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ON THE BEHAVIOR OF STEADY TIME-HARMONIC OСSILLATIONS IN THERMOELASTIC MATERIALS WITH VOIDS

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We study the spatial behavior in a cylinder made of an isotropic and homogeneous thermoelastic material with voids when it is subjected to plane boundary data varying harmonically in time on its lateral surface and on one of the bases. For oscillations with an angular frequency lower than a critical frequency, we show that some appropriate measures associated with the amplitude of the vibration decays exponentially with the distance to the bases.

Keywords time-harmonic oscillations, spatial decay, thermoelastic materials with voids

Nunziato and Cowin [1] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are devoid of material. The linear theory of elastic materials with voids was established by Cowin and Nunziato [2]. Such theories are developed to describe the mechanical behavior of geological materials like rocks and soils and manufactured porous materials.

Iesan [3] has established a linear theory of thermoelastic materials with voids. In such a theory some general theorems (uniqueness, reciprocal, and variational theorems) are also established [3]. Furthermore, Chirita and Scalia [4] and Pompei and Scalia [5] have studied the spatial and temporal behavior of the transient
solutions for the initial-boundary-value problems associated with the linear theory of thermoelastic materials with voids by using the time-weighted surface power function method.

On the other hand, the steady time-harmonic oscillations are an important state in their own right, frequently occurring in practical applications. They are also very significant in the study of full dynamic problems (see, for example, [6]). The harmonic waves in linear elastic materials with voids have been studied by Puri and Cowin [7] and Pompei and Scalia [8], whereas for linear thermoelastic materials with voids the question has been studied by Ciarletta [9].

In this paper we consider steady time-harmonic oscillations within the context of the linear theory of thermoelasticity for materials with voids developed in [3]. We study the spatial behavior of the amplitude of the time-harmonic oscillations in a cylinder made of an isotropic and homogeneous thermoelastic material with voids. The cylinder is subjected to plane boundary data varying harmonically in time on its lateral surface and on one of its bases. When the angular frequency of the oscillation is lower than a certain critical frequency we are able to prove that some appropriate cross-sectional measures associated with the amplitude of the vibration decay exponentially with the distance to the basis.

We have to outline that the spatial behavior in the linear theory of classical thermoelasticity has been studied by Chiriţă [10] by generalizing an idea devised by Flavin and Knops [11] and Flavin et al. [12] for the linear theory of classical elasticity. For a comprehensive bibliography on the work on Saint–Venant’s principle we recommend the review papers by Horgan and Knowles [13] and Horgan [14, 15].

FORMULATION OF THE PROBLEM

Let $x_1, x_2, x_3$ denote rectangular cartesian coordinates and let $t$ denote time. Latin subscripts have the range 1–3, whereas Greek subscripts take only the values 1 and 2. When repeated, subscripts are summed over the appropriate range; when preceded by a comma, they indicate differentiation with respect to the corresponding Cartesian coordinate. As usual, superposed dots denote partial differentiation with respect to the time $t$. Let $R$ denote the interior of a right cylinder of length $L>0$ whose cross section is bounded by one or more piecewise smooth curves. Choose the Cartesian coordinates such that the origin lies in one end of the cylinder and such that the $x_3$-axis is parallel to the generators. Let $D_{x_3}$ denote the cross section of the cylinder corresponding to the axial coordinate $x_3$, and let $\partial D_{x_3}$ denote the cross-sectional boundary. Let $\pi = \partial D \times [0, L]$ denote the lateral surface of the cylinder.

We suppose that the cylinder is made of an isotropic and homogeneous thermoelastic material with voids. In the absence of the body loads, the field equations in terms of the displacement, volume fraction, and temperature are [3]

\begin{align}
\mu u_{i,jj} + (\lambda + \mu)u_{i,ri} + b\phi_{j,j} - \beta \theta_{j} &= \rho \ddot{u}_i \\
\alpha \phi_{j,jj} - bu_{i,r} - \zeta \phi + m\theta &= \rho \kappa \ddot{\phi} \\
k \theta_{j,jj} - \beta T_0 \dot{u}_{i,r} - a T_0 \ddot{\theta} - m T_0 \ddot{\phi} &= 0
\end{align}
In the preceding equations we have used the following notations: \( u_i \) denote the components of the displacement vector; \( \phi \) denotes the change in volume fraction from the reference volume fraction; \( \theta \) is the variation of temperature measured from the absolute temperature \( T_0 > 0 \) of the initial state; \( \kappa > 0 \) is the equilibrated inertia; and \( \lambda, \mu, b, x, \beta, \xi, m, a, \) and \( k \) are constant constitutive coefficients. In what follows we assume that the internal energy density is a positive definite quadratic form. Thus, the constitutive coefficients satisfy the conditions (see [1, 3, 16])

\[
\mu > 0 \quad a > 0 \quad \xi > 0 \quad (3\lambda + 2\mu)\xi > 3b^2 \quad a > 0 \quad k > 0
\]  

(4)

Obviously, it results that

\[
(\lambda + \mu)\xi > b^2 \quad (\lambda + 2\mu)\xi > b^2
\]  

(5)

so that we can introduce the notations

\[
c_1 = 1 - \frac{|b|}{\sqrt{\xi(\lambda + \mu)}} \quad c_2 = \xi(\lambda + 2\mu) - b^2
\]

\[
c_3 = 1 - \frac{|b|}{\sqrt{\frac{\mu}{c_2(\lambda + \mu)}}} \quad M_0 = m(\lambda + 2\mu) - b\beta
\]  

(6)

It follows from Eq. (6) that we have

\[
0 < c_1 < 1 \quad c_2 > 0 \quad 0 < c_3 < 1
\]  

(7)

To Eqs. (1)–(3) we adjoin the following initial conditions:

\[
u_i(x, 0) = u_i^0(x) \quad \dot{u}_i(x, 0) = \ddot{u}_i^0(x) \\
\phi(x, 0) = \phi^*(x) \quad \dot{\phi}(x, 0) = \ddot{\phi}^*(x) \quad \theta(x, 0) = \theta^*(x) \quad x \in \mathbb{R}
\]  

(8)

where \( u_i^0, \ddot{u}_i^*, \phi^*, \ddot{\phi}^*, \) and \( \theta^* \) are prescribed continuous functions. Furthermore, we suppose that all boundary terms are separable with respect to space and time and that the time dependency is periodic. That means that we consider the following lateral boundary conditions

\[
u_i(x, t) = \ddot{u}_i(x) \exp(-i\omega t) \quad \phi(x, t) = \ddot{\phi}(x) \exp(-i\omega t) \\
\theta(x, t) = \ddot{\theta}(x) \exp(-i\omega t) \quad x \in \pi
\]  

(9)

where \( \omega \in \mathbb{R}^+ \) is the frequency of oscillation and \( \ddot{u}_i, \ddot{\phi}, \) and \( \ddot{\theta} \) are prescribed continuous functions. Moreover, we assume the end boundary conditions in the form

\[
u_i(x, t) = u_i^0(x_1, x_2) \exp(-i\omega t) \quad \phi(x, t) = \phi^0(x_1, x_2) \exp(-i\omega t) \\
\theta(x, t) = \theta^0(x_1, x_2) \exp(-i\omega t) \quad \text{on } D_0
\]  

(10)

where \( u_i^0, \phi^0, \theta^0 \) are prescribed continuous functions.
or in the form

\[ \begin{align*}
    u_{x,3}(x,t) &= p_x^0(x_1, x_2) \exp(-i\omega t) \\
    u_3(x,t) &= p_3^0(x_1, x_2) \exp(-i\omega t) \\
    \phi_3(x,t) &= \phi_0^0(x_1, x_2) \exp(-i\omega t) \\
    \theta_3(x,t) &= \theta_0^0(x_1, x_2) \exp(-i\omega t)
\end{align*} \]

on \( D_0 \)

(11)

where \( u_0^0, \phi_0^0, \theta_0^0, p_x^0, \) and \( \theta_0^0 \) are prescribed continuous functions. For a finite cylinder, we have to adjoin null boundary conditions on \( D_L \) of the type described in relations (10) and (11).

It is easy to see that we can write

\[ \begin{align*}
    u_r(x,t) &= U_r(x,t) + v_r(x) \exp(-i\omega t) \\
    \phi(x,t) &= \Phi(x,t) + \chi(x) \exp(-i\omega t) \\
    \theta(x,t) &= \Theta(x,t) + T(x) \exp(-i\omega t)
\end{align*} \]

(12)

where \( \{U_r, \Phi, \Theta\} \) absorbs the initial conditions and satisfies the null boundary conditions and Eqs. (1)–(3), while \( \{v_r, \chi, T\} \) satisfies the boundary-value problem \( (P) \) consisting of the basic equations

\[ \begin{align*}
    &\mu v_{r,ij} + (\lambda + \nu)v_{j,i} + b\chi_{,r} - \beta T_x + \rho \omega^2 v_r = 0 \\
    &x\chi_{,ij} - b v_{r,x} - \xi \chi + mt + \rho \omega^2 \chi = 0 \\
    &k T_{,ij} + i \beta T_0 \omega v_{r,r} + ia T_0 \omega T + im T_0 \omega \chi = 0
\end{align*} \]

(13–15)

in \( R \), the lateral boundary conditions

\[ \begin{align*}
    v_r(x) &= \bar{u}_r(x) \\
    \chi(x) &= \bar{\phi}(x) \\
    T(x) &= \bar{\theta}(x) \\
    x &\in \pi
\end{align*} \]

(16)

and the following end boundary conditions:

\[ \begin{align*}
    v_r(x) &= u_r^0(x_1, x_2) \\
    \chi(x) &= \phi_0^0(x_1, x_2) \\
    T(x) &= \theta_0^0(x_1, x_2)
\end{align*} \]

on \( D_0 \)

(17)

or

\[ \begin{align*}
    v_{x,3}(x) &= p_x^0(x_1, x_2) \\
    v_3(x) &= p_3^0(x_1, x_2) \\
    \chi_3(x) &= q_1^0(x_1, x_2) \\
    T_3(x) &= q_2^0(x_1, x_2)
\end{align*} \]

on \( D_0 \)

(18)

We note that for a finite cylinder we have to adjoin the null end boundary conditions on \( D_L \) of the type described by relations (17) and (18). As regards boundary conditions (11) and their corresponding conditions (18), we have to outline that they are connected with the following mixed boundary conditions:

\[ \begin{align*}
    t_{13}(x) &= t_1^0(x_1, x_2) \\
    t_{23}(x) &= t_2^0(x_1, x_2) \\
    v_3(x) &= v_3^0
\end{align*} \]

(19)
where the components of the stress tensor $t_{ij}$, the components of the equilibrated stress vector $h_i$, and the components of the heat flux vector $q_i$ are defined by [3]

$$
t_{ij} = \lambda \epsilon_{ij} + 2\mu \epsilon_{ij} + b \phi \delta_{ij} - \beta \theta \delta_{ij} \quad h_i = \alpha \phi_i \quad q_i = k \theta_i
$$

(20)

In fact, on basis of relation (20), it is easy to see that boundary conditions (19) can be written in the form expressed by Eq. (11).

Physically, the term $\{U_r, \Phi, \Theta\}$ represents the transient part of the solution and $\{v_r, \chi, T\} \exp(-i\omega t)$ represents the forced oscillation. As we can see, the initial conditions influence the system only through the transient part, whereas the boundary conditions are assigned to the steady-state solution. If sufficient time has elapsed for the transient part of any solution to Eqs. (1)–(3) to have vanished, then we can assume that the system has reached steady state. Thus, Eqs. (13)–(15) constitute the basic equations for the amplitude $\{v_r, \chi, T\}$ of the steady oscillations in thermoelastic materials with voids.

In what follows we study the spatial behavior of the amplitude of the steady thermoelastic oscillations in materials with voids as solution of the boundary-value problem $(\mathcal{P})$ in the following cases: (a) $\tilde{u}_r = 0, \tilde{\phi} = 0, \tilde{\theta} = 0$; and (b) $\tilde{u}_3 = \tilde{u}_3(x_1, x_2)$, $\tilde{u}_3 = 0, \tilde{\phi} = \tilde{\phi}(x_1, x_2), \tilde{\theta} = \tilde{\theta}(x_1, x_2)$.

THERMOELASTIC CYLINDER WITH NULL LATERAL BOUNDARY DATA

Throughout this section we assume that the cylinder is subjected to null lateral boundary data. Therefore, the amplitude $\{v_r, \chi, T\}$ of the steady-state vibration satisfies the problem $(\mathcal{P}_1)$ defined by relations (13)–(15), end boundary conditions (17) or (18), and the lateral boundary conditions

$$
v_r(x) = 0 \quad \chi(x) = 0 \quad T(x) = 0 \quad x \in \pi
$$

(21)

and for a finite cylinder, we adjoin null end boundary conditions of type (17) or (18) on $D_L$.

In this section we study the spatial behavior of the amplitude of the steady-state oscillations in the problem $(\mathcal{P}_1)$. In this aim we associate with the solution $\{v_r, \chi, T\}$ of the aforementioned problem $(\mathcal{P}_1)$ the following cross-sectional measure:

$$
K(x_3) = -\int_{D_3} \left\{ \mu (\tilde{v}_r v_{r,3} + v_r \tilde{v}_{r,3}) + (\lambda + \mu)(\tilde{v}_r \chi v_3 + v_r \tilde{\chi} v_3) + b(\tilde{\chi} v_3 + \chi \tilde{v}_3)

+ \alpha (\tilde{\chi} \chi_3 + \chi \tilde{\chi}_3) - \beta (v_3 \tilde{T} + \tilde{v}_3 T) + \frac{1}{T_0} \delta k (T \tilde{T}_3 + \tilde{T} T_3) \right\} dA
$$

(22)

where the superposed bar denotes complex conjugate and $\delta$ is a positive constant of time unit dimension that will be chosen later.
By a direct differentiation into relation (22), we get

\[
K'(x_3) = - \int_{D_{x_3}} \left\{ 2 \mu v_{r,2} \bar{v}_{r,3} + (\lambda + \mu) (v_{r,2} \bar{v}_{3,3} + \bar{v}_{r,3} v_{3,3}) + 2 \mu \chi_{,3} \bar{\chi}_{,3} \\
+ \frac{2}{T_0} \delta k T_3 \bar{T}_3 \right\} dA - H(x_3) - G(x_3) \tag{23}
\]

where

\[
H(x_3) = \int_{D_{x_3}} \left\{ - \beta (v_{3,3} \bar{T} + \bar{v}_{3,3} T + v_3 \bar{T}_3 + \bar{v}_3 T_3) \\
+ b (\chi_{,3} \bar{v}_3 + \bar{\chi}_{,3} v_3 + \chi \bar{v}_{3,3} + \bar{\chi} v_{3,3}) \right\} dA \tag{24}
\]

\[
G(x_3) = \int_{D_{x_3}} \left\{ \mu (\bar{v}_{r,33} + v_r \bar{v}_{r,3}) + (\lambda + \mu) (v_{3} \bar{v}_{r,3} + \bar{v}_3 v_{r,3}) \\
+ \frac{1}{T_0} \delta k \left( T \bar{T}_{,33} + \bar{T} T_{,33} \right) \right\} dA \tag{25}
\]

By using Eqs. (13)–(15) in relation (25), we deduce

\[
G(x_3) = \int_{D_{x_3}} \left\{ \epsilon_r [ - \mu \bar{v}_{r,\rho} - (\lambda + \mu) v_{s,r} - b \chi_{,\rho} - \beta T_{,\rho} - \rho \omega^2 v_r] \right. \\
+ v_r [ - \mu \bar{v}_{r,\rho} - (\lambda + \mu) \bar{v}_{s,r} - b \bar{\chi}_{,\rho} + \beta \bar{T}_{,\rho} - \rho \omega^2 \bar{v}_r] \\
+ \bar{\chi} (- \xi \chi_{,\rho} + b v_{r,\rho} + \xi \bar{\chi} - m \bar{T} - \rho \omega^2 \bar{\chi}) \\
+ \chi (- \xi \bar{\chi}_{,\rho} + b \bar{v}_{r,\rho} + \xi \chi - m \bar{T} - \rho \omega^2 \chi) \\
+ \frac{1}{T_0} \delta T (- k T_{,\rho} + i \beta T_0 \omega \bar{v}_{r,\rho} + ia T_0 \omega T + im T_0 \omega \bar{\chi}) \\
+ \frac{1}{T_0} \delta \bar{T} (- k \bar{T}_{,\rho} - i \beta T_0 \omega v_{r,\rho} - ia T_0 \omega T - im T_0 \omega \chi) \\
+ \left. (\lambda + \mu) (v_3 \bar{v}_{r,3} + \bar{v}_3 v_{r,3}) \right\} dA \tag{26}
\]

so that, by applying the divergence theorem and lateral boundary conditions (21), we obtain

\[
G(x_3) = \int_{D_{x_3}} \left\{ 2 \mu v_{r,\rho} \bar{v}_{r,\rho} + (\lambda + \mu) (v_{r,\rho} \bar{v}_{r,\rho} + \bar{v}_{r,\rho} v_{r,\rho}) + 2 \mu \chi_{,\rho} \bar{\chi}_{,\rho} + 2 \xi \xi \bar{\chi} \\
+ \frac{2}{T_0} \delta k T_{,\rho} \bar{T}_{,\rho} - 2 \rho \omega^2 v_r \bar{v}_r - 2 \rho \omega^2 \bar{\chi} \chi - \beta (v_r \bar{T} + \bar{v}_r T) \\
- b (v_r \bar{\chi} + \bar{v}_r \chi - \chi \bar{v}_{r,\rho} - \bar{\chi} v_{r,\rho}) - m (T \bar{\chi} + \bar{T} \chi) \\
+ i \beta \omega (T \bar{v}_{r,\rho} - \bar{T} v_{r,\rho}) + i \beta \omega (T \bar{\chi} - \bar{T} \chi) \right\} dA \tag{27}
\]
Furthermore, we have

\[
\int_{D_3} \left\{ \beta (v_r T_r + \ddot{v}_r T_r) - b (v_r \ddot{x}_r + \ddot{v}_r \ddot{x}_r - \ddot{x} v_r - \ddot{x} v_r) \right\} \, dA
\]

\[
= \int_{D_3} \left\{ \beta (v_3 T_3 + \ddot{v}_3 T_3 - v_{\rho, \rho} \ddot{T} - \ddot{v}_{\rho, \rho} T) + b (\ddot{\chi} v_{3,3} + \ddot{x} v_{3,3}
\right.
\]

\[- v_3 \ddot{\chi}_3 - \ddot{v}_3 \ddot{x}_3) + 2b (\ddot{x} v_{\rho, \rho} + \ddot{\chi} v_{\rho, \rho}) \right\} \, dA
\]

(28)

so that we can write relation (27) in the form

\[
G(x_3) = \int_{D_3} \left\{ 2 \mu v_r v_r + (\lambda + \mu) (v_r v_{\rho, \rho} + \ddot{v}_r v_{\rho, \rho}) + 2 \chi \ddot{\chi} + 2 \ddot{\chi} \chi 
\right.
\]

\[+ \frac{2}{T_0} \delta k T_3 \ddot{T} - 2 \rho \omega^2 v_r \ddot{v}_r - 2 \rho \kappa \omega^2 \ddot{\chi} \chi + \beta (v_3 T_3 + \ddot{v}_3 T_3 - v_{\rho, \rho} \ddot{T} - \ddot{v}_{\rho, \rho} T)
\]

\[- \ddot{v}_{\rho, \rho} T) + b (\ddot{\chi} v_{3,3} + \ddot{x} v_{3,3} - v_3 \ddot{\chi}_3 - \ddot{v}_3 \ddot{x}_3) + 2b (\ddot{x} v_{\rho, \rho} + \ddot{\chi} v_{\rho, \rho})
\]

\[- m (T \ddot{\chi} + \ddot{T} \chi) + i \delta \omega (T \ddot{v}_r - \ddot{T} v_r) + i \delta m \omega (T \ddot{\chi} - \ddot{T} \chi) \right\} \, dA
\]

(29)

Thus, relations (24) and (29) give

\[
H(x_3) + G(x_3) = \int_{D_3} \left\{ 2 \mu v_r v_r + (\lambda + \mu) (v_r v_{\rho, \rho} + \ddot{v}_r v_{\rho, \rho}) + 2 \chi \ddot{\chi} + 2 \ddot{\chi} \chi 
\right.
\]

\[+ \frac{2}{T_0} \delta k T_3 \ddot{T} - 2 \rho \omega^2 v_r \ddot{v}_r - 2 \rho \kappa \omega^2 \ddot{\chi} \chi
\]

\[- \beta (v_3 T_3 + \ddot{v}_3 T_3 - v_{\rho, \rho} \ddot{T} - \ddot{v}_{\rho, \rho} T) + b (\ddot{\chi} v_{3,3} + \ddot{x} v_{3,3} - v_3 \ddot{\chi}_3 - \ddot{v}_3 \ddot{x}_3) + 2b (\ddot{x} v_{\rho, \rho} + \ddot{\chi} v_{\rho, \rho})
\]

\[- m (T \ddot{\chi} + \ddot{T} \chi) + i \delta \omega (T \ddot{v}_r - \ddot{T} v_r) + i \delta m \omega (T \ddot{\chi} - \ddot{T} \chi) \right\} \, dA
\]

(30)

and, therefore, relation (23) becomes

\[
K'(x_3) = - \int_{D_3} \left\{ 2 \mu v_r v_r + 2 (\lambda + \mu) v_r v_{\rho, \rho} + 2 \ddot{\chi} \chi + 2 \ddot{\chi} \chi + 2 \ddot{\chi} \chi
\right.
\]

\[+ \frac{2}{T_0} \delta k T_r \ddot{T} - 2 \rho \omega^2 v_r \ddot{v}_r - 2 \rho \kappa \omega^2 \ddot{\chi} \chi
\]

\[- \beta (v_r T + \ddot{v}_r T) + b (\ddot{\chi} v_{3,3} + \ddot{x} v_{3,3} - v_3 \ddot{\chi}_3 - \ddot{v}_3 \ddot{x}_3) + 2b (\ddot{x} v_{\rho, \rho} + \ddot{\chi} v_{\rho, \rho})
\]

\[- m (T \ddot{\chi} + \ddot{T} \chi) - \beta (1 - i \delta \omega) T \ddot{v}_r + (1 + i \delta \omega) \ddot{T} v_r \right\} \, dA
\]

(31)
In view of relations (6) and (7), the result is
\[
(\lambda + \mu)v_{r,s} + b(\chi v_{r,s} + \bar{\chi} v_{r,s}) + \xi \bar{\chi} \geq c_1[(\lambda + \mu)v_{r,s} + \xi \bar{\chi}]
\]  
(32)

Let us consider \(\epsilon \in (0, 1)\). On the basis of relations (31) and (32), we can write
\[
-K'(x_3) \geq \int_{D_{x_3}} \left\{ 2\mu v_{r,s}\bar{v}_{r,s} + 2x_{r,s}\bar{x}_{r,s} + \frac{2}{T_0}\delta k T_{r} T_{r} + 2c_1(\lambda + \mu)(1 - \epsilon)v_{r,s}\bar{v}_{r,s} + 2c_1 \xi \bar{\chi} - 2\rho \omega^2 v_{r,s}\bar{v}_{r,s} - 2\rho \omega^2 \bar{\chi} \bar{\chi} + 2c_1(\lambda + \mu)\epsilon \right. \\
\times \left[ v_{r,s} - \frac{\beta}{2\epsilon c_1(\lambda + \mu)}(1 - i\delta \omega)\xi T \right] \left[ \bar{v}_{r,s} - \frac{\beta}{2\epsilon c_1(\lambda + \mu)}(1 + i\delta \omega)\xi T \right] \\
+ 2c_1 \xi \left[ \chi - \frac{m}{2\epsilon c_1 \xi}(1 - i\delta \omega)\xi T \right] \left[ \bar{\chi} - \frac{m}{2\epsilon c_1 \xi}(1 + i\delta \omega)\xi T \right] \\
\left. - m_0(1 + \delta^2 \omega^2)TT \right\} dA
\]  
(33)

where
\[
m_0 = \frac{1}{2\epsilon c_1(\lambda + \mu + \frac{m^2}{\xi})}
\]  
(34)

In view of lateral boundary conditions (21), we have
\[
\int_{D_{x_3}} v_{r,\rho}\bar{v}_{r,\rho} dA \geq \lambda_1 \int_{D_{x_3}} v_{r}\bar{v}_{r} dA
\]  
(35)
\[
\int_{D_{x_3}} \chi_{r}\bar{\chi}_{r} dA \geq \lambda_1 \int_{D_{x_3}} \chi \bar{\chi} dA
\]  
(36)
\[
\int_{D_{x_3}} T_{r}\bar{T}_{r} dA \geq \lambda_1 \int_{D_{x_3}} TT dA
\]  
(37)

where \(\lambda_1\) is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross section \(D_{x_3}\). Then by using relations (35)–(37) in relation (33), we get
\[
-K'(x_3) \geq \int_{D_{x_3}} \left\{ 2(\mu \lambda_1 - \rho \omega^2)v_{r}v_{r} + 2\mu v_{r,3}\bar{v}_{r,3} + 2[\chi \lambda_1 + c_1 \xi (1 - \epsilon) \\
- \rho \omega^2 \bar{\chi}] + 2\delta k T_{3} T_{3} + \frac{2}{T_0} \delta k \lambda_1 - m_0(1 + \delta^2 \omega^2) \right\} TT dA
\]  
(38)
At this stage of our analysis we choose the arbitrary positive parameter $\delta$ in such a way that we have

$$\delta > \frac{T_0 m_0}{2 k \lambda_1}$$

(39)

and then we set

$$\omega_m^2 = \min \left\{ \frac{\lambda_1 \mu}{\rho}, \frac{\alpha \lambda_1 + c_1 \xi (1 - \varepsilon)}{\rho \kappa}, \frac{2 \delta k \lambda_1 - m_0 T_0}{m_0 T_0 \delta^2} \right\}$$

(40)

Throughout the following we assume that

$$\omega < \omega_m$$

(41)

and, therefore, relation (38) implies that

$$-K'(x_3) \geq \int_{D_{x_3}} \left\{ M_1 v_r \tilde{v}_r + M_2 v_{r,3} \tilde{v}_{r,3} + M_3 \tilde{z} \tilde{x} + M_4 \tilde{x} \tilde{x}_3 + M_5 v_{r,r} \tilde{v}_{r,s} + M_6 T_3 \tilde{T}_3 + M_7 T T \right\} dA$$

(42)

where

$$M_1 = 2 \lambda_1 \mu \left( 1 - \frac{\omega_m^2}{\omega_m^2} \right), \quad M_2 = 2 \mu$$

$$M_3 = 2 \left[ \alpha \lambda_1 + c_1 \xi (1 - \varepsilon) \right] \left( 1 - \frac{\omega_m^2}{\omega_m^2} \right), \quad M_4 = 2 \alpha$$

$$M_5 = 2 c_1 (\lambda + \mu) (1 - \varepsilon), \quad M_6 = \frac{2}{T_0} \delta k$$

$$M_7 = \left( \frac{2}{T_0} \delta k \lambda_1 - m_0 \right) \left( 1 - \frac{\omega_m^2}{\omega_m^2} \right)$$

(43)

Thus, under the assumption that the angular frequency of the vibration is lower than the critical value $\omega_m$ defined by relation (40), relation (42) proves that $K(x_3)$ is a nonincreasing function with respect to $x_3$.

We proceed now to obtain an appropriate estimate for $|K(x_3)|$ in terms of the integral from the right-hand side of relation (42). In this aim we use Schwarz’s inequality and the arithmetic–geometric mean inequality to obtain

$$|K(x_3)| \leq \int_{D_{x_3}} \left\{ M_1^2 v_r \tilde{v}_r + M_2^2 v_{r,3} \tilde{v}_{r,3} + M_3^2 \tilde{z} \tilde{x} + M_4^2 \tilde{x} \tilde{x}_3 + M_5^2 v_{r,r} \tilde{v}_{r,s} + M_6^2 T_3 \tilde{T}_3 + M_7^2 T T \right\} dA$$

(44)
where

\[ M_1' = (\lambda + 2\mu)\sqrt{\lambda_1} + \frac{|b|}{\sqrt{\kappa}} + |\beta|T_0\sqrt{\lambda_1} \]
\[ M_2' = \frac{\mu}{\sqrt{\lambda_1}}, \quad M_3' = |b|\sqrt{\kappa} + \alpha\sqrt{\lambda_1} \]
\[ M_4' = \frac{\alpha}{\sqrt{\lambda_1}}, \quad M_5' = \frac{\lambda + \mu}{\sqrt{\lambda_1}}, \quad M_6' = \frac{\delta k}{T_0\sqrt{\lambda_1}} \]
\[ M_7' = \frac{1}{T_0} \left( \frac{|b|}{\sqrt{\lambda_1}} + \delta k\sqrt{\lambda_1} \right) \] (45)

From relations (42) and (44) we obtain the following first-order differential inequality

\[ v_1|K(x_3)| + K'(x_3) \leq 0 \] (46)

where \( v_1 \) is defined by

\[ \frac{1}{v_1} = \max \left\{ \frac{M_1'}{M_1}, \frac{M_2'}{M_2}, \frac{M_3'}{M_3}, \frac{M_4'}{M_4}, \frac{M_5'}{M_5}, \frac{M_6'}{M_6}, \frac{M_7'}{M_7} \right\} \] (47)

We further integrate the preceding first-order differential inequality to obtain the spatial behavior of the cross-sectional measure \( K(x_3) \). We consider first the case of a finite cylinder, that is, \( L < \infty \). Then, on the basis of our assumption concerning null end boundary conditions on \( D_L \) of the type described by relations (17) or (18), from relation (22) it follows that

\[ K(L) = 0 \] (48)

Since \( K(x_3) \) is a nonincreasing function of \( x_3 \), it results that, for all \( x_3 \in [0, L] \), we have

\[ K(x_3) \geq 0 \] (49)

Thus, relation (46) becomes

\[ v_1K(x_3) + K'(x_3) \leq 0 \] (50)

so that via integration we get

\[ 0 \leq K(x_3) \leq K(0)\exp(-v_1x_3) \quad x_3 \in [0, L] \] (51)

We outline here that the Saint–Venant decay estimate (51) holds true provided the excitation frequency \( \omega \) is below to the critical level described by relations (40) and (41).
Let us now consider the case of a semi-infinite cylinder, that is, $L = \infty$. We introduce the volume measure of the amplitude of steady-state vibration by

$$E(z) = \int_{R_z} \left\{ M_1^r v_r \bar{e}_r + M_2^r v_{r,3} \bar{e}_{r,3} + M_3^r \bar{\chi} \bar{\chi} + M_4^r \bar{z} \bar{z}_{,3} + M_5^r v_{r,3} \bar{e}_{r,3} + M_6^r T_{,3} \bar{T}_{,3} + M_7^r T \bar{T}_o \right\} dA$$

(52)

where

$$R_z = \{ x = (x_1, x_2, x_3) \in R | x_3 > z \}, \quad z \geq 0$$

(53)

Then, in the class of steady-state vibrations whose amplitudes make $E(z)$ bounded, it follows that

$$K(\infty) \equiv \lim_{x_3 \to \infty} K(x_3) = 0$$

(54)

Since $K(x_3)$ is a nonincreasing function of $x_3$, it follows that relation (49) holds true for all $x_3 \in [0, \infty)$ and, therefore, the preceding analysis proves that spatial decay estimate (51) remains valid for all $x_3 \in [0, \infty)$, again provided that the excitation frequency is lower than the critical value described by relation (40).

**CYLINDER WITH PLANE TIME–HARMONIC LATERAL BOUNDARY DATA**

Let us consider the cylinder to be subjected to plane time-harmonic lateral boundary data. Therefore, we consider amplitudes $\{ v_r, \chi, T \}$ satisfying the problem $(P_2)$ consisting of basic Eqs. (13)–(15), end boundary conditions (18), and the following lateral boundary conditions:

$$v_2(x) = \bar{u}_2(x_1, x_2) \quad v_3(x) = 0$$

$$\chi(x) = \bar{\phi}(x_1, x_2) \quad T(x) = \bar{\theta}(x_1, x_2) \quad x \in \pi$$

(55)

and, for a finite cylinder, we have to adjoin null boundary conditions on $D_L$ of the type described by relation (18). Obviously, for any solution of $(P_2)$, we have

$$v_{2,3}(x) = 0 \quad v_{3,3}(x) = 0 \quad v_{3,33}(x) = 0$$

$$\chi_{,3}(x) = 0 \quad T_{,3}(x) = 0 \quad x \in \pi$$

(56)

To study the spatial behavior of the amplitude solution $\{ v_r, \chi, T \}$ of the problem $(P_2)$ we introduce the following cross-sectional measure:
\[ F(x_3) = \int_{D_{x_3}} \left\{ \mu (v_{\rho,3} \dot{\rho} + v_{3,\rho} \dot{\rho}) + \alpha \chi_{3,3} \ddot{x}_3 + \frac{1}{T_0} \delta k \big(T_{3,3} \ddot{T}_{3,3}\big) \right\} \, dA \quad (57) \]

where \( \delta \) is a positive parameter of time unit dimension that will be chosen later.

We differentiate relation (57) in order to get

\[ F'(x_3) = \int_{D_{x_3}} \left\{ \mu (v_{\rho,33} \dot{\rho} + v_{3,\rho,3} \dot{\rho} + v_{3,3,\rho} \dot{\rho} + v_{3,3,\rho,3}) \right\} + \alpha (\chi_{3,33} \ddot{x}_3 + \chi_{3,33} \ddot{x}_3) + \frac{1}{T_0} \delta k \big(T_{3,33} \ddot{T}_{3,33}\big) \, dA \quad (58) \]

and

\[ F''(x_3) = 2 \int_{D_{x_3}} \left\{ \mu (v_{\rho,333} \dot{\rho} + v_{3,3,\rho,3} \dot{\rho} + v_{3,3,3,\rho} \dot{\rho}) + \alpha (\chi_{3,333} \ddot{x}_3 + \chi_{3,333} \ddot{x}_3) + \frac{1}{T_0} \delta k \big(T_{3,333} \ddot{T}_{3,333}\big) + \mu (v_{3,33,\rho} \ddot{v}_{3,\rho} + v_{3,3,\rho} \ddot{v}_{3,3,\rho}) \right\} \, dA + I(x_3) \quad (59) \]

where

\[ I(x_3) = \int_{D_{x_3}} \left\{ \mu (v_{\rho,333} \dot{\rho} + v_{3,3,\rho,3} \dot{\rho} + v_{3,3,3,\rho} \dot{\rho}) + \alpha (\chi_{3,333} \ddot{x}_3 + \chi_{3,333} \ddot{x}_3) \right\} + \frac{1}{T_0} \delta k \big(T_{3,333} \ddot{T}_{3,333}\big) + \mu (v_{3,33,\rho} \ddot{v}_{3,\rho} + v_{3,3,\rho} \ddot{v}_{3,3,\rho}) \, dA \quad (60) \]

We now use Eqs. (13) for \( r = 1, 2, (14), \) and (15), and relation (56) and the divergence theorem, to obtain

\[ I(x_3) = \int_{D_{x_3}} \left\{ \frac{2 \mu v_{2,3} \ddot{v}_{2,3} + 2 \alpha \chi_{3,3} \ddot{x}_3 + \frac{2 \delta k}{T_0} T_{3,3} \ddot{T}_{3,3} + 2 (\lambda + \mu) v_{3,3} \ddot{v}_{3,\rho} \dot{\rho} \right\} + 2 b (\chi_{3} \ddot{x}, \chi_{3} \ddot{x}) + 2 \alpha (\chi_{3} \ddot{x}, \chi_{3} \ddot{x}) - \beta (T_{3} \ddot{v}_{3,\rho} + T_{3} \ddot{v}_{3,\rho}) \right\} + \frac{1}{T_0} \delta k \big(T_{3,3} \ddot{T}_{3,3}\big) - m (T_{3} \ddot{\chi}_{3} + T_{3} \ddot{\chi}_{3}) - 2 \rho \omega^2 v_{3,3} \ddot{v}_{3,3} - 2 \rho \omega^2 \chi_{3,3} \ddot{\chi}_{3} \right\} + i \delta \omega \left\{ \beta (T_{3} \ddot{v}_{3,\rho} - T_{3} \ddot{v}_{3,\rho}) + m (T_{3} \ddot{\chi}_{3} - T_{3} \ddot{\chi}_{3}) \right\} \, dA + J(x_3) \quad (61) \]

where

\[ J(x_3) = \int_{D_{x_3}} \left\{ (\lambda + \mu) (v_{3,33} \ddot{v}_{3,\rho} + \ddot{v}_{3,33} v_{3,\rho}) + b (v_{3,33} \ddot{x}_3 + \ddot{v}_{3,33} \ddot{x}_3) + i \delta \beta \omega (T_{3} \ddot{v}_{3,33} - T_{3} \ddot{v}_{3,33}) - \mu (v_{3,33} \ddot{v}_{3,\rho} + \ddot{v}_{3,33} v_{3,\rho,33}) \right\} \, dA \quad (62) \]
Furthermore, we use relation (13) for \( r = 3 \) in relation (62), so that we obtain

\[
J(x_3) = \frac{1}{\lambda + 2\mu} \int_{D_{x_3}} \left\{ 2\mu^2 v_{x,3z} \bar{v}_{3,\rho} - 2(\lambda + \mu)^2 v_{x,3z} \bar{v}_{\rho,\rho} - 2\beta^2 \lambda \bar{\chi}_{,3} \right. \\
+ 2b(\lambda + \mu)(\chi_{,3} \bar{v}_{3,\rho} + \bar{\chi}_{,3} v_{x,3z}) - 2\beta \mu (T_{,3} \bar{v}_{3,\rho} + T_{3} v_{3,\rho}) \\
+ \beta(\lambda + \mu)(T_{,3} \bar{v}_{3,\rho} + T_{3} v_{3,\rho}) + b(\chi_{,3} \bar{\chi}_{,3} - \bar{\chi}_{,3} \bar{T}_{,3}) \\
- \rho \omega^2 [(\lambda + \mu)(v_{3} \bar{v}_{\rho,\rho} + \bar{v}_{3} v_{\rho,\rho}) + b(v_{3} \bar{\chi}_{,3} + \bar{v}_{3} \chi_{,3}) - \mu (v_{3} \bar{v}_{3,\rho} + \bar{v}_{3} v_{3,\rho})] \\
+ \mathbb{I} \beta \omega \left[ (T_{,3} v_{3,\rho} - T_{3} v_{3,\rho}) + (\lambda + \mu)(T_{,3} v_{3,\rho} - T_{3} \bar{v}_{3,\rho}) \\
+ b(T_{,3} \bar{\chi}_{,3} - T_{3} \bar{\chi}_{,3}) + \rho \omega^2 (v_{3} \bar{T}_{,3} - \bar{v}_{3} T_{,3}) \right] \right\} dA
\]

(63)

By combining relations (61) and (63) and by taking into account relation (6), we get

\[
I(x_3) = \int_{D_{x_3}} \left\{ 2\mu v_{x,3z} \bar{v}_{3,\rho} + 2\chi_{,3} \rho v_{3,\rho} + \frac{2\delta k}{T_{0}} T_{3} \bar{T}_{,3} \rho + \frac{2\mu^2}{\lambda + 2\mu} v_{x,3z} \bar{v}_{3,\rho} \\
- 2\rho \omega^2 v_{x,3z} \bar{v}_{3,\rho} - 2\rho \omega^2 \chi_{,3} \bar{\chi}_{,3} \right\} dA
\]

\[
+ \frac{2}{\lambda + 2\mu} \int_{D_{x_3}} \left\{ \mu(\lambda + \mu) v_{x,3z} \bar{v}_{\rho,\rho} \\
+ b(\chi_{,3} \bar{v}_{3,\rho} + \bar{\chi}_{,3} v_{x,3z}) + c_2 \chi_{,3} \bar{\chi}_{,3} \right\} dA
\]

\[
+ \frac{1}{\lambda + 2\mu} \int_{D_{x_3}} \left\{ -\beta \mu (T_{,3} \bar{v}_{3,\rho} + T_{3} v_{3,\rho}) - M_{0} (T_{3} \bar{\chi}_{,3} + T_{3} \chi_{,3}) \\
- \beta \mu (T_{,3} \bar{v}_{3,\rho} + T_{3} v_{3,\rho}) \\
+ i\delta \omega \left[ \beta \mu (T_{,3} \bar{v}_{3,\rho} - T_{3} v_{3,\rho}) + T_{,3} v_{3,\rho} - T_{3} \bar{v}_{3,\rho} \\
+ M_{0} (T_{,3} \bar{\chi}_{,3} - T_{3} \chi_{,3}) + \beta \rho \omega^2 (v_{3} \bar{T}_{,3} - \bar{v}_{3} T_{,3}) \\
- \rho \omega^2 [(\lambda + \mu)(v_{3} \bar{v}_{\rho,\rho} + \bar{v}_{3} v_{\rho,\rho}) \\
+ b(v_{3} \bar{\chi}_{,3} + \bar{v}_{3} \chi_{,3}) - \mu (v_{3} \bar{v}_{3,\rho} + \bar{v}_{3} v_{3,\rho})] \right] \right\} dA
\]

(64)

Relations (6) and (7) allow us to deduce that

\[
\mu(\lambda + \mu) v_{x,3z} \bar{v}_{\rho,\rho} + b(\chi_{,3} \bar{v}_{3,\rho} + \bar{\chi}_{,3} v_{x,3z}) + c_2 \chi_{,3} \bar{\chi}_{,3} \\
\geq c_3 \left[ \mu(\lambda + \mu) v_{x,3z} \bar{v}_{\rho,\rho} + c_2 \chi_{,3} \bar{\chi}_{,3} \right]
\]

(65)
By substituting relation (65) into relation (64), it follows that, for any fixed $\varepsilon \in (0, 1)$, we can write

$$I(x_3) \geq \int_{D_{y_3}} \left\{ \frac{2}{\lambda + 2\mu} v_{x_2, y_3} v_{x_2, y_3} + \frac{2\mu^2 (1 - \varepsilon)}{\lambda + 2\mu} v_{x_3, y_3} v_{y_3, y_3} + \frac{2}{T_0} \delta K T_{y_3} T_{y_3} - 2\rho \omega^2 v_{x_2, y_3} - 2\rho \omega^2 v_{x_3, y_3} - \omega^2 (A_1 + A_2 \omega^2) v_3^2 v_3 - \left[ B_1 + (B_2 + B_3 \delta^2) \omega^2 + (B_2 + B_3) \delta^2 \omega^4 \right] T_3 T_3 \right\} dA$$

$$+ \frac{2}{\lambda + 2\mu} \int_{D_{y_3}} \left\{ \mu^2 v_{x_2, x_2} - \frac{\beta}{2\mu e} (1 + i \omega\omega) T_3 + \frac{1}{2\mu e} \rho \omega^2 v_3 \right\}$$

$$\times \left[ v_{y_3, y_3} - \frac{\beta}{2c_3 (\lambda + \mu)} (1 - i \omega\omega) T_3 - \frac{1}{2c_3 \mu} \rho \omega^2 v_3 \right]$$

$$+ c_3 \mu (\lambda + \mu) \left[ v_{x_2, x_3} - \frac{\beta}{2c_3 (\lambda + \mu)} (1 - i \omega\omega) T_3 - \frac{1}{2c_3 \mu} \rho \omega^2 v_3 \right]$$

$$\times \left[ v_{y_3, y_3} - \frac{\beta}{2c_3 (\lambda + \mu)} (1 - i \omega\omega) T_3 - \frac{1}{2c_3 \mu} \rho \omega^2 v_3 \right]$$

$$+ \frac{\mu}{4\varepsilon} \rho \omega^2 \left[ v_3 \sqrt{\lambda_1} + \frac{\beta}{\mu \sqrt{\lambda_1}} (1 + i \omega\omega) T_3 \right]$$

$$\times \left[ v_3 \sqrt{\lambda_1} + \frac{\beta}{\mu \sqrt{\lambda_1}} (1 + i \omega\omega) T_3 \right]$$

$$+ \frac{\mu}{4c_3} \rho \omega^2 \left[ v_3 \sqrt{\lambda_1} - \frac{1}{\mu \sqrt{\lambda_1}} \left( \beta + \frac{b M_0}{c_2} \right) (1 + i \omega\omega) T_3 \right]$$

$$\times \left[ v_3 \sqrt{\lambda_1} - \frac{1}{\mu \sqrt{\lambda_1}} \left( \beta + \frac{b M_0}{c_2} \right) (1 + i \omega\omega) T_3 \right]$$

$$+ \frac{\mu}{2} \rho \omega^2 \left[ v_3 \sqrt{\lambda_1} - \frac{\beta}{\mu \sqrt{\lambda_1}} i \omega\omega T_3 \right]$$

$$\times \left[ v_3 \sqrt{\lambda_1} + \frac{\beta}{\mu \sqrt{\lambda_1}} i \omega\omega T_3 \right] \right\} dA$$

(66)
where

\[
A_1 = \frac{\rho \mu \lambda_1}{2(\lambda + 2\mu)} \left( \frac{1}{\varepsilon} + \frac{1}{c_3} + 2 \right) \quad A_2 = \frac{\rho^2}{2(\lambda + 2\mu)} \left( \frac{1}{\varepsilon} + \frac{\lambda + \mu}{c_3\mu} + \frac{b^2}{c_2c_3} \right)
\]

\[
B_1 = \frac{1}{2c_3(\lambda + 2\mu)} \left( \frac{\beta c_3}{\varepsilon} + \frac{\mu \beta^2}{\lambda + \mu} + \frac{M_0^2}{c_2} \right)
\]

\[
B_2 = \frac{\rho}{2\lambda_1\mu(\lambda + 2\mu)} \left[ \frac{\beta^2}{\varepsilon} + \frac{1}{c_3} \left( \beta + \frac{bM_0}{c_2} \right)^2 \right] \quad B_3 = \frac{\rho \beta^2}{\lambda_1\mu(\lambda + 2\mu)}
\]

Next, we use boundary conditions (55) and (56) to write

\[
\int_{D_{33}} v_{x,\rho,\rho} \bar{v}_{x,\rho,3} \, dA \geq \lambda_1 \int_{D_{33}} v_{x,3} \bar{v}_{x,3} \, dA \quad (68)
\]

\[
\int_{D_{33}} \chi_{,\rho,\rho} \bar{\chi}_{,\rho,3} \, dA \geq \lambda_1 \int_{D_{33}} \chi_{,3} \bar{\chi}_{,3} \, dA \quad (69)
\]

\[
\int_{D_{33}} T_{3,\rho} \bar{T}_{3,\rho} \, dA \geq \lambda_1 \int_{D_{33}} T_{3} \bar{T}_{3} \, dA \quad (70)
\]

\[
\int_{D_{33}} v_{3,\rho} \bar{v}_{3,\rho} \, dA \geq \lambda_1 \int_{D_{33}} v_{3} \bar{v}_{3} \, dA \quad (71)
\]

Then, by using relations (68)–(72) in relation (66), we deduce the following:

\[
I(\chi_3) \geq \int_{D_{33}} \left\{ 2(\lambda_1 - \rho \omega^2)v_{x,3} \bar{v}_{x,3} + 2(\lambda_1 - \rho M_0^2)\chi_{,3} \bar{\chi}_{,3}
\right.
\]

\[
+ \left[ \frac{2\mu^2(1-\varepsilon)}{\lambda + 2\mu} \lambda_1 \frac{1}{\varepsilon} \omega^2(A_1 + A_2\omega^2) \right] v_{3,\rho} \bar{v}_{3,\rho}
\]

\[
+ \left[ \frac{2}{T_0} \delta k \lambda_1 - B_1 - (B_1 \delta^2 + B_2)\omega^2 - (B_2 + B_3)\delta^2\omega^4 \right] T_{3} \bar{T}_{3} \ \right\} dA \quad (73)
\]

At this stage of our analysis we choose the parameter \( \delta \) in relation (57) so that

\[
\delta > \frac{T_0 B_1}{2\lambda_1 k} \quad (74)
\]
and we set
\[
\omega_{m}^2 = \min \left\{ \frac{\hat{\lambda}_1\mu}{\rho}, \frac{\hat{\lambda}_1\omega}{\rho\kappa}, w_1, w_2 \right\}
\] (75)

where \(w_1\) is the positive root of the algebraic equation
\[
A_2w^2 + A_1w - \frac{2\mu^2(1 - \varepsilon)}{\lambda + 2\mu} \hat{\lambda}_1 = 0
\] (76)

and \(w_2\) is the positive root of the equation
\[
(B_2 + B_3)\delta^2w^2 + (B_1\delta^2 + B_2)w - \left( \frac{2}{T_0} \delta k \hat{\lambda}_1 - B_1 \right) = 0
\] (77)

We also assume for the remainder of this section that
\[
\omega < \omega_m^* \tag{78}
\]

With these assumptions, relation (73) gives
\[
I(x_3) \geq 2v_2^2 F(x_3)
\] (79)

where
\[
2v_2^2 = \min \left\{ 2\left( \frac{\hat{\lambda}_1 - \frac{\rho\omega^2}{\mu}}{\mu} \right), 2\left( \frac{\hat{\lambda}_1 - \frac{\rho\kappa\omega^2}{x}}{\lambda + 2\mu} \right), \frac{2\mu(1 - \varepsilon)}{\lambda + 2\mu} \hat{\lambda}_1 - \frac{1}{\hat{\lambda}_1\mu} \omega^2(A_1 + A_2\omega^2), 2\hat{\lambda}_1 - \frac{T_0}{\delta k} B_1 - \frac{T_0}{\delta k} (B_1\delta^2 + B_2)\omega^2 - \frac{T_0}{\delta k} (B_2 + B_3)\delta^2\omega^4 \right\}.
\] (80)

From relations (57), (58), (59), and (79), we obtain
\[
FF'' - \frac{1}{2} F'^2 \geq 2v_2^2 F^2 + 2\left\{ \int_{D_{33}} \left[ \mu \left( v_{\rho,3}\bar{v}_{\rho,3} + v_{3,\rho}\bar{v}_{3,\rho} \right) + \alpha \chi_{33}\bar{\chi}_{33} + \frac{1}{T_0} \delta k T_{33} \bar{T}_{33} \right] dA \right. \\
&\times \left. \int_{D_{33}} \left[ \mu \left( v_{\rho,33}\bar{v}_{\rho,33} + v_{3,\rho,3}\bar{v}_{3,\rho,3} \right) + \alpha \chi_{33}\bar{\chi}_{33} + \frac{1}{T_0} \delta k T_{33} \bar{T}_{33} \right] dA \right. \\
&- \left( \frac{1}{2} \int_{D_{33}} \left[ \mu \left( v_{\rho,33}\bar{v}_{\rho,33} + v_{3,\rho,3}\bar{v}_{3,\rho,3} + v_{3,\rho}\bar{v}_{3,\rho} + v_{3,\rho}\bar{v}_{3,\rho,3} \right) + \alpha \chi_{33}\bar{\chi}_{33} + \frac{1}{T_0} \delta k (T_{33}\bar{T}_{33} + T_{3,33}\bar{T}_{33}) \right] dA \right)^2 \right\}
\] (81)
By means of Schwarz’s inequality, we can see that the quantity within the brackets in Eq. (81) is nonnegative and, hence, we have

\[ FF'' - \frac{1}{2} F'^2 \geq 2v_2^2 F^2 \tag{82} \]

that is

\[ \left[ F^\frac{1}{2} \right]'' \geq v_2^2 F^\frac{1}{2} \tag{83} \]

On the basis of a standard comparison theorem [17] we further study the consequences of second-order differential inequality (83). Let us first consider a finite cylinder, that is, \( L < \infty \). According with our assumptions we have

\[ v_{x,3}(x) = 0 \quad v_3(x) = 0 \quad \chi_{x,3}(x) = 0 \quad T_{x,3}(x) = 0 \quad \text{on } D_L \tag{84} \]

so that, by means of relation (57), we get

\[ F(L) = 0 \tag{85} \]

Then, by the maximum principle [17], we have

\[ F^\frac{1}{2} \leq G \tag{86} \]

where \( G \) satisfies

\[ G'' - v_2^2 G = 0 \quad G(0) = F^\frac{1}{2}(0) \quad G(L) = 0 \tag{87} \]

Therefore, we deduce that

\[ F^\frac{1}{2}(x_3) \leq F^\frac{1}{2}(0) \exp\left(-v_2 x_3\right) \quad 0 \leq x_3 \leq L \tag{88} \]

which is an exponential decay of the amplitude of the harmonic vibration, provided the frequency of the oscillation is lower than the critical level described by relation (75).

Let us now consider the case of a semi-infinite cylinder. To discuss this case we introduce the class of amplitudes \( \{v_r, \chi, T\} \) for which the following volume measure is bounded:

\[ E_1(z) = \int_{R_3} \left\{ \mu (v_{x,3} \bar{v}_{x,3} + v_{3,3} \bar{v}_{3,3}) + \alpha \chi_{x,3} \bar{\chi}_{x,3} + \frac{1}{T_0} \delta k T_{x,3} \bar{T}_{x,3} \right\} dv \tag{89} \]

In this class we have \( F(\infty) = 0 \) and, hence, the preceding procedure proves that spatial decay estimate (88) holds true for all \( x_3 \in [0, \infty) \).
CONCLUSIONS

Within the context of the linear theory of thermoelastic materials with voids developed by Ieşan [3] we are able to derive spatial decay results for the amplitude of harmonic vibrations in a cylinder subjected to plane boundary data varying harmonically in time on its lateral surface and on one of its bases, provided the angular frequency is lower than a critical value. As one expects, our results prove that the decay rates of solutions as well as the critical values of the angular frequencies contain new terms characterizing the influence of the material porosity, and their values are therefore modified from the values predicted by Chirită [10] for the classical theory of thermoelasticity.

REFERENCES