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Structural stability in porous elasticity

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We consider the linearized system of equations for an elastic body with voids as derived by Cowin & Nunziato. We demonstrate that the solution depends continuously on changes in the coefficients, which couple the equations of elastic deformation and of voids. It is also shown that the solution to the coupled system converges, in an appropriate measure, to the solutions of the uncoupled systems as the coupling coefficients tend to zero.

Keywords: porous materials; energy bounds; structural stability; convergence

1. Introduction

Porous materials have applications in almost all fields of engineering, e.g. soil mechanics, petroleum industry, material science as well as in biomechanics. A review of the historical development of the porous media theories as well as reference to various contributions may be found in the monographs by Bowen (1976) and Ieșan (2004) and in the articles by Bedford & Drumheller (1983) and De Boer (1998). The theory of elastic materials with voids is one of the simple extensions of the classical theory of elasticity for the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. Such theory seems to be an adequate tool to describe the behaviour of granular materials like rock, soils and manufactured porous bodies. The theory of elastic materials with voids has been developed by Nunziato & Cowin (1979) and Cowin & Nunziato (1983) and has received considerable interest in recent years. In fact, Goodman & Cowin (1972) have developed a continuum theory of granular materials with interstitial voids. The basic concept underlying this theory is that of a material for which the bulk density is written as the product of two fields, the density field of the matrix material and the volume fraction field. Such a representation was used by Nunziato & Cowin (1979) in order to develop a nonlinear theory of elastic materials with voids. The intended applications of the theory are to elastic bodies with small voids or vacuum pores which are distributed throughout the material. The linear theory of elastic materials with voids has been established by Cowin & Nunziato (1983).

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In the present paper, we study the structural stability of the mathematical model of the linear elastic material with voids. One of the most important tasks in the study of the structural stability is to prove that the solutions of problems depend continuously on the constitutive quantities, which may be subjected to error or perturbations in the mathematical modelling process. Concerning the structural stability, we emphasize the continuous dependence on changes in the model itself rather than on the initial and given data. That means changes in coefficients in the partial differential equations and changes in the equations and may be reflected physically by changes in constitutive parameters (as, for example, the coefficients obtained in a small deformation superposed to a finite one). Estimates of continuous dependence play a central role in obtaining numerical approximations to these kinds of problems.

Since many physical phenomena can be modelled by idealized approaches, the derivation of continuous dependence inequalities on various types of data has attracted considerable attention. The structural stability was a subject of great interest in recent years. In this connection, we point out that many studies of this type have been inspired by the fundamental paper of Knops & Payne (1969), where such investigations were initiated in elasticity, cf. also Knops & Wilkes (1973, sections 73, 74) and Knops & Payne (1988). We also recall structural stability analyses in the book by Ames & Straughan (1997) and in a series of papers by Ames & Payne (1995), Franchi & Straughan (1996), Payne & Straughan (1996, 1998), Payne & Song (1997) and Quintanilla (2003).

Throughout this paper we restrict our attention to the case of a centrosymmetric inhomogeneous linear elastic material with voids and discuss the continuous dependence of solution with respect to the coupling coefficients of the model. In §2, we formulate the corresponding initial-boundary value problem in question and present some constitutive assumptions. Section 3 is devoted to obtain a priori estimates for some auxiliary static problems. While §4 gives some a priori bounds for the solution of the dynamic problem. The continuous dependence of the solution with respect to the coupling coefficients in question is established in §5. In §6, we investigate how the solution of the basic initial-boundary value problem behaves as the coupling coefficients tend to zero.

2. Equations of motion

Throughout this paper we will consider, for convenience, a centrosymmetric elastic material with voids. Consequently, the general linear equations, from Ciarletta & Ieşan (1993; eqns (7.3.7), (7.3.12), (7.3.13)) are (see also Cowin (1985))

\[ q\ddot{u}_i = (a_{ijkl}u_{k,l})_{,j} + (b_{ij}\phi)_{,j}, \]

\[ qx\dot{\phi} = (a_{ij}\phi_{,j})_{,i} - \tau\dot{\phi} - b_{ij}u_{i,j} - \xi\phi, \]

in \(\Omega \times (0,T)\), with boundary condition on \(\Gamma\)

\[ u_i = g_i(x, t), \quad \phi = h(x, t), \]

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and initial data

\[
\begin{align*}
    u_i(x,0) &= u^0_i(x), & \phi(x,0) &= \phi^0(x), \\
    \dot{u}_i(x,0) &= v^0_i(x), & \dot{\phi}(x,0) &= \zeta^0(x).
\end{align*}
\]  

(2.4)

The mass density \(\rho\) and the equilibrated inertia \(x\) and the constitutive coefficients \(a_{ij}, b_{ij}, a_{ij}, \tau, \text{ and } \xi\) are prescribed functions of \(x\), supposed to be as smooth as required in our subsequent analysis. Moreover, we require

\[
a_{ij} = a_{kk}, \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji},
\]

\(q, x > 0, \quad \tau \geq 0,\)  

(2.5)

\[
a_{ij} \xi_j \xi_k \geq a_1 |\xi|^2, \quad a_1 > 0,
\]

(2.6)

\[
a_{ij} \xi_j \xi_j \geq a_2 |\xi|^2, \quad a_2 > 0.
\]

(2.7)

We also assume that the energy density \(E\) is a positive definite quadratic form, i.e.

\[
E \equiv \frac{1}{2} a_{ijk} u_{i,j} u_{k,h} + \frac{1}{2} \xi \phi^2 + \frac{1}{2} a_{ij} \phi_i \phi_j + b_{ij} u_{i,j} \phi \geq a_0 \left( |\nabla u|^2 + |\phi|^2 \right)
\]

\[
+ \frac{a_2}{2} |\nabla \phi|^2, \quad a_0 > 0.
\]

(2.8)

In the above relations \(a_0, a_1\) and \(a_2\) are appropriate constants.

The surface of the region is supposed to be as smooth as required, and smooth solutions are envisaged throughout.

3. Rellich identities

Before proceeding to derive an a priori estimate for a solution to equations (2.1)–(2.4), we need some bounds for a solution to certain auxiliary problems. These are achieved via the use of Rellich-like identities, cf. Payne & Weinberger (1958) and Bramble & Payne (1961, 1963).

Let \(H\) be a solution to the problem

\[
\begin{align*}
    (a_{ij} H_{j,i})_i &= 0, & \text{in } \Omega, \\
    H &= q, & \text{on } \Gamma.
\end{align*}
\]  

(3.1)

Under appropriate regularity assumptions, the existence of the solution \(H\) is assured by means of the existence theory presented by Fichera (1972).

We form the identity

\[
0 = \int_\Omega x_k H_{,k} (a_{ij} H_{j,i})_i \, dx,
\]

and then integrate by parts in succession to find

\[ 0 = -\int_Q x_k H_i a_{ij} H_j d\mathbf{x} - \int_Q x_k H_{i;j} H_j d\mathbf{x} + \oint_I n_i x_k H_k a_{ij} H_j dA \]

\[ = -\int_Q a_{ij} H_i H_j d\mathbf{x} - \frac{1}{2} \int_Q x_k a_{ij} (H_i H_j)_{;k} d\mathbf{x} + \oint_I n_i x_k H_k a_{ij} H_j dA \]

\[ = \frac{1}{2} \int_Q a_{ij} H_i H_j d\mathbf{x} + \frac{1}{2} \int_Q x_k a_{ij} H_i H_j d\mathbf{x} + \oint_I n_i x_k H_k a_{ij} H_j dA \]

\[ - \frac{1}{2} \oint_I x_k n_k a_{ij} H_i H_j dA. \]

(3.2)

We now write \( H_i \) on the surface \( \Gamma \) as \( H_i = n_i \partial H / \partial n + s_i \nabla_s H \), where \( n_i \) denotes the unit normal to \( \Gamma \) and \( s_i \) is the tangential vector. In addition, \( \partial / \partial n \) is the normal derivative and \( \nabla_s \) is the tangential derivative, i.e. \( \nabla_s H = x_k a^{\alpha \beta} H_{\beta \gamma} \), where \( a^{\alpha \beta} \) are the coefficients of the first fundamental form of the surface \( \Gamma \), and the subscript \( ';\alpha' \) denotes differentiation with respect to the surface variables \( \theta^\alpha \).

Suppose now also that \( \Gamma \) is star shaped with respect to an origin so that

\[ x_k n_k \geq h_0 > 0, \quad \text{on} \quad \Gamma, \]  

and

\[ |x_k s_k| \leq \delta_0, \quad |x_k a_{ij;k}| \leq a_s < \mu a_2, \]

for \( 0 < \mu < 1 \), where \( \delta_0 \), \( a_s \) and \( \mu \) are constants. Obviously (3.3) places a restriction on the size of the gradients of \( a_{ij} \), but this is not a severe restriction. Then, from (3.2), we may find

\[ \frac{1}{2} (1 - \mu) \int_Q a_{ij} H_i H_j d\mathbf{x} + \frac{h_0 a_2}{2} \oint_I \left( \frac{\partial H}{\partial n} \right)^2 dA \]

\[ \leq \oint_I \left( \frac{1}{2} x_k n_k s_i s_j - n_i s_j s_k x_k \right) a_{ij} |\nabla_s q|^2 dA \]

\[ + \delta_0 \alpha \oint_I a_{ij} n_i n_j \left( \frac{\partial H}{\partial n} \right)^2 dA + \frac{\delta_0}{2\alpha} \oint_I a_{ij} n_i n_j |\nabla_s q|^2 dA, \]

for \( \alpha > 0 \) at our disposal.

If we now let \( a_3 = \max_{\Gamma} |n_i n_j a_{ij}| \), then we pick \( \alpha \) such that

\[ h_0 a_2 > \delta_0 \alpha a_3, \quad \text{e.g.} \quad \alpha = \frac{h_0 a_2}{2\delta_0 a_3}. \]

Then we find
\[
\frac{1-\mu}{2} \int_Q a_{ij} H_i H_j dx + \frac{h_0 a_2}{4} \oint_r \left( \frac{\partial H}{\partial n} \right)^2 dA \\
\leq \oint_r \left[ \left( \frac{1}{2} x_k n_k s_{ij} - n_i s_j s_k x_k \right) + \frac{\delta_0^2 a_3}{h_0 a_2} n_i n_j \right] a_{ij} |\nabla_s q|^2 dA. \tag{3.4}
\]
We deduce from the Poincaré inequality that
\[
\lambda_1 \int_Q H^2 dx \leq \int_Q H_i H_i dx + C_1 \oint_r H^2 dA. \tag{3.5}
\]
Together, inequalities (3.4) and (3.5) furnish a priori bounds for \( \int_Q a_{ij} H_i H_j dx, ||H||^2 \) and \( \oint_r (\partial H/\partial n)^2 dA \) in terms of the data function \( q \).

We also need an equivalent bound for a vector version of (3.4). So, let \( G_i \) be a solution to the problem
\[
(a_{ijkl} G_{k,h})_j = 0 \quad \text{in} \quad Q, \quad \left\{ \begin{array}{l}
G_i = \tilde{g}_i \quad \text{on} \quad \Gamma.
\end{array} \right.
\]
We remark that existence of the function \( G_i \) (as that of \( H \) in equation (3.1)) follows, for example, from the work of Fichera (1972).

We commence with the identity
\[
0 = \int_Q (a_{ijkl} G_{k,h})_j x_r G_{i,r} dx.
\]
After integration by parts, we find
\[
0 = \frac{1}{2} \int_Q a_{ijkl} G_{i,j} G_{k,h} dx + \frac{1}{2} \int_Q x_r a_{ijkl,r} G_{i,j} G_{k,h} dx + \oint_r n_j x_r G_{i,r} a_{ijkl} G_{k,h} dA \\
- \frac{1}{2} \oint_r x_r n_r a_{ijkl} G_{i,j} G_{k,h} dA. \tag{3.7}
\]
We now write \( G_{i,j} \) on \( \Gamma \) as \( G_{i,j} = n_j \partial G_i / \partial n + s_j \nabla_s G_i \) and then from (3.7), we deduce that
\[
0 = \frac{1}{2} \int_Q a_{ijkl} G_{i,j} G_{k,h} dx + \frac{1}{2} \int_Q x_r a_{ijkl,r} G_{i,j} G_{k,h} dx \\
+ \frac{1}{2} \oint_r x_r n_r a_{ijkl,n} \frac{\partial G_i}{\partial n} \frac{\partial G_k}{\partial n} dA + \oint_r x_r s_r a_{ijkl,n} \nabla_s G_i \frac{\partial G_k}{\partial n} dA \\
- \frac{1}{2} \oint_r x_r n_r a_{ijkl} s_j \nabla_s \tilde{g}_i \nabla_s \tilde{g}_k dA + \oint_r x_r s_r a_{ijkl} n_j \nabla_s \tilde{g}_i s_h \nabla_s \tilde{g}_k dA. \tag{3.8}
\]
If we now suppose \( |x_r a_{ijkl,r}| \leq A_0 < a_1, |a_{ijkl}| \leq A_1 \) for constants \( A_0, A_1 \), then from (3.8) we may derive
\[
\frac{a_1 - A_0}{2} \int_Q G_{i,j} G_{i,j} dx + a_1 h_0 \oint_r \frac{\partial G_i}{\partial n} \frac{\partial G_i}{\partial n} dA \leq \frac{(\delta_0 A_1)^2}{a_1 h_0} \oint_r \nabla_s \tilde{g}_i \nabla_s \tilde{g}_k dA \\
+ \frac{1}{2} \oint_r x_r n_r a_{ijkl} s_j \nabla_s \tilde{g}_i s_h \nabla_s \tilde{g}_k dA - \oint_r x_r s_r a_{ijkl} n_j \nabla_s \tilde{g}_i s_h \nabla_s \tilde{g}_k dA \equiv D_1. \tag{3.9}
\]
The right-hand side $D_1$ is a data term and then equation (3.9) represents an a priori estimate for $\int_\Omega a_{ijkl} G_{ij} G_{kl} dx, \phi_I(\partial G_{ij}/\partial n)(\partial G_{ij}/\partial n) dA$. Upon using the Poincaré inequality, we have

$$\lambda_1 \int_\Omega G_{ij} dx \leq \int_\Omega G_{ij} G_{ij} dx + C_1 \int_\Omega \hat{g}_i \hat{g}_j dx. \quad (3.10)$$

In this way, we may use equation (3.9) to also obtain an a priori estimate for $\|G\|^2$.

4. A priori estimates

Let $u_i$ and $\phi$ satisfy the system

$$\begin{cases} -\varrho \ddot{u}_i + (a_{ijkl} u_{kl})_{,j} + (\gamma_{ij} \phi),_j = 0, \\ -\varrho \ddot{\phi} - \tau \phi - \xi \phi - \varrho \chi \phi - \varrho \chi \phi, H = 0, \end{cases} \quad (4.1)$$

in $\Omega \times (0, T)$ with $u_i = g_i$, $\phi = h$ on $\Gamma$, $u_i$, $\phi$ given at $t = 0$ as in equation (2.4). In order to derive structural stability theorems, we first need to produce a priori estimates for $\int_\Omega \|\nabla \phi\|^2 dx$ and $\int_\Omega \|\nabla u_i\|^2 dx$.

Let now $H$ be the function which solves equation (3.1) with $q = h$ on $\Gamma$. We commence with the identity

$$\int_0^t ((a_{ijk} \phi, j,i) - \gamma_{ij} u_{ij} - \tau \phi - \xi \phi - \varrho \chi \phi, H), dx ds = 0.$$ 

After some integrations by parts in space and in time, we may show that

$$\begin{align*} \frac{1}{2} \int_\Omega \xi \phi^2(t) dx + \frac{1}{2} \int_\Omega \varrho \chi \phi^2(t) dx + \int_0^t \tau \phi^2(s) dx ds &+ \frac{1}{2} \int_\Omega a_{ijkl} \phi_{ijkl} dx + \int_0^t \gamma_{ij} u_{ij} \phi dx ds \\
&= \frac{1}{2} \int_\Omega a_{ijkl} \phi_{ijkl} dx + \int_0^t \int_\Gamma n_i n_j a_{ij} \frac{\partial H}{\partial n} dA ds + \int_0^t \int_\Gamma n_j s_i a_{ij} h \nabla s_i dA ds \\
&\quad + \int_0^t (\xi \phi, H), dx ds + \frac{1}{2} (\varrho \chi \phi, \phi^0) + \int_0^t (\gamma_{ij} u_{ij}, H), ds + (\varrho \chi \phi, H) \\
&\quad - (\varrho \chi \phi, H(0)) - \int_0^t (\varrho \chi \phi, H), ds + \int_0^t (\tau \phi, H), ds + \frac{1}{2} (\varrho \chi \phi, \phi^0). \quad (4.2) \end{align*}$$

The idea is now to use the arithmetic-geometric mean inequality on the right-hand side of equation (4.2) and bound the $\phi$, $\phi$, $\phi$ terms using the left-hand side and leave only data terms and terms involving $H$ on the right. The $H$ terms may then be bounded by using the Rellich identity estimates of §3. With

$$\begin{align*} \gamma_M = \max_{\Omega} |\gamma_{ij}|, \quad \varrho_M = \max_\Omega |\varrho|, \quad \varrho_M = \max_\Omega |\varrho|, \quad \varrho_M = \max_\Omega |\varrho|, \quad \xi_M = \max_\Omega \xi, \quad \tau_M = \max_\Omega \tau, \end{align*}$$

we use the arithmetic-geometric mean inequality for \( \mu_1, ..., \mu_5 > 0 \) arbitrary constants at our disposal to derive from equation (4.2),

\[
\frac{1}{2} \int_Q \xi \phi^2(t) \, dx + \frac{1}{2} \int_Q \varrho \chi \phi^2(t) \, dx + \int_0^t \int_Q \tau \phi^2(s) \, dx \, ds + \frac{1}{2} \int_Q a_{ij} \phi_i \phi_j \, dx \\
+ \int_0^t \int_Q \gamma_{ij} u_{ij} \phi \, dx \, ds \leq \frac{\mu_1}{2} \int_0^t \| \phi \|^2 \, ds + \frac{\xi M}{2 \mu_1} \int_0^t \| H \|^2 \, ds + \frac{\mu_2}{2} \int_0^t \| u_{ij} \|^2 \, ds \\
+ \frac{\gamma^2}{2 \mu_2} \int_0^t \| H \|^2 \, ds + \frac{\mu_3}{2} \left( \frac{\varrho \chi \phi \phi}{2 \mu_3} \right) \| H \|^2 + \frac{\mu_4}{2} \int_0^t \| \phi \|^2 \, ds \\
+ \frac{(\varrho \chi \phi \phi)^2}{2 \mu_4} \int_0^t \| H \|^2 \, ds + \frac{\mu_5}{2} \int_0^t \| \phi \|^2 \, ds + \frac{\tau^2}{2 \mu_5} \int_0^t \| H \|^2 \, ds + \frac{1}{2} \int_Q a_{ij} \phi_i^0 \phi_j^0 \, dx \\
+ \int_0^t \int_Q \left( n_i a_{ij} g_i \right) \left( n_j g_j \right) \, dA \, ds + \int_0^t \int_Q n_j a_{ij} \nabla s_i \, dA \, ds + \frac{1}{2} (\xi \phi^0, \phi^0) \\
- (\varrho \chi \xi^0, H(0)) + \frac{1}{2} (\varrho \chi \xi^0, \xi^0). \tag{4.3}
\]

To handle the \( u_i \) equation, we commence with the identity

\[
\int_0^t \left( (a_{ijkl} u_{k,l})_j + (\gamma_{ij} \phi)_j - \varrho \ddot{u}_i, G_i - \dot{u}_i \right) \, ds = 0.
\]

After some integration by parts, we may show that

\[
\frac{1}{2} \int_0^t \frac{d}{ds} \left[ \int_Q \varrho \ddot{u}_i \, dx + \int_Q a_{ijkl} u_{j,k} \, dx \right] \, ds + \int_0^t \int_Q \gamma_{ij} \phi \dot{u}_i \, dx \, ds \\
= \int_0^t \int_Q \left( n_i a_{ijkl} g_k G_{i,j} \right) \, dA \, ds + \int_0^t \int_Q \gamma_{ij} \phi G_{i,j} \, dx \, ds + \int_Q \varrho \ddot{u}_i \, dx \\
- \int_Q \varrho G_i(0) v_i^0 \, dx - \int_0^t \int_Q \varrho G_i \dot{u}_i \, dx \, ds. \tag{4.4}
\]

We next use the arithmetic-geometric mean inequality on the right-hand side of equation (4.4) to find for \( \alpha_1, \alpha_2, \alpha_3 > 0 \) at our disposal,

\[
\frac{1}{2} \int_Q \varrho \ddot{u}_i \, dx + \frac{1}{2} \int_Q a_{ijkl} u_{j,k} \, dx + \int_0^t \int_Q \gamma_{ij} \phi \dot{u}_i \, dx \, ds \\
\leq \frac{1}{2} \int_Q \varrho v_i^0 v_i^0 \, dx + \frac{1}{2} \int_Q a_{ijkl} u_{j,k}^0 \, dx + \frac{\alpha_1}{2} \int_0^t \| \phi \|^2 \, ds + \frac{\gamma^2}{2 \alpha_1} \int_0^t \| G_{i,j} \|^2 \, ds \\
+ \int_0^t \int_Q \left( n_i a_{ijkl} g_k \right) \left( n_j g_j \right) \, dA \, ds + \frac{\alpha_2}{2} \int_Q \varrho \ddot{u}_i \, dx \, ds \\
+ \int_0^t \int_Q \varrho G_i \, dx - \int_Q \varrho G_i(0) v_i^0 \, dx + \frac{\alpha_3}{2} \int_0^t \int_Q \varrho \ddot{u}_i \, dx \, ds \\
+ \frac{1}{2 \alpha_3} \int_0^t \int_Q \varrho G_i \dot{G}_i \, dx \, ds. \tag{4.5}
\]

The next step is to add equations (4.3) and (4.5) to see that

\[
\frac{1}{2} \int_\Omega \dot{u}_i \dot{u}_i \, dx + \frac{1}{2} \int_\Omega a_{ijkl} u_{ij,k} u_{kl,i} \, dx + \frac{1}{2} \int_\Omega (\xi \phi, \phi) + \frac{1}{2} (q \chi \phi, \phi) + \int_0^t (\gamma \dot{\phi}, \phi) \, ds \\
+ \frac{1}{2} \int_\Omega a_ijkl \phi_i \phi_j \, dx + \int_\Omega \gamma_{ij} \phi \phi \, dx \leq \frac{\mu_1}{2} \int_0^t \|\phi\|^2 \, ds + \frac{\mu_2}{2} \int_0^t \|u_{ij}\|^2 \, ds \\
+ \frac{\mu_3}{2} (q \chi \phi, \phi) + \frac{\mu_4}{2} + \frac{\mu_5}{2} \int_0^t \|\phi\|^2 \, ds + \frac{\mu_1}{2} \int_0^t \|\phi\|^2 \, ds + \frac{\mu_2}{2} \int_\Omega \dot{u}_i \dot{u}_i \, dx \\
+ \frac{\mu_3}{2} \int_\Omega \dot{u}_i \dot{u}_i \, dx + F_1, 
\]

where \( F_1 \) is a term involving data or terms which are easily estimated in terms of data using the bounds in $\S 3$. In fact,

\[
F_1 = \int_\Omega \gamma_{ij} u_{ij}^0 \phi^0 \, dx + \frac{1}{2} \int_\Omega q v_i^0 v_i^0 \, dx + \frac{1}{2} \int_\Omega a_{ijkl} u_{ij,k}^0 u_{kl,i}^0 \, dx + \frac{\gamma_M}{2\alpha_1} \int_0^t \|G_{ij}\|^2 \, ds \\
+ \frac{1}{2} \int_0^t \int_\Omega n_h a_{ijkh} g_{k_{ij}} \nabla_i \nabla_j \, dA \, ds + \int_0^t \int_\Omega n_{ij} s_{ij} a_{ij} \nabla_i \nabla_j \, dA \, ds \\
+ \frac{1}{2} \int_0^t \int_\Omega |a_{ijkh}| |g|^2 \, dA \, ds + \frac{1}{2} \int_0^t \int_\Omega \phi \frac{\partial G_i}{\partial n} - \frac{\partial G_i}{\partial n} \, dA \, ds + \frac{1}{2\alpha_2} \int_\Omega \dot{G}_i \dot{G}_i \, dx \\
- \int_\Omega q G_i(0) v_i^0 \, dx + \frac{1}{2\alpha_3} \int_0^t \int_\Omega q G_i \dot{G}_i \, dx \, ds + \frac{\tau_M^2}{2\mu_1} \int_0^t \|H\|^2 \, ds + \frac{\gamma_M}{2\mu_2} \int_0^t \|H\|^2 \, ds \\
+ \frac{\gamma_M}{2\mu_3} \|H\|^2 + \frac{(\gamma_M \gamma_M)}{2\mu_4} \int_0^t \|H\|^2 \, ds + \frac{\tau_M^2}{2\mu_5} \int_0^t \|H\|^2 \, ds + \frac{1}{2} \int_\Omega a_{ij} \phi_{ij}^0 \phi_{ij}^0 \, dx \\
+ \frac{1}{2} \int_0^t \int_\Omega |a_{ij}| h^2 \, dA \, ds + \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial H}{\partial n} \right)^2 \, dA \, ds + \frac{1}{2} (\xi \phi, \phi) \\
- (q \xi \phi, H(0)) + \frac{1}{2} (q \chi \phi, \phi), 
\]

where \( a_M = \max_{\Gamma} |a_{ijkh}| \). On the basis of our assumptions (2.5) and (2.8), it follows that the left-hand side of equation (4.6) is a positive definite measure. We may choose \( \mu_3 = 1/2 \), \( \alpha_2 = 1/2 \). Then, we select as our energy measure

\[
\mathcal{E}(t) = \frac{1}{2} \int_\Omega (q \dot{u}_i \dot{u}_i + q \chi \dot{\phi}^2) \, dx + \frac{1}{2} \int_\Omega a_{ijkl} u_{ij,k} u_{kl,i} \, dx \\
+ \int_\Omega \gamma_{ij} u_{ij} \phi \, dx + \frac{1}{2} \left( \xi \phi(t), \phi(t) \right) + \frac{1}{2} \int_\Omega a_{ij} \phi_i, \phi_j \, dx. 
\]
a computable constant $K>0$, we may obtain from equation (4.6)

$$E(t) \leq K \int_0^t E(s) ds + \mathcal{F}. \quad (4.8)$$

We multiply the inequality (4.8) by $e^{-Kt}$ and then integrate the result over $[0,t]$ to obtain the a priori bound

$$\int_0^t E(s) ds \leq \frac{e^{KT}}{K} \mathcal{F}. \quad (4.9)$$

Inequality (4.9) leads directly to a priori estimates for $\int_0^t \|\nabla u\|^2 ds$ and $\int_0^t \|\nabla \phi\|^2 ds$.

5. Continuous dependence on the coupling coefficient $b_{ij}$

To study continuous dependence on the coefficient $b_{ij}$ in equations (2.1)–(2.4), we let $(u_i, \phi)$ and $(v_i, \psi)$ be solutions to these equations for the same boundary and initial data, but for different coupling coefficients $\gamma_{ij}$ and $\delta_{ij}$ respectively. The coefficients $\varrho, \tau, \alpha_{ijkh}, \tau, \xi$ and $\alpha_{ij}$ are the same for both systems.

Define the difference variables $w_i, \theta, \beta_{ij}$ as

$$w_i = u_i - v_i, \quad \theta = \phi - \psi, \quad \beta_{ij} = \gamma_{ij} - \delta_{ij}. \quad (5.1)$$

One finds $(w_i, \theta)$ satisfies the system

$$\begin{align*}
(a_{ijkh} w_{k,h})_j + (\delta_{ij} \theta)_j - \varrho w_i &= - (\beta_{ij} \phi)_j, \\
(a_{ij} \theta)_i - \xi \theta - \tau \theta - \delta_{ij} w_i - \varrho x \theta &= \beta_{ij} u_i,
\end{align*} \quad (5.2)$$

in $\Omega \times (0,T)$. The boundary and initial conditions are:

$$w_i = 0, \quad \theta = 0, \quad \text{on } \Gamma \times (0,T), \quad (5.4)$$

$$\begin{align*}
w_i(x,0) &= 0, \quad \dot{w}_i(x,0) = 0, \\
\theta(x,0) &= 0, \quad \dot{\theta}(x,0) = 0.
\end{align*} \quad (5.5)$$

To establish continuous dependence, we multiply equation (5.2) by $\dot{w}_i$ and integrate over $\Omega$, then multiply equation (5.3) by $\dot{\theta}$ and similarly integrate over $\Omega$, to find

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ \int_\Omega \varrho \dot{w}_i \dot{w}_i dx + \int_\Omega a_{ijkh} w_{k,h} \dot{w}_i dx \right] + \int_\Omega \delta_{ij} \theta \dot{w}_i dx &= \int_\Omega (\beta_{ij} \phi)_j \dot{w}_i dx, \\
\frac{1}{2} \frac{d}{dt} \left[ \int_\Omega (\varrho x \dot{\theta}) \dot{\theta} dx + \int_\Omega a_{ij} \theta_i \theta_j dx + (\xi \theta, \theta) \right] + (\tau \dot{\theta}, \dot{\theta}) + \int_\Omega \delta_{ij} \dot{w}_i \dot{\theta} dx &= - \int_\Omega \beta_{ij} \dot{\theta} u_i, \dot{\theta} dx.
\end{align*} \quad (5.6)$$

Upon addition of these equations, we may see that
\[
\frac{d}{dt} \left\{ \frac{1}{2} \left[ \int_{\Omega} q \dot{w}_i \dot{w}_i \,dx + \int_{\Omega} a_{ijkl} w_{ij,k,l} \,dx + (q \dot{\theta}, \dot{\theta}) + (\xi, \theta) \right] + \int_{\Omega} \delta_{ij} w_{ij} \theta \,dx \right\} + (r \dot{\theta}, \dot{\theta})
\]
\[
= \int_{\Omega} (\beta_{ij} \phi)_{,j} \dot{w}_i \,dx - \int_{\Omega} \beta_{ij} \theta \dot{u}_{ij} \,dx
\]
\[
\leq \frac{1}{2 \varepsilon_m} \int_{\Omega} q \ddot{w}_i \ddot{w}_i \,dx + \frac{1}{2} \int_{\Omega} \beta^2 \phi, \phi, \,dx + \frac{1}{2} \int_{\Omega} q \dot{w}_i \dot{w}_i \,dx
\]
\[
+ \frac{1}{2} \int_{\Omega} \beta^2 \phi, \phi, \,dx + \int_{\Omega} \frac{\beta^2}{2a_1} a_{ijkl} u_{ij,kh} \,dx + \frac{1}{2} \| \partial \|^2,
\]
(5.8)
where \( \beta^2 = \beta_{ij} \beta_{ij} \), \( \beta^2 = \beta_{ir} \beta_{is} \) and \( \varepsilon_m = \min_{\Omega} q \). If we put \( \beta_M^2 = \max_{\Omega} \beta^2 \), then define
\[
E_1(t) = \frac{1}{2} \left[ \int_{\Omega} q \ddot{w}_i \ddot{w}_i \,dx + \int_{\Omega} a_{ijkl} w_{ij,k,l} \,dx + (q \dot{\theta}, \dot{\theta}) + (\xi, \theta) + \int_{\Omega} a_{ij} \theta, \theta \,dx \right]
\]
\[
+ \int_{\Omega} \delta_{ij} w_{ij} \theta \,dx,
\]
and then for \( C = \max \{2/\varepsilon_m, 1/\varepsilon_m a_1 \} \), we find
\[
\frac{dE_1}{dt} \leq C E_1 + \beta_M^2 \left( \int_{\Omega} \frac{1}{2a_1} a_{ijkl} u_{ij,kh} \,dx + \frac{1}{2} \int_{\Omega} \phi, \phi, \,dx \right) + \frac{1}{2} \beta^2 \| \phi \|^2.
\]
This inequality is integrated to see that
\[
E_1(t) \leq C \int_0^t E_1(s) \,ds + \beta_M^2 \left[ \int_0^t \left( \int_{\Omega} \frac{1}{2a_1} a_{ijkl} u_{ij,kh} \,dx \,ds + \frac{1}{2} \int_{\Omega} \phi, \phi, \,dx \,ds \right) + \frac{1}{2} \beta^2 \int_0^t \| \phi \|^2 \,ds \right]
\]
(5.9)
The last term of equation (5.9) is bounded using the \( a \) priori estimate (4.9) to find
\[
E_1(t) \leq C \int_0^t E_1(s) \,ds + \beta_M^2 F_1 + \beta_M^2 F_2,
\]
(5.10)
where \( F_1 \) and \( F_2 \) are some data terms. Upon integration of equation (5.10), we may arrive at
\[
\int_0^t E_1(s) \,ds \leq (\beta_M^2 F_1 + \beta_M^2 F_2) \frac{1}{C} \left( e^{CT} - 1 \right).
\]
(5.11)
Inequality (5.11) is truly \( a \) priori and demonstrates the solution to (2.1)–(2.4) depends continuously on changes in the coupling coefficient \( b_{ij} \) in the measure

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By combining equations (5.10) and (5.11), we may also show that
\begin{equation}
\mathcal{E}_1(t) \leq \left( \beta_M^2 \mathcal{F}_1 + \tilde{\beta}_M^2 \mathcal{F}_2 \right) e^{CT}.
\end{equation}

Thus, continuous dependence on \( \beta_{ij} \) in the measure \( \mathcal{E}_1(t) \) also follows.

6. Convergence

We now investigate how the solution to (2.1)–(2.4) behaves as the coefficient \( b_{ij} \to 0 \). Thus, let \((u, \phi)\) be a solution to (2.1)–(2.4), where now we assume \( b_{ij} \) are constants. Let also \((v_i, \psi)\) be a solution to (2.1)–(2.4) with \( b_{ij} \equiv 0 \) (so the equations are uncoupled). The difference solution \( w_i = u_i - v_i, \theta = \phi - \psi \) is seen to satisfy the boundary-initial value problem,
\begin{align}
\dot{q} w_i &= (a_{ijkh} w_{k,h})_{,j} + (b_{ij} \phi)_{,j}, \\
q x \dot{\theta} &= (a_{ij} \theta_{,j})_{,i} - \tau \dot{\theta} - \xi \theta - b_{ij} u_{i,j},
\end{align}
in \( Q \times (0, T) \), with
\begin{align}
w_i &= 0, \quad \theta = 0, \quad \text{on} \quad T \times (0, T), \quad (6.3)
\end{align}
and
\begin{align}
w_i &= 0, \quad \dot{w}_i = 0, \quad \theta = 0, \quad \dot{\theta} = 0, \quad \text{at} \quad t = 0. \quad (6.4)
\end{align}

Multiply equation (6.1) by \( \dot{w}_i \) and integrate over \( Q \), and likewise multiply equation (6.2) by \( \dot{\theta} \) and integrate over \( Q \) to find
\begin{equation}
\frac{d}{dt} \left[ \int_Q q \dot{w}_i \dot{w}_i dx + \int_Q a_{ijkh} w_{i,j} w_{k,h} dx \right] = \int_Q \dot{w}_i (b_{ij} \phi)_{,j} dx, \quad (6.5)
\end{equation}
\begin{equation}
\frac{d}{dt} \left[ \int_Q q x \dot{\theta} \dot{\theta} + \int_Q a_{ij} \theta_{,i} \theta_{,j} dx \right] + \frac{d}{dt} \int_0^t (\tau \dot{\theta}, \dot{\theta}) ds = -\int_Q b_{ij} u_{i,j} \dot{\theta} dx. \quad (6.6)
\end{equation}

We next add these equations and use the arithmetic–geometric mean inequality on the right-hand side of the resulting equation to arrive at
\begin{align}
\frac{d}{dt} &\left[ \int_Q q \dot{w}_i \dot{w}_i dx + \int_Q a_{ijkh} w_{i,j} w_{k,h} dx + (q x \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) + \int_Q a_{ij} \theta_{,i} \theta_{,j} dx \right] \\
+ (\tau \dot{\theta}, \dot{\theta}) &\leq \frac{1}{2} \| \dot{\theta} \|^2 + \frac{1}{2b_m} \int_Q q \dot{w}_i \dot{w}_i dx + \frac{b_M^2}{2} (\| \nabla \phi \|^2 + \| \nabla u \|^2), \quad (6.7)
\end{align}
where \( b_M = \max_Q |b_{ij}| \). If we denote by
\begin{align}
\mathcal{E}_2(t) &= \frac{1}{2} \left[ \int_Q q \dot{w}_i \dot{w}_i dx + \int_Q a_{ijkh} w_{i,j} w_{k,h} dx + (q x \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) + \int_Q a_{ij} \theta_{,i} \theta_{,j} dx \right],
\end{align}

then from equation (6.7), we deduce that for $\lambda = \max\{a^{-1}, b^{-1}\},$

$$\mathcal{E}_2(t) \leq \lambda \int_0^t \mathcal{E}_2(s) ds + b_M^2 \int_0^t (\|\nabla \phi\|^2 + \|\nabla u\|^2) ds.$$  (6.8)

Thanks to the a priori bound (4.9), $\int_0^t (\|\nabla \phi\|^2 + \|\nabla u\|^2) ds$ is bounded in terms of a known data constant, $2k_1$ say. Thus from equation (6.8), we find that

$$\mathcal{E}_2(t) \leq \lambda \int_0^t \mathcal{E}_2(s) ds + k_1 b_M^2.$$  (6.9)

After integration, equation (6.9) yields

$$\int_0^t \mathcal{E}_2(s) ds \leq \frac{k_1}{\lambda} (e^{\lambda t} - 1) b_M^2.$$  (6.10)

Upon using equation (6.9), we may also show that

$$\mathcal{E}_2(t) \leq k_1 e^{\lambda T} b_M^2.$$  (6.11)

Estimates (6.10) and (6.11) demonstrate convergence in the measures $\mathcal{E}_2(t)$ and $\int_0^t \mathcal{E}_2(s) ds$, as $b_{ij} \to 0$.

**Remark**

We have only dealt with continuous dependence and convergence questions involving the coupling coefficients $b_{ij}$. We could also establish similar results for other coefficients, such as $\xi$ and $\tau$.

**References**


