\]

In view of the hypotheses (3.4)–(3.8), (3.12) and (3.13), we can conclude that the first three integral terms in (3.17) are non-negative. Further, from the relation (3.17) it results necessary to have

\[
\sigma \int_B T^2(t) \, dv \geq x_1 \int_0^t \int_B T^2(s) \, dv \, ds, \quad \forall t \in [0, \infty),
\]

that is

\[
\int_B T^2(t) \, dv \geq \omega^2 \int_0^t \int_B T^2(s) \, dv \, ds, \quad \forall t \in [0, \infty), \quad \omega^2 = \frac{x_1}{\sigma}.
\]

We further set

\[
\chi^2(t) = \int_0^t \int_B T^2(s) \, dv \, ds, \quad t \in [0, \infty),
\]

and note that the relation (3.19) becomes

\[
2 \chi(t) \dot{\chi}(t) \geq \omega^2 \chi^2(t), \quad t \in [0, \infty).
\]

If \( \chi(t) = 0 \) for all \( t \in [0, \infty) \), then it results that \( T(x, t) = 0 \) for all \( (x, t) \in \overline{B} \times [0, \infty) \) and the problem is reduced to the above case, that is we arrive to the uniqueness result. To this end, we can assume that there exists \( \tau \in [0, \infty) \) so that \( \chi(\tau) > 0 \) and hence we have

\[
\chi(t) > 0 \quad \text{for all} \ t \in [\tau, \infty).
\]
Then the relation (3.21) implies that
\[
\frac{d}{dt} \left\{ \chi(t)e^{-(\omega^2/2)t} \right\} \geq 0, \quad \forall t \in [\tau, \infty).
\] (3.23)

Thus, we deduce that
\[
\chi(\tau)e^{-(\omega^2/2)\tau} \leq \chi(t)e^{-(\omega^2/2)t} \leq \lim_{t \to \infty} \left[ \chi(t)e^{-(\omega^2/2)t} \right].
\] (3.24)

If we assume that
\[
\left( \int_0^t \int_B T^2(s) dv ds \right)^{1/2} \leq M^2e^{at}, \quad \forall t \in [0, \infty), \quad M = \text{const.}, \quad 0 \leq a < \frac{\omega^2}{2},
\] (3.25)

then the relation (3.24) implies that \( \chi(t) = 0 \) for all \( t \in [0, \infty) \) and this contradicts our initial assumption (3.22). Thus we can conclude that
\[
T(x, t) = 0 \quad \text{for all} \quad (x, t) \in \overline{B} \times [0, \infty).
\] (3.26)

If we further use the relation (3.26) into relation (3.15), then, on the basis of the assumptions (3.4)–(3.8), we deduce that
\[
\dot{u}_s(x, t) = 0, \quad \dot{u}_f(x, t) = 0, \quad \dot{u}_g(x, t) = 0, \quad \forall (x, t) \in B \times [0, \infty).
\] (3.27)

If we use the initial conditions (3.3), from the relation (3.27) we deduce
\[
u_s^0(x, t) = 0, \quad \nu_f^0(x, t) = 0, \quad \nu_g^0(x, t) = 0, \quad \forall (x, t) \in \overline{B} \times [0, \infty),
\] (3.28)

and therefore, we have the relation (3.16), that is the uniqueness of solution.

Thus, we can conclude that, under the assumptions (3.4)–(3.8), in the class of solutions \( \{u_s^0, u_f^0, u_g^0, T\} \) for which the condition (3.25) holds true, we have the uniqueness for solutions of the initial boundary value problem \( \mathcal{P} \).

3.2. Second uniqueness result

Throughout this subsection we shall assume the following hypotheses upon the thermoelastic coefficients of mixture:
\[
\rho_s^0 > 0, \quad \rho_f^0 > 0, \quad \rho_g^0 > 0,
\] (3.29)

and the following quadratic form is positive semi-definite for all values of \( u_{ij}^0, u_i^0 - u_j^0, \nu_i^0 - \nu_j^0, \int_0 T_s(z) dz \) and \( \mathcal{T} \):
appropriate null initial conditions we obtain

\[
\hat{A}(U, T) = \mu_i u_{i,j}^e u_{i,j} + (\lambda_v + \mu_i) u_{i,j} + \varepsilon^{ff}(u_{i,j}^e - u_i) (u_{i,j}^e - u_i) + \varepsilon^{gg}(u_{i,j}^e - u_i) (u_{i,j}^e - u_i) \\
+ 2\xi^{gg}(u_{i,j}^e - u_i) (u_{i,j}^e - u_i) + \frac{k}{T_0} \left( \int_0^t T_i(z) \, dz \right) \left( \int_0^t T_i(z) \, dz \right) + \sigma \hat{T}^2.
\] (3.30)

The last quadratic form will be non-negative, for all values of \( u_{i,j}^e, u_i^e - u_i, \int_0^t T_i(z) \, dz \) and \( \hat{T} \), if and only if

\[
\mu_i \geq 0, \quad \lambda_v + \frac{3}{\beta} \mu_i \geq 0, \quad \xi^{ff} \geq 0, \quad \xi^{gg} - (\xi^{gg})^2 \geq 0, \quad k \geq 0, \quad \sigma \geq 0.
\] (3.31)

First of all we integrate Eq. (3.1)_4 with respect to time on the interval \([0, t], t \in [0, \infty)\) and then use the initial conditions (3.3) to obtain

\[
0 = \dot{x}_1 T - \sigma \hat{T} + (\gamma^e + \alpha^e) u_{i,j}^e + (\gamma^f + \alpha^f) u_{i,j} + (\gamma_0 - \gamma^e - \gamma^f) u_{i,j} - \frac{k}{T_0} \int_0^t T_i(z) \, dz.
\] (3.32)

Now we start with the following identities

\[
\frac{\partial}{\partial s} \{ \kappa[V(t+s) V(t-s)] \} = \kappa \left[ \dot{V}(t+s) V(t-s) - V(t+s) \dot{V}(t-s) \right],
\] (3.33)

\[
\frac{\partial}{\partial s} \left\{ \beta[\tilde{W}(t+s) W(t-s) + W(t+s) \tilde{W}(t-s)] \right\} = \beta \left[ \dot{W}(t+s) W(t-s) - W(t+s) \dot{W}(t-s) \right]
\] (3.34)

for all \( s \in [0, t] \), so that by an integration with respect to \( s \) over \([0, t] \) and by taking into account appropriate null initial conditions we obtain

\[
\int_B \kappa V^2(t) \, dv = \int_0^t \int_B \kappa \left[ \dot{V}(t-s) V(t+s) - V(t-s) \dot{V}(t+s) \right] \, dv \, ds,
\] (3.35)

\[
2 \int_B \beta W(t) \dot{W}(t) \, dv = \int_0^t \int_B \beta \left[ \dot{W}(t-s) W(t+s) - W(t-s) \dot{W}(t+s) \right] \, dv \, ds.
\] (3.36)

By setting \( V(x, t) = T(x, t) \) and \( \kappa = \sigma \) into the identity (3.35) and then by using the basic equation (3.32) and the null boundary conditions (3.2), we get
\[ \sigma \int_B T^2(t) \, dv + \frac{k}{T_0} \int_B \left( \int_0^t T_s(z) \, dz \right) \left( \int_0^t T_z(z) \, dz \right) \, dv \]
\[ = (\gamma^g + \alpha^g) \int_B \left[ T(t + s) u^g_{ij}(t - s) - T(t - s) u^g_{ij}(t + s) \right] \, dv \, ds \]
\[ + (\gamma^f + \alpha^f) \int_B \left[ T(t + s) u^f_{ij}(t - s) - T(t - s) u^f_{ij}(t + s) \right] \, dv \, ds \]
\[ + (\alpha_0 - \gamma^g - \gamma^f) \int_B \left[ T(t + s) u^g_{ij}(t - s) - T(t - s) u^g_{ij}(t + s) \right] \, dv \, ds. \] (3.37)

We now set \( \beta = \rho_0^g \) and \( W(x, t) = u^g_i(x, t) \) in (3.36) and then use Eq. (3.1)_1 and the null boundary conditions (3.2) in order to obtain

\[ 2 \int_B \rho_0^g u_i^g(t) \, \ddot{u}_i^g(t) \, dv = \int_0^t \int_B \left\{ \sigma^g \left[ u^g_{ij}(t - s) u^g_{ij}(t + s) - u^g_{ij}(t + s) u^g_{ij}(t - s) \right] \right. \]
\[ + \sigma^f \left[ u^f_{ij}(t - s) u^f_{ij}(t + s) - u^f_{ij}(t + s) u^f_{ij}(t - s) \right] + (\alpha_0 - \gamma^f - \gamma^g) \]
\[ \times \left[ T(t - s) u^g_{ij}(t + s) - T(t + s) u^g_{ij}(t - s) \right] + (\xi^g + \xi^g) [\dot{u}^g_i(t - s) u^g_i(t + s) \]
\[ - \dot{u}^g_i(t + s) u^g_i(t - s) + \ddot{u}^g_i(t - s) u^g_i(t + s) - \ddot{u}^g_i(t + s) u^g_i(t - s) \]
\[ + (\xi^g + \xi^f) [\dot{u}^f_i(t - s) u^f_i(t + s) - \dot{u}^f_i(t + s) u^f_i(t - s)] \]
\[ - \ddot{u}^f_i(t - s) u^f_i(t + s) + \ddot{u}^f_i(t + s) u^f_i(t - s) \] \( \ddots \) \( (3.38) \)

Further, we set \( \beta = \rho_0^f \) and \( W(x, t) = u^f_i(x, t) \) in (3.36) and then use Eq. (3.1)_2 and the null boundary conditions (3.2) in order to obtain

\[ 2 \int_B \rho_0^f u_i^f(t) \, \ddot{u}_i^f(t) \, dv = \int_0^t \int_B \left\{ \sigma^f \left[ u^g_{ij}(t - s) u^g_{ij}(t + s) - u^g_{ij}(t + s) u^g_{ij}(t - s) \right] \right. \]
\[ + \sigma^f \left[ u^f_{ij}(t - s) u^f_{ij}(t + s) - u^f_{ij}(t + s) u^f_{ij}(t - s) \right] + (\alpha_0 - \gamma^f - \gamma^g) \]
\[ \times \left[ T(t - s) u^g_{ij}(t + s) - T(t + s) u^g_{ij}(t - s) \right] + (\xi^g + \xi^g) [\dot{u}^g_i(t - s) u^g_i(t + s) \]
\[ - \dot{u}^g_i(t + s) u^g_i(t - s) + \ddot{u}^g_i(t - s) u^g_i(t + s) - \ddot{u}^g_i(t + s) u^g_i(t - s) \]
\[ + (\xi^g + \xi^f) [\dot{u}^f_i(t - s) u^f_i(t + s) - \dot{u}^f_i(t + s) u^f_i(t - s)] \]
\[ - \ddot{u}^f_i(t - s) u^f_i(t + s) + \ddot{u}^f_i(t + s) u^f_i(t - s) \] \( \ddots \) \( (3.39) \)

Finally, we set \( \beta = \rho_0^g \) and \( W(x, t) = u^g_i(x, t) \) in (3.36) and then use Eq. (3.1)_3 and the null boundary conditions (3.2) in order to obtain
In view of the hypotheses described by the relations (3.29) and (3.31), from the relation (3.42) we obtain the following identity

\[
2 \int_B \rho_0^g u^g_i(t) \dot{u}^g_i(t) \, dv = \int_0^t \int_B \left\{ \sigma^g \left[ u^g_{j,i}(t-s)u^g_{j,i}(t+s) - u^g_{j,i}(t+s)u^g_{j,i}(t) \right] \\
+ \sigma^{gr} \left[ u^r_{j,i}(t-s)u^g_{j,i}(t+s) - u^r_{j,i}(t+s)u^g_{j,i}(t) \right] \\
+ (\varepsilon^g + \gamma^g) [T(t-s)u^g_{j,i}(t+s) - T(t+s)u^g_{j,i}(t)] \\
+ \xi^{rg} [\dot{u}^g_i(t+s)u^g_i(t-s) - \dot{u}^r_i(t+s)u^g_i(t) + \dot{u}^r_i(t-s)u^g_i(t) + \dot{u}^g_i(t+s)u^g_i(t-s)] \\
+ \dot{u}^r_i(t+s)u^g_i(t+s) - \dot{u}^r_i(t+s)u^g_i(t)] \right\} ds \, dv.
\]  

(3.40)

Therefore, by summing the relations (3.37)–(3.40) and by using the identities (3.33) and (3.34) and the initial conditions (3.3), we obtain the following identity

\[
2 \int_B \left[ \rho_0^g u^g_i(t) \dot{u}^g_i(t) + \rho_0^r u^r_i(t) \dot{u}^r_i(t) + \rho_0^g u^g_i(t) \dot{u}^g_i(t) \right] \, dv + \sigma \int_B T^2(t) \, dv + \int_B \left\{ \mu_{ij} u^f_{ij}(t) \dot{u}^f_{ij}(t) \\
+ (\varepsilon + \mu_{ij}) u^f_{ij}(t) \dot{u}^f_{ij}(t) + \frac{k}{T_0} \left( \int_0^t T_i(z) \, dz \right) \left( \int_0^t T_j(z) \, dz \right) + \xi [u^f_i(t) - u^g_i(t)] [u^f_i(t) \\
- u^g_i(t)] \right\} \, dv \\
= 0.
\]  

(3.41)

Thus, by an integration with respect to \( s \) over \([0, t]\) and by taking into account the null initial conditions (3.3), from (3.41) we deduce that

\[
I(t) = 0, \quad \forall t \in [0, \infty),
\]  

(3.42)

where

\[
I(t) = \int_B \left[ \rho_0^g u^g_i(t) u^g_i(t) + \rho_0^r u^r_i(t) u^r_i(t) + \rho_0^g u^g_i(t) u^g_i(t) \right] \, dv + \sigma \int_0^t \int_B T^2(z) \, dv \, dz \\
+ \int_0^t \int_B \left\{ \mu_{ij} u^f_{ij}(z) u^f_{ij}(z) + (\varepsilon + \mu_{ij}) u^f_{ij}(z) u^f_{ij}(z) + \frac{k}{T_0} \left( \int_0^z T_i(s) \, ds \right) \left( \int_0^z T_j(s) \, ds \right) \\
+ \xi [u^f_i(z) - u^g_i(z)] [u^f_i(z) - u^g_i(z)] + \xi [u^g_i(z) - u^r_i(z)] [u^f_i(z) - u^g_i(z)] + 2 \xi [u^f_i(z) \\
- u^g_i(z)] [u^f_i(z) - u^g_i(z)] \right\} \, dv \, dz.
\]  

(3.43)

In view of the hypotheses described by the relations (3.29) and (3.31), from the relation (3.42) we deduce that

\[
u^g_i(x, t) = 0, \quad u^r_i(x, t) = 0, \quad u^g_i(x, t) = 0, \quad \forall (x, t) \in \overline{B} \times [0, \infty)
\]  

(3.44)
and
\[
\sigma T^2(x,t) + \frac{k}{T_0} \left( \int_0^t T_s(x,z) \, dz \right) \left( \int_0^t T_s(x,z) \, dz \right) = 0, \quad \forall (x,t) \in \mathcal{B} \times [0, \infty). \tag{3.45}
\]

If \( \sigma > 0 \) then the relation (3.45) implies that
\[
T(x,t) = 0, \quad \forall (x,t) \in \mathcal{B} \times [0, \infty). \tag{3.46}
\]

Thus, in this case the relations (3.44) and (3.46) imply the requested uniqueness result.

Let us now consider the case \( \sigma = 0 \). Then, in view of the fact that we have \( k \geq 0 \), we can consider the following cases: (a) \( k > 0 \); (b) \( k = 0 \). In the first case the relation (3.45) gives
\[
T_s(x,t) = 0, \quad \forall (x,t) \in B \times [0, \infty), \tag{3.47}
\]

and hence, in view of the null boundary conditions, we deduce the relation (3.46), that is the uniqueness result. For the second case, that is for \( k = 0 \), the relations (3.32) and (3.44) imply that
\[
\alpha T(x,t) = 0, \quad \forall (x,t) \in B \times [0, \infty), \tag{3.48}
\]

and hence we have uniqueness result if we assume that \( \alpha \neq 0 \).

4. Continuous data dependence

In this section we discuss how the above procedures can be extended to be used to study the continuous data dependence of solutions of the initial-boundary value problem \( \mathcal{P} \).

4.1. An estimate based on an integro-differential inequality

In this subsection we assume that the hypotheses described by the relations (3.4), (3.5) and (3.9)–(3.11) hold true. Our aim in this subsection is to establish an estimate describing the continuous dependence of solutions of the initial-boundary value problem \( \mathcal{P} \) with respect to supply terms. To this end, in view of the linearity of problem, in what follows we shall assume that \( \{u_i^s, u_i^f, u_i^g, T\} \) is the solution of the problem \( \mathcal{P} \) with null initial and boundary data and the supply terms \( \{f_i^s, f_i^f, f_i^g, h\} \). With this solution we associate the following measure
\[
\mathcal{G}(t) = \int_0^t \mathcal{E}(s) \, ds + \int_0^t \int_0^s \mathcal{D}(z) \, dz \, ds + \frac{\sigma}{2} \int_0^t \int_0^s \int_B \mathcal{T}^2(z) \, dv \, dz \, ds, \quad \forall t \in [0, \infty), \tag{4.1}
\]

where \( \mathcal{E}(t) \) and \( \mathcal{D}(t) \) are defined by the relations (3.12) and (3.13), respectively. Under our constitutive assumptions we can see that we have
\[
\mathcal{G}(t) \geq 0, \quad \forall t \in [0, \infty), \tag{4.2}
\]
and \( \mathcal{G}(t) = 0, \forall t \in [0, \infty) \), implies that \( \{u^i, u^f, u^g, T\} = \{0, 0, 0, 0\} \). Moreover, by a direct differentiation, from (4.1) we deduce

\[
\dot{\mathcal{G}}(t) \geq \frac{1}{2} \alpha_1 \int_B T^2(t) \, dv, \quad \forall t \in [0, \infty). \tag{4.3}
\]

In the above conditions, the fundamental identity (3.15) becomes

\[
\mathcal{G}(t) = \frac{\sigma}{2} \int_B T^2(t) \, dv + \int_0^t \int_B \left\{ \rho^s_0 f^s_i(z) \dot{u}^i_s(z) + \rho^f_0 f^f_i(z) \dot{u}^f_i(z) + \rho^g_0 f^g_i(z) \dot{u}^g_i(z) \\
+ \frac{\rho_0}{T_0} h(z) T(z) \right\} \, dv \, dz \, ds, \quad \forall t \in [0, \infty). \tag{4.4}
\]

By means of the Cauchy–Schwarz inequality and by using the relations (4.1) and (4.3), from (4.4) we obtain

\[
\mathcal{G}(t) \leq \frac{\sigma}{\alpha_1} \int_0^t \mathcal{G}(s)^{1/2} g(s) \, ds, \quad \forall t \in [0, \infty), \tag{4.5}
\]

where

\[
g(t) = \left( 2 \int_0^t \int_B \left[ \rho^s_0 f^s_i(z) f^s_i(z) + \rho^f_0 f^f_i(z) f^f_i(z) + \rho^g_0 f^g_i(z) f^g_i(z) + \frac{\rho_0^2}{T_0^2} h^2(z) \right] \, dv \, dz \right)^{1/2}. \tag{4.6}
\]

We now define the function \( \Psi(t) \) by

\[
\Psi(t) = [\mathcal{G}(t)]^{1/2}, \quad \forall t \in [0, \infty), \tag{4.7}
\]

and note that it is nondecreasing with respect to \( t \). Then the integro-differential inequality (4.5) can be written in the form

\[
\Psi^2(t) \leq \frac{2\sigma}{\alpha_1} \Psi(t) \Psi(t) + \int_0^t \Psi(s) g(s) \, ds, \quad \forall t \in [0, \infty). \tag{4.8}
\]

We further proceed to integrate the integro-differential inequality (4.8). To this end we shall treat separately the cases in which \( \sigma = 0 \) and \( \sigma > 0 \).

Let us first consider the case \( \sigma = 0 \). Then the relation (4.8) becomes

\[
\Psi^2(t) \leq \int_0^t \Psi(s) g(s) \, ds, \quad \forall t \in [0, \infty). \tag{4.9}
\]

By the Gronwall’s lemma (see, for example, [12]), from the relation (4.9) we obtain the estimate

\[
\Psi(t) \leq \frac{1}{2} \int_0^t g(s) \, ds, \quad \forall t \in [0, \infty), \tag{4.10}
\]
and hence, by (4.7) we have

$$[\mathcal{G}(t)]^{1/2} \leq \frac{1}{2} \int_0^t g(s) \, ds, \quad \forall t \in [0, \infty).$$

(4.11)

Such an estimate describe the continuous dependence of the solution \(\{u_i, u_i^f, u_i^f, T\}\) with respect to the supply term \(\{f^i, f^i_f, f^i_f, h\}\).

Let us now consider the case \(\sigma > 0\). Since \(\mathcal{Y}(s) \leq \mathcal{Y}(t)\) for all \(s \leq t\), from the relation (4.8) we deduce that

$$\mathcal{Y}(t) \leq \mathcal{Y}(t) \frac{2\sigma}{\lambda_1} \mathcal{Y}(t) + \int_0^t \mathcal{Y}(s) g(s) \, ds \leq \mathcal{Y}(t) \left[ \frac{2\sigma}{\lambda_1} \mathcal{Y}(t) + \int_0^t g(s) \, ds \right],$$

(4.12)

so that we get

$$\mathcal{Y}(t) \leq \frac{2\sigma}{\lambda_1} \mathcal{Y}(t) + \int_0^t g(s) \, ds, \quad \forall t \in [0, \infty).$$

(4.13)

Further, we can write

$$\frac{d}{dt} \left\{ e^{-(\lambda_1/2\sigma)t} \mathcal{Y}(t) + \frac{\lambda_1}{2\sigma} \int_0^t e^{-(\lambda_1/2\sigma)s} \left( \int_0^s g(z) \, dz \right) \, ds \right\} \geq 0, \quad \forall t \in [0, \infty),$$

(4.14)

so that, by an integration, we obtain

$$0 \leq e^{-(\lambda_1/2\sigma)t} \mathcal{Y}(t) + \frac{\lambda_1}{2\sigma} \int_0^t e^{-(\lambda_1/2\sigma)s} \left( \int_0^s g(z) \, dz \right) \, ds \leq \lim_{t \to -\infty} e^{-(\lambda_1/2\sigma)t} \mathcal{Y}(t)$$

$$+ \frac{\lambda_1}{2\sigma} \int_0^t e^{-(\lambda_1/2\sigma)s} \left( \int_0^s g(z) \, dz \right) \, ds.$$  

(4.15)

Thus, if we suppose that the given data are so that there exists the integral

$$\int_0^\infty e^{-(\lambda_1/2\sigma)s} \left( \int_0^s g(z) \, dz \right) \, ds < \infty,$$

(4.16)

then, in the class of solutions of Eq. (2.1) for which

$$\lim_{t \to -\infty} \left\{ e^{-(\lambda_1/2\sigma)t} \mathcal{Y}(t) \right\} = 0,$$

(4.17)

we have the following estimate

$$0 \leq \mathcal{Y}(t) \leq \frac{\lambda_1}{2\sigma} \int_t^\infty e^{-(\lambda_1/2\sigma)(s-t)} \left( \int_0^s g(z) \, dz \right) \, ds, \quad \forall t \in [0, \infty).$$

(4.18)
Thus, the estimate (4.18) describe the continuous dependence of the solution \( \{ u^s_t, u^f_t, u^g_t, T \} \), in the class satisfying (4.17), with respect to the supply term \( \{ f^s_t, f^f_t, f^g_t, h \} \) for which (4.16) holds true, in the case when \( \sigma > 0 \).

4.2. Lagrange identity method

In the remainder of this paper we shall use the Lagrange–Brun identity in order to obtain estimates describing the continuous data dependence of solutions of the initial-boundary value problem \( \mathcal{P} \). In this aim we assume that the constitutive constants satisfy the conditions (3.29) and (3.31).

Let us suppose that \( \{ u^s_t, u^f_t, u^g_t, T \} \) is the solution of the initial-boundary value problem \( \mathcal{P} \), corresponding to the supply term \( \{ f^s_t, f^f_t, f^g_t, h \} \) and to null initial and boundary data. Then the identity (3.42) becomes

\[
I(t) = \int_0^t \int_B \left\{ \rho_0^s f^s_t(t-s)u^s_t(t+s) - f^s_t(t+s)u^s_t(t-s) + \rho_0^f f^f_t(t-s)u^f_t(t+s) - f^f_t(t+s)u^f_t(t-s) + \rho_0^g f^g_t(t-s)u^g_t(t+s) - f^g_t(t+s)u^g_t(t-s) \right. \\
+ \left. \frac{T(t-s)}{T_0} \int_0^{t-s} h(z) \, dz - T(t+s) \int_0^{t-s} h(z) \, dz \right\} \, dv \, ds, \quad \forall t \in [0, \infty), \tag{4.19}
\]

where \( I(t) \) is defined by the relation (3.43). This identity allows us to establish estimates describing the continuous dependence of solutions. As an example we consider the continuous dependence of solution with respect to heat source. In this aim we define the class of solutions to (2.1) for which there is

\[
\int_0^\infty \int_B T^2(z) \, dv \, dz \leq M^2, \quad M = \text{const.} \tag{4.20}
\]

We further note that in this class, by means of the Cauchy–Schwarz inequality, the relation (4.19) implies the following estimate

\[
I(t) \leq \frac{\rho_0 M}{T_0} \left\{ \left( \int_0^t \int_B \left[ \int_0^z h(\eta) \, d\eta \right]^2 \, dv \, dz \right)^{1/2} + \left( \int_0^t \int_B \left[ \int_0^z h(\eta) \, d\eta \right]^2 \, dv \, dz \right)^{1/2} \right\}, \quad \forall t \in [0, \infty). \tag{4.21}
\]

As it can be seen, the relation (4.21) furnishes the continuous dependence with respect to heat sources of the solutions of the problem \( \mathcal{P} \) in the class described by the relation (4.20).

We outline that, by the same manner as in the above, we can individuate classes of solutions where we can establish estimates describing the continuous dependence with respect to other body forces.
Restrictions of the type described by the relation (4.20) are usually assumed when the problem of continuous dependence is considered (see, for example, Knops and Payne [13] and Rionero and Chiriţă [14]).

5. An extension

In the above analysis we have used the null boundary conditions (3.2) to establish some uniqueness and continuous dependence results under assumptions like (3.9), (3.10) or (3.31) and we have outlined that such assumptions are weaker than those reported in [1] as consequences of the dissipation inequality. This was possible because we have used an integration by parts based on an identity of the type

\[ u_i^f u_i^f = (u_i^f u_i^f)_j - u_j^f u_i^f \]

(5.1)

combined with the null boundary conditions (3.2). In this section we use a different integration by parts based on an identity of type

\[ u_i^f u_i^f = (u_i^f u_i^f)_j - u_j^f u_i^f \]

(5.2)

coupled with the null boundary conditions (3.2).

In this way we will be able to establish the uniqueness result of the Subsection 3.1 by substituting the hypotheses (3.4)–(3.8) by appropriate ones. Thus, throughout this subsection we shall assume that

\[ \rho_0^e > 0, \quad \rho_0^g > 0, \quad \rho_0^g > 0, \quad \alpha_1 > 0; \]

(5.3)

moreover, we shall assume that the following four quadratic forms

\[ \bar{\Phi}(U) = \mu u_i^e u_i^e + (\lambda + \mu)u_i^e u_i^e - \sigma_{i, j} u_i^e u_j^e - \sigma_{i, j} u_i^e u_j^e - 2\sigma_{i, j} u_i^e u_j^e - 2\sigma_{i, j} u_i^e u_j^e, \]

\[ U = \{ u_i^e, u_j^e, u_i^e \}, \]

(5.4)

\[ \bar{A}_1(U_1) = \mu u_i^e u_i^e + (\lambda + \mu)u_i^e u_i^e, \quad U_1 = \{ u_i^e \}, \]

(5.5)

\[ \bar{A}_2(U_2) = \zeta (u_i^e - \bar{u}_i^e)(u_i^e - \bar{u}_i^e) + \zeta (u_i^e - \bar{u}_i^e)(u_i^e - \bar{u}_i^e) + 2\xi (u_i^e - \bar{u}_i^e)(u_i^e - \bar{u}_i^e), \]

\[ U_2 = \{ u_i^e - \bar{u}_i^e, \bar{u}_i^e - \bar{u}_i^e \}, \]

(5.6)

\[ \bar{A}_3(T) = \frac{k}{T_0} T_0 T_1 + \frac{1}{2} \sigma T^2 \]

(5.7)

are positive semi-definite.

It is worth compare the above hypotheses with those of the Subsection 3.1. To this end we note that the quadratic form \( \bar{A}_1(U_1) \) will be non-negative, for all values of \( \bar{u}_i^e \), if and only if

\[ \mu_v \geq 0, \quad \lambda_v + 2\mu_v \geq 0. \]

(5.8)
As it can be seen, such a hypothesis is weaker than that described by the relation (3.9). Analog assertions can be formulated by expliciting the hypotheses (3.5) and (5.4), while (3.6)–(3.8) are equivalent to (5.5)–(5.7).

On the basis of the above discussion we can define

\[
\widehat{\mathcal{S}}(t) = \frac{1}{2} \int_B \left[ \rho_{0}^{2} \ddot{u}_{i}^{2}(t) \ddot{u}_{i}^{2}(t) + \rho_{0}^{2} \ddot{u}_{i}^{2}(t) \ddot{u}_{i}^{2}(t) + \rho_{0}^{2} \ddot{u}_{i}^{2}(t) \ddot{u}_{i}^{2}(t) + x_{1} T^2(t) + \Phi(U) \right] dv,
\]

(5.9)

\[
\widehat{\mathcal{D}}(t) = \int_B [\widehat{\mathcal{A}}_1(U_1) + \widehat{\mathcal{A}}_2(U_2) + \widehat{\mathcal{A}}_3(T)] dv,
\]

(5.10)

and the identity (3.15) becomes

\[
\int_{0}^{\tau} \widehat{\mathcal{S}}(s) ds + \int_{0}^{\tau} \int_{0}^{\tau} \widehat{\mathcal{D}}(z) dz ds + \overline{\sigma} \int_{0}^{\tau} \int_{0}^{\tau} \int_{B} T^2(z) dv dz ds - \overline{\sigma} \int_{B} T^2(t) dv = 0,
\]

\[\forall t \in [0, \infty).\]

(5.11)

Thus, we can exploit this identity to establish the uniqueness result under the assumptions described by the relations (5.3)–(5.7) which are milder than the corresponding set described by (3.4)–(3.8).

Further, we can substitute the hypotheses described by the relations (3.29) and (3.31) with the following set described by

\[
\rho_{0}^{a} > 0, \quad \rho_{0}^{a} > 0, \quad \rho_{0}^{a} > 0, \quad \mu_{v} \geq 0, \quad \lambda_{v} + 2 \mu_{v} \geq 0, \quad \xi_{v} \geq 0, \quad \xi_{v} > 0, \quad \xi_{v} \geq 0, \quad \xi_{v} = 0, \quad \overline{\sigma} \geq 0.
\]

(5.12)

By introducing the measure

\[
\widehat{I}(t) = \int_B [\rho_{0}^{a} u_{i}^{2}(t) u_{i}^{2}(t) + \rho_{0}^{a} u_{i}^{2}(t) u_{i}^{2}(t) + \rho_{0}^{a} u_{i}^{2}(t) u_{i}^{2}(t) + k T^2(t)] dv + \overline{\sigma} \int_{0}^{T} \int_{0}^{T} T^2(z) dv dz
\]

\[
+ \int_{0}^{T} \int_{0}^{T} \left\{ \mu_{0} u_{i,j}^{2}(z) u_{i,j}^{2}(z) + (\lambda_{v} + 2 \mu_{v}) u_{i,j}^{2}(z) u_{i,j}^{2}(z) + k \frac{T^2(s)}{T_0} \left( \int_{0}^{T} T_{i}(s) ds \right) \left( \int_{0}^{T} T_{i}(s) ds \right)
\]

\[
+ \xi_{v} [u_{i}^{2}(z) - u_{i}^{2}(z)] [u_{i}^{2}(z) - u_{i}^{2}(z)] + \xi_{v}^{2} [u_{i}^{2}(z) - u_{i}^{2}(z)] [u_{i}^{2}(z) - u_{i}^{2}(z)] + 2 \xi_{v} [u_{i}^{2}(z)
\]

\[- u_{i}^{2}(z) [u_{i}^{2}(z) - u_{i}^{2}(z)] \right\} dv dz,
\]

(5.13)

we can discuss the uniqueness question as in Subsection 3.2.

Further we have to outline that it is possible to modify the arguments of the Section 4 in order to discuss the continuous dependence question under the hypotheses of this section.
6. Conclusion

With the presence of the time derivative of temperature in the set of constitutive variables in the theory of swelling porous thermoelastic soils and under mild assumptions upon the constitutive constants, we study the uniqueness and continuous data dependence of solutions. In treating the uniqueness of solutions we use first a method based on an integral inequality and then we individuate the class of temperature–displacements in which one can establish uniqueness result. We also use a method based on the Lagrange–Brun identity in order to establish the uniqueness under weaker assumptions upon the constitutive constants than those used in the first method.

When we treat the continuous data dependence problem we use a method based on an integro-differential inequality which when integrated leads to estimates valid in an appropriate class of temperature–displacements. We further use the Lagrange–Brun identity to individuate classes of temperature–displacements in which we can establish various estimates describing continuous dependence.

It should be emphasized that we can obtain uniqueness and continuous data dependence results by using the logarithmic convexity method by means of a measure concerned to $I(t)$.

Summarising, we conclude that the present paper reaches the following points: (a) is concerned with the general approach developed by Eringen in [1] (by considering the case $\sigma > 0$) and therefore covers an important case not treated in the paper by Quintanilla [8]; (b) relaxes the restrictions imposed on the constitutive constants in [2,8] for assuring uniqueness and continuous dependence results; (c) proves that the general linearised approach of the swelling porous thermoelastic soils is well posed.

Acknowledgement

The author is grateful to the anonymous referee for his helpful comments which led to improvements in this paper.

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