Strong ellipticity for tetragonal system in linearly elastic solids

Stan Chirită a,*, Alexandre Danescu b

a Faculty of Mathematics, Al. I. Cuza University, Blvd. Carol I, No. 11, 700506 Iaşi, Romania
b Department Mécanique des Solides, Génie Mécanique, Génie Civil, Ecole Centrale de Lyon, Av. Guy de Collongue, BP 163-69134, Ecully Cedex, France

Abstract

The present paper studies the strong ellipticity for all crystal classes of tetragonal system in a linearly elastic material. Explicit conditions characterizing the strong ellipticity of the elasticity tensor are established for the tetragonal system with six elasticities (that is tetragonal–scalenohedral, ditetragonal–pyramidal, tetragonal–trapezohedral, ditetragonal–dipyramidal crystal classes) as well as for the tetragonal system with seven elasticities (tetragonal–disphenoidal, tetragonal–pyramidal and tetragonal–dipyramidal crystal classes).

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1. Introduction

For a linearly anisotropic elastic solid the components $C_{ijkl}$ of the tensor of elastic moduli satisfy the symmetries

$$C_{ijkl} = C_{klij} = C_{ijlk},$$

(1.1)

while the indices $i, j, k, l$ take values 1, 2, 3. The strong ellipticity condition states that

$$C_{ijkl} n_i n_j m_k m_l > 0,$$

(1.2)

for all non-zero vectors $\mathbf{m} = (m_1, m_2, m_3)$ and $\mathbf{n} = (n_1, n_2, n_3)$, where Einstein summation convention is assumed. The strong ellipticity condition is of importance in discussing uniqueness, wave propagation (see e.g. Gurtin, 1972, p. 86), loss of ellipticity in the context of the non-linear elasticity of fibre-reinforced materials (see Merodio and Ogden, 2003) and also in the study of spatial behaviour of the constrained anisotropic cylinders (see Chirită and Carletta, 2006).

In this paper we consider the tetragonal system and use the standard notation

$$c_{ij} = C_{11i1j}, \quad c_{2i} = C_{2ii2}, \quad c_{3i} = C_{3ii3}, \quad c_{ij} = C_{11i2j}, \quad c_{ij} = C_{11i3j},$$

$$c_{ij} = C_{22i1j}, \quad c_{ij} = C_{22i2j}, \quad c_{ij} = C_{22i3j},$$

$$c_{ij} = C_{33i1j}, \quad c_{ij} = C_{33i2j}, \quad c_{ij} = C_{33i3j},$$

(1.3)

in order to write the strong ellipticity condition (1.2). Restrictions placed by symmetries in the seven crystal classes of the tetragonal system are covered by the following two situations: (1) four classes, i.e. tetragonal–scalenohedral, ditetragonal–scalenohedral, tetragonal–trapezohedral, ditetragonal–dipyramidal, and (2) five classes, i.e. tetragonal–disphenoidal, tetragonal–pyramidal, tetragonal–dipyramidal, anisotropic elastic material.
pyramidal, tetragonal–trapezohedral, ditetragonal–dipyramidal providing an elasticity tensor with six elastic coefficients and (2) three classes, i.e. tetragonal–disphenoidal, tetragonal–pyramidal, tetragonal–dipyramidal providing an elasticity tensor with seven elastic coefficients. Thus, for the tetragonal system $C_3$ (Gurtin, 1972) generated by $\mathbb{R}^{e_1^2}$, $\mathbb{R}^{e_1}$ (as usual, $\mathbb{R}^e_4$ represents the orthogonal tensor corresponding to a right-handed rotation through the angle $\alpha$ about an axis in the direction of the unit vector $u$) the strong ellipticity condition (1.2) takes the following form:

\[
c_{11}(n_1m_1 + n_2m_2)^2 + c_{66}(n_1m_2 + n_2m_1)^2 + c_{33}n_3m_3^2 + 2(c_{12} - c_{11})n_1m_1n_2m_2 + 2c_{13}(n_1m_1 + n_2m_2)n_3m_3 + c_{55}(n_3m_1 + n_1m_3)^2 + (n_3m_2 + n_2m_3)^2 > 0,
\]

(1.4)

for all non-zero vectors $m = (m_1, m_2, m_3)$ and $n = (n_1, n_2, n_3)$, while for the tetragonal system $C_4$ generated by $\mathbb{R}^{e_1^2}$, it becomes

\[
c_{11}(n_1^2m_1^2 + n_2^2m_2^2) + 2c_{12}n_1m_1n_2m_2 + c_{66}(n_1^2m_2^2 + n_2^2m_1^2 + 2n_1m_2n_2m_1) + c_{33}(n_3m_3^2 + n_3^2m_3^2 + n_1^2m_2^2 + n_2^2m_1^2 + n_3^2m_1^2 + n_3^2m_2^2) + 2(c_{13} + c_{55})(n_1m_1 + n_2m_2)n_3m_3 + c_{16}[n_1n_2m_2^2 - m_1m_2(n_3^2 - n_2^2)] > 0,
\]

(1.5)

for all non-zero vectors $m = (m_1, m_2, m_3)$ and $n = (n_1, n_2, n_3)$. In this paper we establish explicit necessary and sufficient conditions in terms of the elasticities only in order to have fulfilled the above strong ellipticity conditions. The results are expressed by the following two theorems.

**Theorem 1.** Suppose that the symmetry of a linearly elastic material is represented by the tetragonal system $C_3$ generated by $\mathbb{R}^{e_1^2}$, $\mathbb{R}^e$. Then the elasticity tensor is strongly elliptic if and only if

\[
c_{11} > 0, \quad c_{33} > 0, \quad c_{55} > 0, \quad c_{66} > 0, \quad -2c_{66} - c_{11} < c_{12} < c_{11}, \quad -2c_{55} - \sqrt{c_{11}c_{33}} < c_{13} < \sqrt{c_{11}c_{33}},
\]

(1.6)

and, for $(c_{13} + c_{55})^2 > \frac{1}{2} \max(2c_{33}^2, c_{11}(c_{11} + c_{12}))$, we have

\[
\left| \frac{c_{13} + c_{55}}{c_{33}} - c_{55} \right| < \frac{1}{2} c_{33} [c_{11} + c_{12} + \min(c_{11} - c_{12}, 2c_{66})].
\]

(1.7)

**Theorem 2.** Suppose that the symmetry of a linearly elastic material is represented by the tetragonal system $C_4$ generated by $\mathbb{R}^{e_1^2}$. Then the elasticity tensor is strongly elliptic if and only if

\[
c_{11} > 0, \quad c_{33} > 0, \quad c_{55} > 0, \quad c_{66} > 0.
\]

(1.9)

\[
c_{16}^2 < \frac{1}{2} \min(c_{66}(c_{11} - c_{12}), c_{11}(c_{11} + c_{12} + 2c_{66})) < c_{11}c_{66},
\]

(1.10)

and, for $(c_{13} + c_{55})^2 > \frac{1}{2} \max(2c_{33}^2, c_{31}(c_{11} + c_{22}))$, we have

\[
\left| \frac{c_{13} + c_{55}}{c_{33}} - c_{55} \right| < \frac{1}{4} c_{33} \sqrt{(c_{11} - c_{12} - 2c_{66})^2 + 16c_{16}^2} < \frac{1}{4} c_{33} (3c_{11} + c_{12} + 2c_{66}).
\]

(1.11)

**Remark 1.** It is a straightforward task to verify that when $c_{16} \to 0$ the relations (1.9)–(1.12) reduce to the set described by relations (1.6)–(1.8). In fact, relation (1.10) gives

\[
c_{11} - c_{12} > 0, \quad c_{11} + c_{12} + 2c_{66} > 0.
\]

(1.13)

that is relation (1.7). While the relation (1.12) leads to (1.8).

**Remark 2.** If we consider the case of cubic system, that is we make $c_{33} = c_{11}, c_{13} = c_{12}$ and $c_{55} = c_{66}$, then the relations (1.6)–(1.8) reduce to

\[
c_{11} > 0, \quad c_{66} > 0, \quad -2c_{66} - c_{11} < c_{12} < c_{11}.
\]

(1.14)

In fact, the relation (1.8) is identically satisfied.

**Remark 3.** For an isotropic material we have $c_{11} = \lambda + 2\mu$, $c_{66} = \mu$ and $c_{12} = \lambda$ ($\lambda$ and $\mu$ the Lamé coefficients) and therefore, the strong ellipticity condition (1.14) reduces to

\[
\lambda + 2\mu > 0, \quad \mu > 0,
\]

(1.15)

which coincides with the condition given in Gurtin (1972).

We can prove Theorem 2 by using a direct procedure like that used in the proof of Theorem 1 or by using the results established in Theorem 1. It is this the reason for which we find convenient to treat separately the above two classes of elastic crystals.
We have to outline that, recently Chiriță et al. (2007) have developed a method for obtaining necessary and sufficient conditions for strong ellipticity in several classes of anisotropic linearly elastic materials. Previous results on the strong ellipticity of transversely isotropic materials have been obtained by Padovani (2002), Merodio and Ogden (2003) and Chiriță (2006).

2. Strong ellipticity for tetragonal system $\mathcal{C}_5$ generated by $R_0^{(2)}$, $R_1^{(6)}$ (six elasticities)

In what follows we shall describe a procedure for establishing the restrictions placed upon the elastic coefficients of the tetragonal system in discussion by the strong ellipticity condition (1.4). To this end we first rewrite (1.4) as

$$
\begin{align*}
&c_{35}(m_1^2 + m_2^2) + c_{33}m_3^2 + 2(c_{13} + c_{35})(m_1n_1 + m_2n_2)m_3n_3 + c_{55}(n_1^2 + n_2^2)m_3^2 + c_{11}(n_1^2m_1^2 + n_2^2m_2^2) \\
&+ c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 > 0.
\end{align*}
$$

(2.1)

Furthermore, we regard (2.1) as a quadratic equation in $n_3 \in \mathbb{R}$ in order to deduce

$$
\begin{align*}
c_{35}(m_1^2 + m_2^2) + c_{33}m_3^2 > 0 \quad &\text{for all } m = (m_1, m_2, m_3) \neq 0, \\
c_{55}(n_1^2 + n_2^2)m_3^2 + c_{11}(n_1^2m_1^2 + n_2^2m_2^2) + c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 > 0,
\end{align*}
$$

(2.2)

and

$$
\begin{align*}
(c_{13} + c_{35})^2(n_1m_1 + n_2m_2)^2m_3^2 &< (c_{35}(m_1^2 + m_2^2) + c_{33}m_3^2)(c_{55}(n_1^2 + n_2^2)m_3^2 + c_{11}(n_1^2m_1^2 + n_2^2m_2^2)) \\
&+ c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2,
\end{align*}
$$

(2.4)

for all non-zero vectors $m = (m_1, m_2, m_3)$ and $n = (n_1, n_2, n_3)$.

As a direct consequence, from (2.2) we obtain

$$
\begin{align*}
c_{33} > 0, & \quad c_{55} > 0,
\end{align*}
$$

(2.5)

while the relation (2.3) implies

$$
\begin{align*}
(c_{11}m_1^2 + c_{66}m_2^2)n_1^2 &+ 2[(c_{12} + c_{66})m_1m_2n_1]n_1 + (c_{66}m_1^2 + c_{11}m_2^2)n_2^2 > 0.
\end{align*}
$$

(2.6)

By considering (2.6) as a quadratic in $n_1 \in \mathbb{R}$, we deduce

$$
\begin{align*}
c_{11}m_1^2 + c_{66}m_2^2 > 0, \\
c_{66}m_1^2 + c_{11}m_2^2 > 0 \quad &\text{for all } (m_1, m_2) \neq 0,
\end{align*}
$$

(2.7)

and

$$
\begin{align*}
|c_{12} + c_{66}|m_1m_2|^2 &< (c_{11}m_1^2 + c_{66}m_2^2)(c_{66}m_1^2 + c_{11}m_2^2) \quad &\text{for all } (m_1, m_2) \neq 0.
\end{align*}
$$

(2.8)

Consequently, the relation (2.7) implies

$$
\begin{align*}
c_{11} > 0, & \quad c_{66} > 0,
\end{align*}
$$

(2.9)

and inequality (2.8) becomes

$$
\begin{align*}
c_{11}c_{66}(m_1^2 - m_2^2)^2 + [(c_{11} + c_{66})^2 - (c_{12} + c_{66})^2]m_1^2m_2^2 > 0.
\end{align*}
$$

(2.10)

Therefore, if we choose $m_1 = m_2$, then we obtain the restriction

$$
|c_{12} + c_{66}| < c_{11} + c_{66}.
$$

(2.11)

Let us now consider the relation (2.4) and note that it can be written as

$$
\begin{align*}
c_{35}c_{55}(n_1^2 + n_2^2)m_3^2 + (c_{35}(m_1^2 + m_2^2)(n_1^2 + n_2^2) + c_{11}c_{33}(n_1^2m_1^2 + n_2^2m_2^2) + c_{33}c_{66}(n_1^2m_2^2 + n_2^2m_1^2) \\
+ 2c_{33}(c_{12} + c_{66})n_1m_1m_2m_2 - (c_{13} + c_{55})^2(n_1m_1 + n_2m_2)^2m_3^2 + c_{55}(m_1^2 + m_2^2)(c_{11}(n_1^2m_1^2 + n_2^2m_2^2)) \\
+ c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 > 0,
\end{align*}
$$

(2.12)

for all $m_3 \in \mathbb{R}$ and for all non-zero vectors $(m_1, m_2)$ and $(n_1, n_2)$. Therefore, if we set

$$
m_3 = \frac{1}{\sqrt{c_{33}}}
\left[\sqrt{m_1^2 + m_2^2}c_{11}(n_1^2m_1^2 + n_2^2m_2^2) + c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2\right]^{1/2},
$$

(2.13)

into relation (2.12), we deduce

$$
\begin{align*}
|c_{13} + c_{55}||n_1m_1 + n_2m_2| &< c_{55}\sqrt{m_1^2 + m_2^2}(n_1^2 + n_2^2) + \sqrt{c_{33}}(c_{11}(n_1^2m_1^2 + n_2^2m_2^2)) \\
&+ c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 \right]^{1/2},
\end{align*}
$$

(2.14)
for all non-zero vectors \((m_1, m_2)\) and \((n_1, n_2)\).

Let us first note that, by setting \(n_1 = \lambda m_1\), \(\lambda \in \mathbb{R}\), and \(n_2 = m_2 = 0\), the relation (2.14) implies
\[
|c_{13} + c_{55}| < c_{55} + \sqrt{c_{11}c_{33}}.
\]  
(2.15)

Moreover, we can write the relations (2.11) and (2.15) as
\[
-c_{11} - 2c_{66} < c_{12} < c_{11},
\]
(2.16)
\[
-2c_{55} - \sqrt{c_{11}c_{33}} < c_{13} < \sqrt{c_{11}c_{33}}.
\]
(2.17)

Furthermore, we write the inequality (2.14) as
\[
|c_{13} + c_{55}| \frac{|n_1 m_1 + n_2 m_2|}{\sqrt{(m_1^2 + m_2^2)(n_1^2 + n_2^2)}} < c_{55} + \sqrt{c_{11}c_{33}} \left[\sigma_m + \frac{1}{2} (c_{11} + c_{12}) \frac{(n_1 m_1 + n_2 m_2)^2}{(m_1^2 + m_2^2)(n_1^2 + n_2^2)} \right]^{1/2},
\]
and hence, we have
\[
|c_{13} + c_{55}| \frac{|n_1 m_1 + n_2 m_2|}{\sqrt{(m_1^2 + m_2^2)(n_1^2 + n_2^2)}} < c_{55} + \sqrt{c_{11}c_{33}} \left[\sigma_m + \frac{1}{2} (c_{11} + c_{12}) \frac{(n_1 m_1 + n_2 m_2)^2}{(m_1^2 + m_2^2)(n_1^2 + n_2^2)} \right]^{1/2},
\]
(2.19)

where
\[
\sigma_m = \frac{1}{2} \min(c_{11} - c_{12}, 2c_{66}) > 0.
\]
(2.20)

Thus, if we set
\[
y = \frac{|n_1 m_1 + n_2 m_2|}{\sqrt{(m_1^2 + m_2^2)(n_1^2 + n_2^2)}},
\]
(2.21)
then (2.18) is equivalent with
\[
|c_{13} + c_{55}| y < c_{55} + \sqrt{c_{11}c_{33}} \left[\sigma_m + \frac{1}{2} (c_{11} + c_{12}) y^2 \right]^{1/2} \quad \text{for all } y \in [0, 1].
\]
(2.22)

We proceed now to obtain the restrictions on the elastic coefficients imposed by the inequality (2.22). To this end we first observe that for
\[
|c_{13} + c_{55}| y < c_{55},
\]
(2.23)
the relation (2.22) is trivially satisfied. Thus, without losing the generality, we shall assume
\[
\frac{c_{55}}{|c_{13} + c_{55}|} \leq y \leq 1.
\]
(2.24)

In view of (2.17), this last assumption is possible if and only if
\[-2c_{55} - \sqrt{c_{11}c_{33}} < c_{13} < -2c_{55} \quad \text{or} \quad 0 < c_{13} < \sqrt{c_{11}c_{33}}.
\]
(2.25)

With this in mind, the relation (2.22) is equivalent with
\[
f(y) > 0 \quad \text{for all } y \in \left[\frac{c_{55}}{|c_{13} + c_{55}|}, 1\right],
\]
(2.26)
where
\[
f(y) = \frac{1}{2} c_{33}(c_{11} + c_{12}) - (c_{13} + c_{55})^2 y^2 + 2c_{55} \frac{|c_{13} + c_{55}| y + c_{33} \sigma_m - c_{55}^2}{|c_{13} + c_{55}|^2}.
\]
(2.27)

Moreover, we note that
\[
f\left(\frac{c_{55}}{|c_{13} + c_{55}|}\right) = \frac{1}{2} c_{33}(c_{11} + c_{12}) - \frac{c_{55}^2}{(c_{13} + c_{55})^2} + c_{33} \sigma_m,
\]
(2.28)
and prove that
\[
f\left(\frac{c_{55}}{|c_{13} + c_{55}|}\right) > 0.
\]
(2.29)

To this end, we first note that (2.29) is trivially satisfied when \(c_{11} + c_{12} \geq 0\). So, in what follows, we shall assume \(c_{11} + c_{12} < 0\), that is
\[-2c_{66} < c_{11} + c_{12} < 0.
\]
(2.30)
In view of relations (2.24), (2.28) and (2.30) we deduce
\[ f\left(\frac{c_{55}}{c_{13} + c_{55}}\right) \geq \frac{1}{2} c_{33} (c_{11} + c_{12}) + c_{33} \sigma_m. \] (2.31)

Furthermore, if \( c_{11} \leq c_{12} + 2c_{66} \) then we have \( \sigma_m = \frac{1}{2} (c_{11} - c_{12}) \) and hence (2.31) implies
\[ f\left(\frac{c_{55}}{c_{13} + c_{55}}\right) \geq c_{11} c_{33} > 0, \] (2.32)
while when \( c_{11} > c_{12} + 2c_{66} \), we have \( \sigma_m = c_{66} \) and (2.31) implies
\[ f\left(\frac{c_{55}}{c_{13} + c_{55}}\right) \geq \frac{1}{2} c_{33} (c_{11} + c_{12} + 2c_{66}) > 0, \] (2.33)
in view of the assumption (2.30). Thus, we can conclude that the relation (2.29) is always trivially satisfied. If \( \frac{1}{2} c_{33} (c_{11} + c_{12}) > (c_{11} + c_{55})^2 > 0, \) (2.34)
then \( f(y) \) represents a quadratic function having a minimum at \( y = y_m \), with
\[ y_m = -\frac{c_{55} | c_{13} + c_{55} |}{\frac{1}{2} c_{33} (c_{11} + c_{12}) - (c_{11} + c_{55})^2} < 0. \] (2.35)

In view of the relation (2.29) and by recalling that
\[ \frac{c_{55} | c_{13} + c_{55} |}{\frac{1}{2} c_{33} (c_{11} + c_{12}) - (c_{11} + c_{55})^2} < \frac{c_{55}}{|c_{13} + c_{55}|} < 1, \] (2.36)
it follows that (2.26) is satisfied when (2.34) holds true. If \( \frac{1}{2} c_{33} (c_{11} + c_{12}) - (c_{11} + c_{55})^2 = 0, \) (2.37)
then \( f(y) \) represents a non-decreasing linear function and the relation (2.26) is satisfied in view of the relation (2.29).

Finally, let us consider the case
\[ \frac{1}{2} c_{33} (c_{11} + c_{12}) - (c_{11} + c_{55})^2 < 0. \] (2.38)
Then \( f(y) \) represents a quadratic function having a maximum at \( y = y_M \), with
\[ y_M = \frac{c_{55} | c_{13} + c_{55} |}{(c_{13} + c_{55})^2 - \frac{1}{2} c_{33} (c_{11} + c_{12})} > 0. \] (2.39)
and, in view of the relation (2.29), it follows that (2.26) is satisfied if and only if
\[ f(1) > 0. \] (2.40)

Since
\[ f(1) = c_{33} \left[ \frac{1}{2} (c_{11} + c_{12}) + \sigma_m \right] - (|c_{13} + c_{55}| - c_{55})^2, \] (2.41)
we can conclude that the condition (2.26) is equivalent with
\[ (|c_{13} + c_{55}| - c_{55})^2 < \frac{1}{2} c_{33} (c_{11} + c_{12} + 2\sigma_m), \] (2.42)
provided the conditions (2.25) and (2.38) hold true. Summarizing, we can see that the relations (2.5), (2.9), (2.16), (2.17) and (2.42) lead to the relations (1.6)–(1.8).

Conversely, starting with relations (1.6)–(1.8), by means of relations (2.27)–(2.40), we can conclude that (2.22) holds true. Furthermore, this implies (2.14) and moreover, (2.12) and (2.4) hold true. Consequently, the strong ellipticity condition (2.1) is satisfied and the proof of Theorem 1 is complete.

3. Strong ellipticity for tetragonal system \( C_4 \) generated by \( \mathbb{R}^{3/2}_3 \) (seven elasticities)

In order to prove the Theorem 2 we have to note that, by setting \( n_i = -n_i \) and \( m_1 = -m_1 \) the inequality (1.5) becomes
\[
\begin{align*}
c_{11} (n_1^2 m_1^2 + n_2^2 m_2^2) + 2c_{12} n_1 m_1 n_2 m_2 + c_{66} (n_1^2 m_2^2 + n_2^2 m_1^2 + 2n_1 m_2 n_1 m_1) + c_{55} (n_1^2 m_1^2 + n_2^2 m_1^2 + n_2^2 m_2^2) \\
+ 2c_{13} (c_{35}/(n_1 m_1 + n_2 m_2) n_2 m_1 + c_{33} n_1^2 m_2^2 + 2c_{16} [n_1 m_2 (m_1^2 - m_2^2)] + m_1 m_2 (n_1^2 - n_2^2)) > 0.
\end{align*}
\]
for all non-zero vectors \( m = (m_1, m_2, m_3) \) and \( n = (n_1, n_2, n_3) \).
Further, we shall follow a procedure similar with that developed in the above section for establishing the restrictions placed upon the elastic coefficients of the tetragonal system in discussion by the strong ellipticity condition (3.1). Therefore, regarding (3.1) as a quadratic equation in \( n_3 \in \mathbb{R} \) we deduce

\[
c_{55}(m_1^2 + m_2^2) + c_{13}m_3^2 > 0 \quad \text{for all } \mathbf{m} = (m_1, m_2, m_3) \neq 0,
\]

\[
c_{55}(m_1^2 + n_2^2)m_3^2 + c_{11}(n_1^2m_1^2 + n_2^2m_2^2) + c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 \\
+ 2c_{16}[n_1n_2(m_1^2 - m_2^2) + m_1m_2(n_1^2 - n_2^2)] > 0,
\]

and

\[
(c_{13} + c_{55})^2(n_1 + n_2m_2)^2m_3^2 < [c_{55}(m_1^2 + m_2^2) + c_{13}m_3^2]\{(c_{55}(n_1^2 + m_2^2)n_3^2 + c_{11}(n_1^2m_1^2 + n_2^2m_2^2) \\
+ c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1m_1n_2m_2 \pm 2c_{16}[n_1n_2(m_1^2 - m_2^2) + m_1m_2(n_1^2 - n_2^2)]\},
\]

for all non-zero vectors \( \mathbf{m} = (m_1, m_2, m_3) \) and \( \mathbf{n} = (n_1, n_2, n_3) \).

As a direct consequence, from (3.2) we obtain

\[
c_{13} > 0, \quad c_{55} > 0,
\]

while the relation (3.3) implies

\[
(c_{11}m_1^2 + c_{66}m_2^2 \pm 2c_{16}m_1m_2)n_3^2 + 2[(c_{12} + c_{66})n_1m_2 \pm c_{16}(m_1^2 - m_2^2)]n_1n_2 + (c_{66}m_1^2 + c_{11}m_2^2 \pm 2c_{16}m_1m_2)n_3^2 > 0.
\]

By considering (3.6) as a quadratic in \( n_1 \in \mathbb{R} \), we deduce

\[
c_{11}m_1^2 + c_{66}m_2^2 \pm 2c_{16}m_1m_2 > 0,
\]

\[
c_{66}m_1^2 + c_{11}m_2^2 \pm 2c_{16}m_1m_2 > 0 \quad \text{for all } (m_1, m_2) \neq 0,
\]

and

\[
[(c_{12} + c_{66})m_1m_2 \pm c_{16}(m_1^2 - m_2^2)]^2 < (c_{11}m_1^2 + c_{66}m_2^2 \pm 2c_{16}m_1m_2)(c_{66}m_1^2 + c_{11}m_2^2 \pm 2c_{16}m_1m_2) \quad \text{for all } (m_1, m_2) \neq 0.
\]

Consequently, the relation (3.7) implies

\[
c_{11} > 0, \quad c_{66} > 0,
\]

\[
|c_{16}| < \sqrt{c_{11}c_{66}},
\]

and inequality (3.8) can be written as

\[
(c_{11}c_{66} - c_{16}^2)(m_1^2 - m_2^2)^2 \pm 2c_{16}c_{11}(c_{11} + c_{12})(m_1^2 - m_2^2)m_1m_2 + [(c_{11} + c_{66})^2 - 4c_{16}^2 - (c_{12} + c_{66})^2]m_1^2m_2^2 > 0.
\]

Furthermore, in view of the relation (3.10), we can rewrite (3.11) in the following form:

\[
\left[ \sqrt{c_{11}c_{66} - c_{16}^2}(m_1^2 - m_2^2) \pm \frac{c_{16}(c_{11} + c_{12})}{\sqrt{c_{11}c_{66} - c_{16}^2}}m_1m_2 \right]^2 + \left[ (c_{11} + c_{66})^2 - 4c_{16}^2 - (c_{12} + c_{66})^2 - \frac{c_{16}^2(c_{11} + c_{12})^2}{c_{11}c_{66} - c_{16}^2} \right]m_1^2m_2^2 > 0,
\]

for all \( (m_1, m_2) \neq 0 \).

Therefore, if we choose \( (m_1, m_2) \) so as to have

\[
(c_{11}c_{66} - c_{16}^2)(m_1^2 - m_2^2) \pm c_{16}(c_{11} + c_{12})m_1m_2 = 0,
\]

as it is possible, then we obtain the restriction

\[
(c_{11} + c_{66})^2 - 4c_{16}^2 - (c_{12} + c_{66})^2 - \frac{c_{16}^2(c_{11} + c_{12})^2}{c_{11}c_{66} - c_{16}^2} > 0.
\]

and hence we have

\[
c_{16}^2 \left[ 4 + \frac{(c_{11} + c_{12})^2}{c_{11}c_{66} - c_{16}^2} \right] < (c_{11} - c_{12})(c_{11} + c_{12} + 2c_{66}).
\]

For later convenience we note that (3.15) implies \( c_{11} - c_{12})(c_{11} + c_{12} + 2c_{66}) > 0 \) and hence, by recalling (3.9), we obtain

\[
-c_{11} - 2c_{66} < c_{12} < c_{11}.
\]

Moreover, we can write (3.15) in the following equivalent form

\[
g(\zeta) > 0,
\]
where
\[ g(\xi) \equiv 4\xi^2 - 2[c_{11}(c_{11} + c_{12}) + c_{66}(3c_{11} - c_{12})]\xi + c_{11}c_{66}(c_{11} - c_{12})(c_{11} + c_{12} + 2c_{66}), \]
and
\[ \xi = c_{16}^2. \]

Further, we observe that
\[ g(c_{11}c_{66}) = -c_{11}c_{66}(c_{11} + c_{12})^2 < 0, \]
and \( g(\xi) = 0 \) for
\[ \xi_1 = \frac{1}{2}c_{66}(c_{11} - c_{12}) > 0, \quad \xi_2 = \frac{1}{2}c_{11}(c_{11} + c_{12} + 2c_{66}) > 0. \]

Thus, in view of the relations (3.10) and (3.20), we can see that the inequality (3.17) is equivalent with
\[ c_{16}^2 < \frac{1}{2}\min(c_{66}(c_{11} - c_{12}), \ c_{11}(c_{11} + c_{12} + 2c_{66})). \]

Concluding, we can note that the relation (3.14) is equivalent with (3.22). Obviously, we have
\[ \frac{1}{2}\min(c_{66}(c_{11} - c_{12}), \ c_{11}(c_{11} + c_{12} + 2c_{66})) \leq c_{11}c_{66}. \]

Let us now consider the relation (3.4) and note that it can be written as
\[
\begin{align*}
c_{33}c_{55}(n_1^2 + n_2^2)m_3^4 + (c_{35}^2(m_1^2 + m_2^2)(n_1^2 + n_2^2) + c_{11}c_{33}n_1^2m_3^4 + n_2^2m_3^4) + c_{33}c_{66}(n_1^2m_3^2 + n_2^2m_3^2) \\
+ 2c_{33}(c_{12} + c_{66})m_1n_1n_2m_2 \pm 2c_{33}c_{16}[n_1n_2(n_1^2 - m_2^2) + m_1m_2(n_1^2 - n_2^2)] - (c_{13} + c_{55})^2(n_1m_1 + n_2m_2)^2m_3^2 \\
+ c_{55}(n_1^2 + m_2^2)(c_{11}(n_1m_1^2 + n_2m_2^2) + c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1n_2n_1n_2m_2 \pm 2c_{16}[n_1n_2(m_1^2 - m_2^2) \\
+ m_1m_2(n_1^2 - n_2^2)]) > 0,
\end{align*}
\]
for all \( m_3 \in \mathbb{R} \) and for all non-zero vectors \((m_1, m_2)\) and \((n_1, n_2)\). Therefore, if we set
\[
m_3^2 = \frac{1}{\sqrt{c_{33}c_{55}}}
\sqrt{\frac{m_1^4 + m_2^4}{n_1^4 + n_2^4}(c_{11}(n_1m_1^2 + n_2m_2^2) + c_{66}(n_1^2m_2^2 + n_2^2m_1^2) + 2(c_{12} + c_{66})n_1n_2n_1n_2m_2 \pm 2c_{16}[n_1n_2(m_1^2 - m_2^2) \\
+ m_1m_2(n_1^2 - n_2^2)])^{1/2},
\]
into relation (3.24), we deduce
\[
\begin{align*}
&\ |c_{13} + c_{55}|n_1m_1 + n_2m_2| < \sqrt{55}\sqrt{(m_1^2 + m_2^2)(n_1^2 + n_2^2)} + \sqrt{c_{33}}\{c_{11}(n_1m_1 - n_2m_2)^2 + c_{66}(n_1m_2 + n_2m_1)^2 \\
- 2\ |c_{16}(n_1m_1 - n_2m_2)(n_1m_2 + n_2m_1)| \} + 2|c_{12} + c_{66}|n_1m_1 + n_2m_2)^{1/2},
\end{align*}
\]
for all non-zero vectors \((m_1, m_2)\) and \((n_1, n_2)\).

For later convenience we note that the relation (3.26), with \( n_1 = \lambda m_1, \ \lambda \in \mathbb{R} \) and \( n_2 = m_2 = 0 \), furnishes
\[ -2c_{55} - \sqrt{c_{11}c_{33}} < c_{13} < \sqrt{c_{11}c_{33}}. \]

At this instant we write the inequality (3.26) as
\[
\begin{align*}
&\ |c_{13} + c_{55}|n_1m_1 + n_2m_2| < \sqrt{c_{33}}\{\frac{1}{2}(c_{11} - c_{12})(n_1m_1 - n_2m_2)^2 + c_{66}(n_1m_2 + n_2m_1)^2 \\
&+ \sqrt{c_{33}}\{\{c_{11}(n_1m_1 - n_2m_2)^2 + c_{66}(n_1m_2 + n_2m_1)^2 \} - 2\ |c_{16}(n_1m_1 - n_2m_2)(n_1m_2 + n_2m_1)| \\
&+ \frac{1}{2}(c_{11} + c_{12})(n_1m_1 + n_2m_2)^2\}^{1/2}.
\end{align*}
\]

On the other hand, we have
\[
\omega_m[(n_1m_1 - n_2m_2)^2 + (n_1m_2 + n_2m_1)^2] \leq \frac{1}{2}(c_{11} - c_{12})(n_1m_1 - n_2m_2)^2 + c_{66}(n_1m_2 + n_2m_1)^2 \\
- 2\ |c_{16}(n_1m_1 - n_2m_2)(n_1m_2 + n_2m_1)| \\
\leq \omega_{m_m}(n_1m_1 - n_2m_2)^2 + (n_1m_2 + n_2m_1)^2,
\]
where
\[
\omega_m = \frac{1}{4}\left[ c_{11} - c_{12} + 2c_{66} - \sqrt{(c_{11} - c_{12} - 2c_{66})^2 + 16c_{16}^2} \right],
\]
\[
\omega_{m_m} = \frac{1}{4}\left[ c_{11} - c_{12} + 2c_{66} + \sqrt{(c_{11} - c_{12} - 2c_{66})^2 + 16c_{16}^2} \right].
\]
Therefore, the inequality (3.28) is equivalent with

\[ |c_{13} + c_{55}| \frac{|n_1 m_1 + n_2 m_2|}{\sqrt{(m_1^2 + m_2^2)(n_1^2 + n_2^2)}} < c_{55} + \sqrt{c_{33}} \left[ \omega_m + \frac{1}{2} (c_{11} + c_{12}) \left( \frac{(n_1 m_1 + n_2 m_2)^2}{(m_1^2 + m_2^2)(n_1^2 + n_2^2)} \right) \right]^{1/2}. \]  

(3.32)

Such inequality can be treated in a manner similar with that used in the above section for the inequality (2.19). Concluding, we find that (3.26) is equivalent with

\[ (|c_{13} + c_{55}| - c_{55})^2 + \frac{1}{4} c_{33} \left( c_{11} - c_{12} - 2c_{66} \right)^2 + 16c_{16}^2 < \frac{1}{4} c_{33} (3c_{11} + c_{12} + 2c_{66}), \]  

(3.33)

provided

\[ |c_{13} + c_{55}| \geq c_{55}, \quad 0 \leq c_{11} + c_{12} < \frac{2}{c_{33}} (c_{13} + c_{55})^2. \]  

(3.34)

Thus, by means of the relations (3.5), (3.9), (3.10), (3.22), (3.27), (3.33) and (3.34), we can conclude that the relations (1.9)–(1.12) hold true.

Conversely, we start with the relations (1.9)–(1.12). Then relation (1.12) implies that (3.32) holds true and hence 3.26 is true. On this basis we deduce that (3.24), and hence (3.4), are true. Furthermore, by means of (1.9)–(1.11) we can conclude that (3.2) and (3.3) are true. Consequently, relation (3.1) is true and so the proof of Theorem 2 is complete.

4. Reduction to the tetragonal system with six elasticities

It is well known that, by means of a coordinate transformation, any elastic tensor of tetragonal symmetry with seven elasticities can be always reduced to an elastic tensor of tetragonal symmetry with six independent elasticities. With this idea in mind we proceed in what follows to establish Theorem 2 by using the results described in Theorem 1. To this end we consider a rotation \( \mathbf{R}_e^\theta \) of the Cartesian system and note that the system of non-zero components of the elasticity tensor with respect to this new Cartesian system are as follows:

\[ \begin{align*}
    c'_{11} &= c'_{22} = \frac{1}{4} [3c_{11} + c_{12} + 2c_{66} + (c_{11} - c_{12} - 2c_{66}) \cos 4\theta + 4c_{16} \sin 4\theta],
    
    c'_{12} &= \frac{1}{4} [c_{11} + 3c_{12} - 2c_{66} - (c_{11} - c_{12} - 2c_{66}) \cos 4\theta - 4c_{16} \sin 4\theta],
    
    c'_{66} &= \frac{1}{4} [c_{11} - c_{12} + 2c_{66} - (c_{11} - c_{12} - 2c_{66}) \cos 4\theta - 4c_{16} \sin 4\theta],
    
    c'_{16} &= -c'_{26} = \frac{1}{4} [-(c_{11} - c_{12} - 2c_{66}) \sin 4\theta + 4c_{16} \cos 4\theta],
    
    c'_{13} &= c'_{23} = c_{13}, \quad c'_{33} = c_{33}, \quad c'_{44} = c'_{55} = c_{55}.
\end{align*} \]  

(4.1)

In view of the fact that we assume \( c_{16} \neq 0 \), we can set

\[ c_{11} - c_{12} - 2c_{66} = \varrho \cos \varphi, \quad 4c_{16} = \varrho \sin \varphi, \]  

(4.2)

where

\[ \varrho = \sqrt{(c_{11} - c_{12} - 2c_{66})^2 + 16c_{16}^2}. \]  

(4.3)

Then we have

\[ \begin{align*}
    c'_{11} &= c'_{22} = \frac{1}{4} [3c_{11} + c_{12} + 2c_{66} + \varrho \cos (4\theta - \varphi)],
    
    c'_{66} &= \frac{1}{4} [c_{11} - c_{12} + 2c_{66} - \varrho \cos (4\theta - \varphi)], \quad c'_{12} = \frac{1}{4} [c_{11} + 3c_{12} - 2c_{66} - \varrho \cos (4\theta - \varphi)],
    
    c'_{16} &= -c'_{26} = -\frac{1}{4} \varrho \sin (4\theta - \varphi).
\end{align*} \]  

(4.4)

(4.5)

Now, we choose the angle of rotation \( \theta \) in such a way that \( c'_{16} = 0 \), that is we have only six elastic coefficients and so we can reduce the problem of characterization of the strong ellipticity for the class with seven coefficients to that with six coefficients. This means we have

\[ 4\theta - \varphi = \kappa \pi, \quad \kappa \in \{0, 1\}. \]  

(4.6)
and hence we obtain
\[ c'_{11} = c'_{22} = \frac{1}{4} (3c_{11} + c_{12} + 2c_{66} + \varrho), \]
\[ c'_{66} = \frac{1}{4} (c_{11} - c_{12} + 2c_{66} - \varrho), \quad c'_{12} = \frac{1}{4} (c_{11} + 3c_{12} - 2c_{66} - \varrho), \]
\[ c'_{16} = 0, \quad c'_{13} = c'_{23} = c_{13}, \quad c'_{33} = c_{33}, \quad c'_{44} = c'_{55} = c_{55}. \]  
(4.7)

for \( \kappa = 0 \) and
\[ c'_{11} = c'_{22} = \frac{1}{4} (3c_{11} + c_{12} + 2c_{66} - \varrho), \]
\[ c'_{66} = \frac{1}{4} (c_{11} - c_{12} + 2c_{66} + \varrho), \quad c'_{12} = \frac{1}{4} (c_{11} + 3c_{12} - 2c_{66} + \varrho), \]
\[ c'_{16} = 0, \quad c'_{13} = c'_{23} = c_{13}, \quad c'_{33} = c_{33}, \quad c'_{44} = c'_{55} = c_{55}. \]  
(4.8)

for \( \kappa = 1 \).

In view of \textit{Theorem 1}, the strong ellipticity is characterized by the conditions
\[ c'_{11} > 0, \quad c'_{33} > 0, \quad c'_{55} > 0, \quad c'_{66} > 0, \]  
(4.9)

\[ -2c'_{66} - c'_{11} < c'_{12} < c_{11}, \]  
(4.10)

\[ -2c'_{55} - \frac{1}{2} c_{11} c'_{33} < c'_{13} < \sqrt{c_{11} c'_{33}}, \]  
(4.11)

and, for \( (c_{13} + c_{55})^2 > \frac{1}{2} \max (2c_{55}, c_{33}(c'_{11} + c'_{12})) \)
\[ (|c'_{13} + c_{55}| - c_{55})^2 < \frac{1}{2} c_{33}(c'_{11} + c'_{12} + \min (c'_{11} - c'_{12}, 2c_{66})). \]  
(4.12)

In what follows we will prove \textit{Theorem 2} by means of the results expressed by \textit{Theorem 1}.

To this end, we first assume the relation (4.7) holds true. Then we note that a direct substitution of the relation (4.7) into (4.12) leads to the relation (1.12) of \textit{Theorem 2}. Furthermore, the substitution of (4.7) into (4.9)–(4.11) read as
\[ c_{33} > 0, \quad c_{55} > 0, \]
\[ 3c_{11} + c_{12} + 2c_{66} + \varrho > 0, \]  
(4.13)
\[ c_{11} - c_{12} + 2c_{66} > \varrho > 0, \]  
(4.14)

\[ c_{11} - c'_{12} = \frac{1}{2} (c_{11} - c_{12} + 2c_{66} + \varrho) > 0, \]  
(4.16)
\[ c_{11} + c_{12} + 2c_{66} = \frac{1}{2} (3c_{11} + c_{12} + 2c_{66} - \varrho) > 0, \]  
(4.17)

\[ -2c_{55} - \frac{1}{2} c_{11} c_{33} < \frac{1}{2} c_{33}(3c_{11} + c_{12} + 2c_{66} + \varrho), \]  
(4.18)
\[ -2c_{55} - \sqrt{\frac{1}{2} c_{11} (3c_{11} + c_{12} + 2c_{66} + \varrho)} < c_{13} < -2c_{55}. \]  
(4.19)

At this stage we have to outline that the relations (4.14) and (4.16) are always satisfied on the basis of the relations (4.15) and (4.17). Moreover, the relations (4.15) and (4.17) become
\[ c'_{16} = \frac{1}{2} c_{66}(c_{11} - c_{12}), \]  
(4.20)
\[ c'_{16} = \frac{1}{2} c_{11}(c_{11} + c_{12} + 2c_{66}). \]  
(4.21)

and hence the relation (1.10) is satisfied. Since \( c_{11} - c_{12} + 2c_{66} > 0 \), the relation (4.20) implies
\[ c_{66} > 0, \]  
(4.22)
\[ c_{11} > c_{12}. \]  
(4.23)

In the same manner, the relation (4.21) furnishes
\[ c_{11} > 0, \]  
(4.24)
\[ c_{11} + c_{12} + 2c_{66} > 0. \]  
(4.25)

The relations (4.18) and (4.19) are identically satisfied in view of the relation (1.12).

Thus, summarizing these results we can see that relations (4.9)–(4.12) lead to the necessary and sufficient conditions (1.9)–(1.12) established in \textit{Theorem 2} and characterizing the strong ellipticity of the elastic tensor in the tetragonal system with seven elasticities.

A similar discussion can be formulated for the case \( \kappa = 1 \).
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