On Saint-Venant’s principle in a poroelastic arch-like region

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In this paper we consider the state of plane strain in an elastic material with voids occupying a curvilinear strip as an arch-like region described by $R: a < r < b, \theta < \omega$, where $r$ and $\theta$ are polar coordinates and $a$, $b$, and $\omega (-2\pi)$ are prescribed positive constants. Such a curvilinear strip is maintained in equilibrium under self-equilibrated traction and equilibrated force applied on one of the edges, whereas the other three edges are traction free and subjected to zero volumetric fraction or zero equilibrated force. In fact, we study the case when one right or curved edge is loaded. Our aim is to derive some explicit spatial estimates describing how some appropriate measures of a specific Airy stress function and volume fraction evolve with respect to the distance to the loaded edge. The results of the present paper prove how the spatial decay rate varies with the constitutive profile. For the problem corresponding to a loaded right edge, we are able to establish an exponential decay estimate with respect to the angle $\theta$. Whereas for the problem corresponding to a loaded curved edge, we establish an algebraical spatial decay with respect to the polar distance $r$, provided the angle $\omega$ is lower than the critical value $\pi\sqrt{2}$.

The intended applications of these results concern various branches of medicine as for example the bone implants. Copyright © 2010 John Wiley & Sons, Ltd.

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1. Introduction

The basic idea of including voids in a continuous body is due to Goodman and Cowin [1], although they developed constitutive theory appropriate to a fluid. Nunziato and Cowin [2, 3] have established a theory of elastic materials with voids for which the bulk density is written as the product of two fields, the density field of the matrix material and the volume fraction field.

The intended applications of the theory are to elastic bodies with small voids or vacuous pores which are distributed throughout material. A further important application area for elastic materials with voids is in the production of building materials like bricks. Modern buildings are usually made with lighter, thinner bricks, often with many voids in the building materials. In seismic areas lighter materials are necessary and much applied research activity is taking place.

The theory of elastic materials with voids was explored in various papers (see, for example Cowin and Puri [4], Cowin [5], Ieşan [6], Straughan [7]). In the linear theory of isotropic elastic materials with voids, the deformation of right cylinders has been the subject of various investigations. Cowin and Nunziato [3] have established the solution of the pure bending of a cylinder made of a homogeneous and isotropic elastic materials with voids. Saint-Venant’s principle in the theory of elastic materials with voids has been studied by Batra and Yang [8]. A general treatment of Saint-Venant’s problem for homogeneous and isotropic porous elastic cylinders has been presented by Dell’Isola and Batra [9], while Ghiba [10] considers the case of anisotropic materials with voids.

On the other hand, estimates reflecting Saint-Venant’s principle for the biharmonic equation have been the subject of considerable interest starting with the basic papers by Knowles [11, 12] and Flavin [13]. Further results on the spatial decay of biharmonic functions were derived by Horgan [14], Vafeades and Horgan [15], and Payne and Schaefer [16]. A comprehensive survey on the Saint-Venant’s principle is given by Horgan and Knowles [17], followed by two updates by Horgan [18, 19].

The spatial behavior of solutions of the biharmonic equation for a homogeneous isotropic elastic material occupying an arch-like region was recently studied by Flavin [20], Flavin and Gleeson [21], Chiriţă [22], and D’Apice and Chiriţă [23, 24].
It is worth to recall that Wu [25] has investigated one problem of this kind for Laplace’s equation in two dimensions (with vanishing normal derivative at \( \theta = 0, \theta = \omega, \) and \( r = b) \). It was shown that the energy stored beyond the radius \( r \) satisfies an algebraic spatial decay estimate similar to the one established in the present paper. Horgan and Payne [26] extended the results on bounded (or unbounded) regions of arbitrary shape. For unbounded wedge-shaped domains, the decay is shown to be of power law type rather than exponential. Issues related to those just described have been established in linear elasticity for cone-like domains by Knops et al. [27].

This paper considers a curvilinear strip in the form of an arch-like region \( R \) described in the polar coordinates \( r \) and \( \theta \) by \( R: a < r < b, 0 < \theta < \omega, \) where \( a, b, \) and \( \omega (< 2\pi) \) are prescribed positive constants. The curvilinear strip is made of an isotropic poroelastic material in an equilibrium state of plane strain under self-equilibrated traction and equilibrated force applied on one of the edges, whereas the other three edges are traction free and subjected to zero volumetric fraction or zero equilibrated force. The formulation of the problem is given in terms of a specific Airy stress function and of the volumetric fraction, that is a differential system of fourth order for the Airy stress function and second order for the volumetric fraction in terms of the polar coordinates with specific boundary conditions. By an appropriate change of independent variables and change of dependent functions, we reduce the problem to a more tractable differential system. Then we are able to define appropriate measures associated to the specific Airy stress function and to the volumetric fraction which satisfy some differential inequalities which when integrated give the description how they evolve with the polar distance and the polar angle, respectively. In fact, when a right edge is loaded we are able to provide an exponential decay with respect to the distance to the loaded edge. Whereas for a curved loaded edge we establish an algebraical decay estimate of edge effects, provided the angle of the arch-like region is assumed to be lower than the critical value \( \pi \sqrt{2} \).

The intended applications of the present results concern the manufacture of building materials and in medicine for bone implants. It is well known that the cancellous bone is considered as a porous system.

2. Formulation of the problems

We consider a curvilinear strip of the form of an arch-like region \( R \), which in polar coordinates \( r \) and \( \theta \) is described by \( R: a < r < b, 0 < \theta < \omega, \) where \( a, b, \) and \( \omega (< 2\pi) \) are prescribed positive constants. The curvilinear strip is made of a homogeneous and isotropic elastic material with voids and is subject to zero body force and zero equilibrated force. One edge is subject to a prescribed self-equilibrated traction and an equilibrated force, whereas the other three edges are traction free and subject to zero volume fraction or zero equilibrated force.

If \( u_r \) and \( u_\theta \) are the radial and transversal components of the plane displacement vector in a plane polar reference frame, then the geometrical measures of deformation are

\[
e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{r\theta} = \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} \right),
\]

\[
e_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right)
\]

By eliminating \( u_r \) and \( u_\theta \) in the above relation, we obtain the Saint-Venant compatibility condition in the form

\[
r \frac{\partial^2}{\partial r^2} (re_{\theta\theta}) + \left( \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \theta^2} \right) e_{rr} - 2 \frac{\partial^2}{\partial r \partial \theta} (re_{r\theta}) = 0
\]

The constitutive equations for a plane strain state \((u_r, u_\theta, \varphi)\), with \( \varphi(r, \theta) \)—the volume fraction field, in an isotropic and homogeneous elastic body are

\[
\tau_{rr} = (\lambda + 2\mu)e_{rr} + \lambda e_{\theta\theta} + \beta \varphi
\]

\[
\tau_{r\theta} = \lambda e_{rr} + (\lambda + 2\mu)e_{\theta\theta} + \beta \varphi
\]

\[
\tau_{\theta\theta} = 2\mu e_{\theta\theta}
\]

\[
h_r = a \frac{\partial \varphi}{\partial r}
\]

\[
h_\theta = \frac{a}{r} \frac{\partial \varphi}{\partial \theta}
\]

\[
g = -\beta (e_{rr} + e_{\theta\theta}) - \zeta \varphi
\]

where \( \tau_{rr}, \tau_{r\theta}, \tau_{\theta\theta} \) are the components of the plane stress in plane polars, \( h_r \) and \( h_\theta \) are the components of the equilibrated stress vector in plane polars, \( g \) is the intrinsic-equilibrated body force, and \( \lambda, \mu, \alpha, \beta, \) and \( \zeta \) are constant constitutive coefficients. Throughout this paper, we suppose that the elastic moduli satisfy the restrictions imposed by the fact that the internal energy is a positive definite quadratic form (see, for example \([3, 5, 6]\), i.e.

\[
\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \alpha > 0, \quad \zeta > 0, \quad (3\lambda + 2\mu) > 3\beta^2
\]
The relations in (3) can be solved in terms of the geometrical measures \( e_{rr}, e_{\theta\theta}, \text{and } e_{r\theta} \), to obtain

\[
e_{rr} = \epsilon \tau_{rr} - \epsilon \tau_{\theta\theta} - \eta \phi
\]
\[
e_{\theta\theta} = -\epsilon \tau_{rr} + \epsilon \tau_{\theta\theta} - \eta \phi
\]
\[
e_{r\theta} = (\epsilon + \mu) \tau_{r\theta}
\]

where

\[
\epsilon = \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)}, \quad \eta = \frac{\beta}{2(\lambda + \mu)}
\]

As a consequence of the restriction (5), we have

\[
\epsilon > 0, \quad \zeta - 2\beta \eta > 0
\]

The equilibrium equations corresponding to a plane strain state \((u_r, u_\theta, \phi)\) reduce to

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0
\]

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r} \tau_{rr} = 0
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{r\theta}) + \frac{1}{r} \tau_{\theta\theta} + g = 0
\]

The equilibrium equations (9) are identically satisfied when the state of plane stress is represented in terms of the Airy stress function \( \phi \) by

\[
\tau_{rr} = 1 \frac{\partial^2 \phi}{\partial r^2} + 1 \frac{\partial \phi}{\partial r}
\]

\[
\tau_{\theta\theta} = \frac{\partial^2 \phi}{\partial \theta^2}
\]

\[
\tau_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)
\]

If we substitute relation (11) into (6) and the result in relation (2), then we get the following equation for the Airy stress function and the volume fraction:

\[
\epsilon \left[ \frac{r^2}{r^2} \left( \frac{\partial^2 \phi}{\partial r^2} \right) + 2r \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} \right] + \frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + 4 \frac{\partial^2 \phi}{\partial \theta^2} - \eta^2 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \phi}{\partial r} \right) = 0
\]

On the other hand, by substituting relations (4), (6), and (11) into equilibrium equation (10), we obtain

\[
x \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \phi}{\partial r} \right) - \eta \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \phi}{\partial r} \right) - (\xi - 2\beta \eta) \phi = 0
\]

In conclusion, we can see that the basic equations for the plane strain state of the curvilinear strip are equivalent with the differential system (12) and (13) for the Airy stress function \( \phi \) and the volume fraction \( \phi \). Concerning the above differential system we will consider two different boundary value problems.

**Problem A.** Here we consider the edge \( \theta = 0 \) to be subjected to a given traction, whereas the other three edges are free of tractions, that is we have the following boundary conditions:

\[
\phi(a, \theta) = \frac{\partial \phi}{\partial r}(a, \theta) = 0, \quad \phi(b, \theta) = \frac{\partial \phi}{\partial r}(b, \theta) = 0, \quad \theta \in [0, \omega]
\]

\[
\phi(r, \omega) = \frac{\partial \phi}{\partial \theta}(r, \omega) = 0, \quad r \in [a, b]
\]

\[
\phi(r, 0) = \int_a^r (r - \xi) \tau_{\theta\theta}(q, 0) dq_0 \quad \frac{1}{r} \frac{\partial \phi}{\partial \theta}(r, 0) = -\int_a^r \tau_{r\theta}(q, 0) dq_0
\]

where the applied self-equilibrated tractions \( \tau_{r\theta}(\rho, 0) \) and \( \tau_{\theta\theta}(\rho, 0) \) satisfy the following conditions of global equilibrium:

\[
\int_a^b \tau_{r\theta}(q, 0) dq_0 = 0, \quad \int_a^b \tau_{\theta\theta}(q, 0) dq_0 = 0, \quad \int_a^b (b - q) \tau_{r\theta}(q, 0) dq_0 = 0
\]
As regards the volumetric fraction we will consider either
\[ \varphi(a, \theta) = \varphi(b, \theta) = 0, \quad \theta \in [0, \omega] \]  
\[ \varphi(r, \omega) = 0, \quad r \in [a, b] \]  
\[ \varphi(r, 0) = f_0(r), \quad r \in [a, b] \]  
(18)

or
\[ \frac{\partial \varphi}{\partial r}(a, \theta) = \frac{\partial \varphi}{\partial r}(b, \theta) = 0, \quad \theta \in [0, \omega] \]  
\[ \varphi(r, \omega) = 0, \quad r \in [a, b] \]  
\[ \frac{\partial \varphi}{\partial \theta}(r, 0) = f_1(r), \quad r \in [a, b] \]  
(21)

where \( f_0(r) \) and \( f_1(r) \) are prescribed functions.

**Problem B.** Here we consider the edge \( r = a \) to be subjected to a prescribed traction, whereas the other three edges are free of loads, that is we associate with the differential system (12), (13) the following boundary conditions:

\[ \mathcal{A}(r, 0) = \frac{\partial \varphi}{\partial r}(r, 0) = 0, \quad \mathcal{A}(r, \omega) = \frac{\partial \varphi}{\partial \theta}(r, \omega) = 0, \quad r \in [a, b] \]  
(22)

\[ \mathcal{A}(b, \theta) = \frac{\partial \varphi}{\partial r}(b, \theta) = 0, \quad \theta \in [0, \omega] \]  
(23)

\[ \mathcal{A}(a, \theta) = a^2 \int_0^\theta \left[ \tau_{rr}(a, s) + \int_0^s \tau_{r\theta}(a, \sigma) d\sigma \right] \sin(\theta - s) ds \]  
\[ + \int_0^\theta \int_0^s \tau_{r\theta}(a, \sigma) d\sigma ds - a \int_0^\theta \tau_{r\theta}(a, \sigma) d\sigma \]  
(24)

where the applied self-equilibrated tractions \( \tau_{rr}(a, \sigma) \) and \( \tau_{\theta\theta}(a, \sigma) \) satisfy the global equilibrium conditions, that is

\[ \int_0^\omega \left[ \tau_{rr}(a, s) + \int_0^s \tau_{r\theta}(a, \sigma) d\sigma \right] \sin(\theta - s) ds = 0, \quad \int_0^\omega \tau_{r\theta}(a, \sigma) d\sigma = 0 \]  
(25)

As regards the volumetric fraction we will consider either
\[ \varphi(r, 0) = \varphi(r, \omega) = 0, \quad r \in [a, b] \]  
\[ \varphi(b, \theta) = 0, \quad \theta \in [0, \omega] \]  
(26)

or
\[ \frac{\partial \varphi}{\partial r}(a, \theta) = \frac{\partial \varphi}{\partial \theta}(a, \theta) = 0, \quad \theta \in [0, \omega] \]  
\[ \varphi(b, \theta) = 0, \quad \theta \in [0, \omega] \]  
(28)

where \( g_0(\theta) \) and \( g_1(\theta) \) are prescribed functions.

### 3. The transformed problems

We proceed now to transform the above boundary value problems to obtain some more tractable forms. To this end, we will use the following change of variable
\[ r = e^t \]  
(30)

and the following change of functions
\[ \mathcal{A}(r, \theta) = e^t \psi(t, \theta), \quad \varphi(r, \theta) = \phi(t, \theta) \]  
(31)
Then the differential equations (12) and (13) become
\[
\varepsilon \left( \frac{\partial^4 \psi}{\partial t^4} + 2 \frac{\partial^2 \psi}{\partial t^2 \partial \theta^2} + \frac{\partial^4 \psi}{\partial \theta^4} - 2 \frac{\partial^2 \psi}{\partial t \partial \theta^2} + 2 \frac{\partial^2 \psi}{\partial \theta^2 \partial \phi} + \psi \right) - \eta e^t \left( \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0
\]
(32)
\[
\alpha \left( \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \psi}{\partial \theta^2} \right) - \eta e^t \left( \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \phi}{\partial \theta^2} + 2 \frac{\partial \psi}{\partial t} + \psi \right) - (\varepsilon - 2 \beta \eta)e^{2t} \phi = 0
\]
(33)
Further, the problems A and B become as follows:

**Problem A** consists of the partial differential system described in (32) and (33) and the boundary value conditions
\[
\psi(a_1, \theta) = \frac{\partial \psi}{\partial \theta} (a_1, \theta) = 0, \quad \psi(b_1, \theta) = \frac{\partial \psi}{\partial \theta} (b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
(34)
\[
\psi(t, \omega) = \frac{\partial \psi}{\partial \theta} (t, \omega) = 0, \quad t \in [a_1, b_1]
\]
(35)
\[
\psi(t, 0) = \int_{a_1}^{t} (1 - e^{t-s})\tau_{10}(e^s, 0) e^s ds, \quad \frac{\partial \psi}{\partial \theta} (t, 0) = - \int_{a_1}^{t} \tau_{10}(e^s, 0) e^s ds
\]
(36)
and
\[
\phi(a_1, \theta) = \phi(b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
\[
\phi(t, \omega) = 0, \quad t \in [a_1, b_1]
\]
(37)
\[
\phi(t, 0) = f_0(e^t), \quad t \in [a_1, b_1]
\]
(38)

or
\[
\frac{\partial \phi}{\partial t} (a_1, \theta) = \frac{\partial \phi}{\partial t} (b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
(39)
\[
\frac{\partial \phi}{\partial \theta} (t, \omega) = 0, \quad t \in [a_1, b_1]
\]
(40)

In the above relations, we have used the notation
\[
a_1 = \ln a, \quad b_1 = \ln b.
\]
(41)

**Problem B** consists of the partial differential system described in (32) and (33) and the boundary value conditions
\[
\psi(t, 0) = \frac{\partial \psi}{\partial \theta} (t, 0) = 0, \quad \psi(t, \omega) = \frac{\partial \psi}{\partial \theta} (t, \omega) = 0, \quad t \in [a_1, b_1]
\]
(42)
\[
\psi(b_1, \theta) = \frac{\partial \psi}{\partial \theta} (b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
(43)
\[
\psi(a_1, \theta) = a \int_0^\theta \left[ \tau_{1r}(a, s) + \int_0^s \tau_{10}(a, \tau) d\tau \right] ds
\]
\[
\frac{\partial \psi}{\partial \theta} (a_1, \theta) = -a \int_0^\theta \tau_{10}(a, s) ds, \quad \theta \in [0, \omega]
\]
(44)
and
\[
\phi(t, 0) = \phi(t, \omega) = 0, \quad t \in [a_1, b_1]
\]
\[
\phi(b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
(45)
\[
\phi(a_1, \theta) = g_0(\theta), \quad \theta \in [0, \omega]
\]
(46)

or
\[
\frac{\partial \phi}{\partial t} (t, 0) = \frac{\partial \phi}{\partial \theta} (t, \omega) = 0, \quad t \in [a_1, b_1]
\]
\[
\phi(t, \omega) = 0, \quad \theta \in [0, \omega]
\]
(47)
\[
\phi(b_1, \theta) = 0, \quad \theta \in [0, \omega]
\]
(48)
On the basis of the equations (32) and (33), we have the following useful identities:

\[
\epsilon \left[ \frac{\partial^2 \psi}{\partial t^2} \right]^2 + 2 \left( \frac{\partial^2 \psi}{\partial t \partial \theta} \right)^2 + 2 \left( \frac{\partial \psi}{\partial t} \right)^2 + \frac{2 (\epsilon \phi)}{\partial t} - 2 \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon ^2 \psi}{\partial \theta^2} + 2 \frac{\partial \psi}{\partial t} + \psi
\]

\[
= \frac{\partial^2}{\partial t \partial \theta} \left[ \epsilon \left( \frac{\partial \psi}{\partial t} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \psi^2 - \psi \frac{\partial^2 \psi}{\partial \theta^2} \right] + \eta \epsilon ^2 \phi \psi
\]

\[
+ \frac{\partial^2}{\partial \theta^2} \left[ \epsilon \left( \frac{\partial \psi}{\partial t} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \psi^2 - \psi \frac{\partial^2 \psi}{\partial \theta^2} \right] + \eta \epsilon ^2 \phi \psi
\]

\[
- 2 \frac{\partial^2}{\partial t \partial \theta} \left( \epsilon \frac{\partial^2 \psi}{\partial t^2 \partial \theta} - \frac{\partial \epsilon \phi}{\partial t} + \frac{\partial \psi}{\partial \theta} \right) - 2 \eta \epsilon ^2 \phi \psi
\]

\[
\psi \left[ a \left( \frac{\partial \phi}{\partial t} \right)^2 + a \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial t} + \psi \right] + (\epsilon - 2 \beta \eta) \epsilon ^2 \phi = \frac{\epsilon ^2}{\epsilon ^2} \left( \frac{\epsilon ^2}{\epsilon ^2} \phi \right)^2 + \epsilon ^2 \frac{\epsilon ^2}{\epsilon ^2} \phi
\]  

(50)

In what follows we will use these identities to define appropriate measures of the solutions \((\psi, \phi)\) of the above problems A and B.

4. Spatial behavior for problem A

Within the context of the problem A the above identities (49) and (50), when coupled with the boundary conditions (34), (37), and (39), give

\[
\int_{a_1}^{b_1} \left[ \epsilon \left( \frac{\partial \psi}{\partial t} \right)^2 + 2 \left( \frac{\partial \psi}{\partial t \partial \theta} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial t} - 2 \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial \theta} + \psi \right] + (\epsilon - 2 \beta \eta) \epsilon ^2 \phi \psi \right] dt = \frac{d^2}{d \theta^2} \int_{a_1}^{b_1} \left( \frac{\partial^2 \psi}{\partial t^2 \partial \theta} - \frac{\partial \epsilon \phi}{\partial t} + \frac{\partial \psi}{\partial \theta} \right) dt - \frac{d}{d \theta} \int_{a_1}^{b_1} 2 \eta \epsilon ^2 \phi \psi \frac{\partial \psi}{\partial \theta} dt
\]

(51)

On this basis we can introduce the following functional:

\[
E(\theta) = \int_{a_1}^{b_1} \left[ \epsilon \left( \frac{\partial \psi}{\partial t} \right)^2 + 2 \left( \frac{\partial \psi}{\partial t \partial \theta} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial t} - 2 \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial \theta} + \psi \right] + (\epsilon - 2 \beta \eta) \epsilon ^2 \phi \psi \right] dt
\]

(52)

and note that, by means of (51), we have

\[
E(\theta) = \frac{d^2}{d \theta^2} (\theta) \quad \text{for all } \theta \in [0, \omega]
\]

(53)

where

\[
I(\theta) = \int_{a_1}^{b_1} \left[ \epsilon \left( \frac{\partial \psi}{\partial t} \right)^2 + 2 \left( \frac{\partial \psi}{\partial t \partial \theta} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial t} - 2 \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{2 \epsilon \phi}{\partial \theta} + \psi \right] dt + \int_{R_{11}} 2 \eta \epsilon ^2 \phi \frac{\partial \psi}{\partial \theta} dt d\sigma
\]

(54)

and \(R_{11} = [a_1, b_1] \times [0, \omega]\). In view of relations (5) and (8), it follows that \(E(\theta)\) can be taken like a possible candidate for an acceptable measure of the couple \((\psi, \phi)\). In fact, since \(\frac{\partial \psi}{\partial t} \epsilon (a_1, \alpha) = \frac{\partial \psi}{\partial t} (b_1, \sigma) = 0\) for all \(\alpha \in [0, \omega]\), it follows that

\[
\int_{a_1}^{b_1} \left( \frac{\partial^2 \psi}{\partial t^2} \right)^2 dt \geq \frac{4 \pi^2}{(b_1 - a_1)^2} \int_{a_1}^{b_1} \left( \frac{\partial \psi}{\partial t} \right)^2 dt
\]

(55)

Moreover, by taking into account that \(\psi(a_1, \alpha) = \psi(b_1, \sigma) = 0\) for all \(\sigma \in [0, \omega]\), it follows that

\[
\int_{a_1}^{b_1} \left( \frac{\partial \psi}{\partial t} \right)^2 dt \geq \frac{\pi^2}{(b_1 - a_1)^2} \int_{a_1}^{b_1} \psi^2 dt
\]

(56)

and

\[
\int_{a_1}^{b_1} \left( \frac{\partial^2 \psi}{\partial t \partial \theta} \right)^2 dt \geq \frac{\pi^2}{(b_1 - a_1)^2} \int_{a_1}^{b_1} \left( \frac{\partial \psi}{\partial \theta} \right)^2 dt
\]

(57)
On this basis we deduce that

\[ E(\theta) \geq \int_{a_1}^{b_1} \left\{ \epsilon \left[ \frac{4\pi^2}{(b_1-a_1)^2} + 2 \left( \frac{\hat{e}_\psi}{\hat{c}} \right)^2 + \frac{2\pi^2}{(b_1-a_1)^2} \left( \frac{\hat{e}_\psi}{\hat{c}} \right)^2 + \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right] + \frac{\epsilon^2}{2 \sqrt{\epsilon}} \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right\} \, dt \quad (58) \]

and, moreover, we have

\[ E(\theta) \geq \int_{a_1}^{b_1} \left\{ \epsilon \left[ \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 + \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right] + (\xi - 2\beta \eta) e^{2\tau} \phi^2 \right\} \, dt \quad (59) \]

where

\[ v = \frac{2\pi^4}{(b_1-a_1)^4} + \frac{\pi^2}{(b_1-a_1)^2} + 1 - \sqrt{\frac{2\pi^4}{(b_1-a_1)^4} + \frac{\pi^2}{(b_1-a_1)^2}} + 1 \quad (60) \]

From (59) we can conclude that \( E(\theta) \) appears as an acceptable measure for the couple \((\psi, \phi)\). Consequently, we can introduce the following measure

\[ \mathcal{E}(\theta) = \int_{0}^{\omega} (\sigma - \theta) E(\sigma) \, d\sigma \quad \text{for} \ \theta \in [0, \omega] \quad (61) \]

and note that, by means of (53), we have

\[ \frac{d^2 \mathcal{E}}{d \theta^2}(\theta) = \frac{d^2 l}{d \theta^2}(\theta) \quad \text{for} \ \theta \in [0, \omega] \quad (62) \]

In view of relation (61), we have

\[ \frac{d \mathcal{E}}{d \theta}(\omega) = 0 \quad (63) \]

whereas relations (35), (37), (39), and (54) give

\[ \frac{dl}{d \theta}(\omega) = 0 \quad (64) \]

Thus, relation (62) gives

\[ \frac{d \mathcal{E}}{d \theta}(\theta) = \frac{dl}{d \theta}(\theta) \quad \text{for} \ \theta \in [0, \omega] \quad (65) \]

Further, relation (61) implies

\[ \mathcal{E}(\omega) = 0 \quad (66) \]

whereas (35), (37), (39), and (54) give

\[ l(\omega) = 0 \quad (67) \]

and hence (65) furnishes

\[ \mathcal{E}(\theta) = l(\theta) \geq 0 \quad \text{for} \ \theta \in [0, \omega] \quad (68) \]

On the other hand, by means of the Cauchy–Schwarz and the arithmetic–geometric mean inequalities, from (54) and (68) we get

\[ \mathcal{E}(\theta) = l(\theta) \leq \int_{a_1}^{b_1} \left\{ \epsilon \left[ \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 + \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right] + \frac{\epsilon^2}{2 \sqrt{\epsilon}} \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right\} + \frac{|\eta|}{2 \sqrt{\epsilon}} \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 + (\xi - 2\beta \eta) e^{2\tau} \phi^2 \right\} \, dt \quad (58) \]

Further, relations (53), (58), (59), and (62) give

\[ \frac{d^2 \mathcal{E}}{d \theta^2}(\theta) = \frac{d^2 l}{d \theta^2}(\theta) = E(\theta) \geq \int_{a_1}^{b_1} \left\{ \epsilon \left[ \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 + \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right] + \frac{\epsilon^2}{2 \sqrt{\epsilon}} \left( \frac{\hat{e}^2 \psi}{\hat{c}^2 \phi} \right)^2 \right\} \, dt \quad (59) \]
Moreover, relation (58) implies
\[
- \frac{d \epsilon}{d \theta} (0) = \int_0^\theta E(\sigma) d\sigma \geq \int_0^\theta \left[ \frac{2 \epsilon \pi^2}{(b_1 - a_1)^2} \left( \frac{\partial \psi}{\partial \theta} \right)^2 + (\zeta - 2 \beta \eta) e^{2 \sigma^2} \right] d\sigma \quad (71)
\]

Now we combine relations (69), (70), and (71) to obtain the following second-order differential inequality:
\[
\epsilon'(\theta) \leq \gamma_1 \frac{d^2 \epsilon}{d \theta^2} (\theta) - \gamma_2 \frac{d \epsilon}{d \theta} (\theta) \quad \text{for all } \theta \in [0, \omega] \quad (72)
\]

where
\[
\gamma_1 = \max \left( \frac{(b_1 - a_1)^2}{\pi^2}, \frac{|\eta| \sqrt{\epsilon}}{r \sqrt{\zeta - 2 \beta \eta}}, \frac{\zeta}{2e^{2 \sigma_1 (\zeta - 2 \beta \eta)}}, \frac{|\eta|}{\sqrt{\epsilon}(\zeta - 2 \beta \eta)} \right)
\]
\[
\gamma_2 = \frac{|\eta|}{\sqrt{\epsilon}(\zeta - 2 \beta \eta)} \left( \frac{(b_1 - a_1)^2}{2\pi^2}, 1 \right)
\]

By the comparison principle, from (72) it follows that \( \epsilon'(\theta) \) is bounded above by \( G(\theta) \), the solution of the differential equation
\[
\frac{d^2 G}{d \theta^2} (\theta) - \gamma_2 \frac{d G}{d \theta} (\theta) - \frac{1}{\gamma_1} G(\theta) = 0 \quad \text{for all } \theta \in [0, \omega] \quad (74)
\]

with the boundary conditions
\[
G(0) = \epsilon(0), \quad G(\omega) = \epsilon(\omega) = 0 \quad (75)
\]

On this basis, we can write
\[
\epsilon'(\theta) \leq \frac{1 - e^{-(\kappa_1 + \kappa_2)(\omega - \theta)}}{1 - e^{-(\kappa_1 + \kappa_2)\omega}} \epsilon'(0) e^{-\kappa_2 \theta} + \frac{1 - e^{-(\kappa_1 + \kappa_2)\omega}}{1 - e^{-(\kappa_1 + \kappa_2)\omega}} \epsilon'(\omega) e^{-\kappa_1 (\omega - \theta)}
\]
\[
\leq \epsilon'(0) e^{-\kappa_1 \theta} + \epsilon'(\omega) e^{-\kappa_1 (\omega - \theta)} = \epsilon'(0) e^{-\kappa_2 \theta} \quad (76)
\]

where
\[
\kappa_1 = \frac{1}{2\gamma_1} \left( \gamma_2 + \sqrt{\gamma_2^2 + 4 \gamma_1} \right), \quad \kappa_2 = \frac{1}{2\gamma_1} \left( -\gamma_2 + \sqrt{\gamma_2^2 + 4 \gamma_1} \right) \quad (77)
\]

In conclusion, we have obtained the following spatial decay estimate:
\[
\epsilon'(\theta) \leq \epsilon'(0) e^{-\kappa_2 \theta} \quad \text{for all } \theta \in [0, \omega] \quad (78)
\]

5. Spatial behavior for problem B

Within the context of the problem B the identities (49) and (50), when coupled with the boundary conditions (42), (45), and (47), furnish
\[
\int_0^\omega \left\{ \epsilon \left[ \left( \frac{\partial^2 \psi}{\partial t^2} \right)^2 + (\frac{\partial \psi}{\partial t})^2 \right] + \left( \frac{\partial \psi}{\partial t} \right)^2 - 2 \left( \frac{\partial \psi}{\partial t} \right)^2 + \psi^2 \right\} d\theta + \frac{d^2}{dt^2} \int_0^\theta \epsilon \left[ \left( \frac{\partial \psi}{\partial t} \right)^2 + \left( \frac{\partial \psi}{\partial t} \right)^2 + \psi^2 \right] d\theta - \frac{d}{dt} \int_0^{\omega} \epsilon \left[ \left( \frac{\partial \psi}{\partial t} \right)^2 + \left( \frac{\partial \psi}{\partial t} \right)^2 + \psi^2 \right] d\theta \quad (79)
\]

Properties concerning the behavior of the solution \((\psi, \phi)\) are established now by means of a functional related to a line integral that enables a differential inequality to be derived. The functional to be considered is
\[
F(t) = \int_0^\omega \left\{ \epsilon \left[ \left( \frac{\partial^2 \psi}{\partial t^2} \right)^2 + (\frac{\partial \psi}{\partial t})^2 \right] + \left( \frac{\partial \psi}{\partial t} \right)^2 - 2 \left( \frac{\partial \psi}{\partial t} \right)^2 + \psi^2 \right\} d\theta + \frac{d}{dt} \int_0^{\omega} \epsilon \left[ \left( \frac{\partial \psi}{\partial t} \right)^2 + (\frac{\partial \psi}{\partial t})^2 \right] d\theta, \quad t \in [a_1, b_1] \quad (80)
\]
In view of the identity (79), we deduce that

\[ F(t) = \frac{d^2 F(t)}{dt^2}, \quad t \in [a_1, b_1] \tag{81} \]

where

\[ J(t) = \int_0^\omega \left\{ \varepsilon \left[ \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \psi^2 - \psi^2 \right] + \frac{\eta}{2} \phi \psi + \frac{\pi}{2} \phi^2 \right\} d\theta + \int_{R_t} 2\eta e^\phi \left( \frac{\partial \psi}{\partial \theta} + \psi \right) d\theta d\phi \tag{82} \]

and \( R_t = [a_1, b_1] \times [0, \omega] \).

We proceed now to study if we can take \( F(t) \) like a possible candidate for an acceptable measure of the couple \((\psi, \phi)\). To this end we note that the boundary conditions (42) allow us to write the inequality

\[ \int_0^\omega \left( \frac{\partial \psi}{\partial \theta} \right)^2 d\theta \geq \frac{4\pi^2}{\omega^2} \int_0^\omega \left( \frac{\partial \psi}{\partial \theta} \right)^2 d\theta \tag{83} \]

and, moreover,

\[ \int_0^\omega \left( \frac{\partial \phi}{\partial \theta} \right)^2 d\theta \geq \frac{\pi^2}{\omega^2} \int_0^\omega \left( \frac{\partial \phi}{\partial \theta} \right)^2 d\theta \tag{84} \]

\[ \int_0^\omega \left( \frac{\partial \psi}{\partial \theta} \right)^2 d\theta \geq \frac{\pi^2}{\omega^2} \int_0^\omega \phi^2 d\theta \tag{85} \]

On applying (83) into (80), we obtain

\[ F(t) \geq \int_0^\omega \left\{ \varepsilon \left[ \left( \frac{\partial^2 \psi}{\partial \theta^2} \right)^2 + 2 \left( \frac{\pi^2}{\omega^2} + 1 \right) \left( \frac{\partial \psi}{\partial \theta} \right)^2 + 2 \left( \frac{2\pi^2}{\omega^2} - 1 \right) \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \psi^2 \right] + 2 \left( \frac{2\pi^2}{\omega^2} - 1 \right) \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \psi^2 \right\} d\theta, \quad t \in [a_1, b_1] \tag{86} \]

Let us assume henceforward that

\[ 0 < \omega < \pi \sqrt{2} \tag{87} \]

In these circumstances, by means of relations (5), (8), and (87), from (86) it follows that \( F(t) \) represents a global measure of the couple \((\psi, \phi)\) in \( R_t \).

So in the remainder of this section, we assume that the angle \( \omega \) of the arch-like region is lower than \( \pi \sqrt{2} \) and then we set

\[ \frac{2\pi^2}{\omega^2} - 1 \equiv \tau > 0 \tag{88} \]

Consequently, relation (86) furnishes

\[ F(t) \geq \int_0^\omega \left\{ \varepsilon \left[ \left( \frac{\partial^2 \psi}{\partial \theta^2} \right)^2 + 2 \left( \frac{\pi^2}{\omega^2} + 1 \right) \left( \frac{\partial \psi}{\partial \theta} \right)^2 + 2 \tau \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \psi^2 \right] + 2 \tau \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \psi^2 \right\} d\theta, \quad t \in [a_1, b_1] \tag{89} \]

Furthermore, we introduce the following measure:

\[ \mathcal{F}(t) = \int_t^{b_1} (s - t) F(s) \, ds \tag{90} \]

and note that, by (81), we have

\[ \frac{d^2 \mathcal{F}(t)}{dt^2} = F(t) = \frac{d^2 F(t)}{dt^2} \quad \text{for all } t \in [a_1, b_1] \tag{91} \]

In view of relation (90), we have

\[ -\frac{d \mathcal{F}(t)}{dt} = \int_t^{b_1} F(s) \, ds \tag{92} \]
and hence
\[
\frac{d\mathcal{F}}{dt}(b_1) = 0, \quad \mathcal{F}(b_1) = 0 \tag{93}
\]

On the other hand, by relation (82) and by taking into account the boundary conditions (43), (45), and (47), we deduce that
\[
J(b_1) = \frac{dJ}{dt}(b_1) = 0 \tag{94}
\]

Consequently, relations (91), (93), and (94) imply that
\[
\mathcal{F}(t) = J(t) \quad \text{for all } t \in [a_1, b_1] \tag{95}
\]

Further, relations (89), (91), and (92) give
\[
\mathcal{F}(t) = J(t) \quad \text{for all } t \in [a_1, b_1]
\]

By applying the Cauchy–Schwarz and the arithmetic–geometric mean inequalities into relation (82) and by using relation (95), we get
\[
0 \leq \mathcal{F}(t) = J(t) \leq \int_0^\infty \left\{ \varepsilon \left[ \left( \frac{\partial \psi}{\partial t} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 + \frac{1}{2} \left( \frac{\partial \psi}{\partial t} \right)^2 \right] + \frac{3\varepsilon}{2} + \frac{|\sigma|}{\sqrt{\varepsilon - 2\beta\eta}} \psi^2 + \left( \frac{|\sigma|}{\sqrt{\varepsilon - 2\beta\eta}} + 2\beta a_1 \right) e^{2\varepsilon \phi} \left( \frac{\partial \psi}{\partial \theta} \right)^2 \right\} d\theta
\]

Finally, we combine relations (96), (97), and (98) to obtain the following second-order differential inequality:
\[
\mathcal{F}(t) \leq \chi_1 \frac{d^2\mathcal{F}}{dt^2}(t) - \chi_2 \frac{d\mathcal{F}}{dt}(t) \quad \text{for all } t \in [a_1, b_1] \tag{99}
\]

where
\[
\chi_1 = \max \left( \frac{1}{2\varepsilon} + \frac{3\varepsilon}{2} + \frac{|\sigma|}{\sqrt{\varepsilon - 2\beta\eta}} + \frac{2\beta a_1}{2\varepsilon \phi} \right)
\]

\[
\chi_2 = \max \left( \frac{2|\sigma|}{\sqrt{\varepsilon - 2\beta\eta}} + \frac{|\sigma|}{\sqrt{\varepsilon - 2\beta\eta}} + \frac{2\beta a_1}{2\varepsilon \phi} \right)
\]

By applying the comparison principle to the differential inequality (99), we obtain the following estimate:
\[
0 \leq \mathcal{F}(t) \leq \mathcal{F}(0) e^{-\chi_2(t-a_1)} \quad \text{for all } t \in [a_1, b_1] \tag{100}
\]

where
\[
\chi_2 = \frac{1}{2\chi_1} (-\chi_2 + \sqrt{\chi_2^2 + 4\chi_1})
\]

In terms of the initial variables \((r, \theta)\), in view of relation (30), the decay estimate (100) can be written as follows
\[
0 \leq \mathcal{F}_1(r) \leq \mathcal{F}_1(0) \left( \frac{a}{r} \right)^{\chi_2} \quad \text{for all } r \in [a, b] \tag{101}
\]

where
\[
\mathcal{F}_1(r) = \int_r^b \frac{1}{s} F_1(b) ds \tag{102}
\]

Consequently, in the case of the problem \(B\) we have a spatial decay of algebraical type as described in relation (103), provided the angle of the arch-like region is lower than the critical value \(\pi \sqrt{2}\).
Let us discuss now the case of a semi-infinite curvilinear rectangle, that is the case when $b \to \infty$. Then

$$ F^*(t) = \int_{t_0}^{\infty} ds \int_{s}^{\infty} F(s) ds $$

is (1) bounded above for all $t \in [0, \infty)$ or (2) infinite for all $t \in [0, \infty)$. In the case (1) we deduce that $0 \leq F^*(t) \leq F(a_1) e^{-\nu_2(t-a_1)}$ for all $t \in [0, \infty)$, that is we have an algebraical spatial decay, whereas in the case (2) the corresponding energetic measure is infinite.

### 6. Concluding remarks

The poroelastic plane strain problem for a rectangle region was studied recently by D’Apice and Chiriţă [28] for a homogeneous and inhomogeneous material. The corresponding boundary value problems for an arch-like region are much more difficult to deal with explicitly than are their analogs for a rectangular region. This is because the polar coordinates lead to the differential system (12) and (13) which has variable coefficients and, moreover, the significance of the two polar variables give rise to two essentially different boundary value problems. Thus, we are led to two different types of spatial behavior: one is an exponential decay as described by (78) for problem A, whereas the other is an algebraical decay as given by (103) for problem B. Although we use the changes of independent variable and of couple $(\alpha, \phi)$ described by (30) and (31) to arrive at the simple form (32) and (33) of the restriction (87), which is a reminiscence coming from the corresponding classical problem for biharmonic equation.

The estimates (78) and (101) can be made fully explicit by establishing, in terms of the given data, some appropriate bounds for the terms $\epsilon(0)$ and $F^*(0)$, respectively. We will proceed here to obtain an appropriate bound for $\epsilon(0)$. To this end, we follow an idea developed by Flavin and Gleeson [21] and consider two arbitrary smooth couples $(\psi^{(1)}, \phi^{(1)}), (\psi^{(2)}, \phi^{(2)})$ satisfying the same boundary conditions like $(\psi, \phi)$ and introduce the following scalar product:

$$ \langle (\psi^{(1)}, \phi^{(1)}), (\psi^{(2)}, \phi^{(2)}) \rangle = \int_0^\infty d\xi \int_{\mathbb{R}^2} \left\{ \epsilon \left[ \frac{\partial^2 \psi^{(1)}(\xi)}{\partial t^2} \frac{\partial^2 \psi^{(2)}}{\partial t^2} + 2 \frac{\partial^2 \psi^{(1)}(\xi)}{\partial t \partial \xi} \frac{\partial^2 \psi^{(2)}}{\partial t \partial \xi} + \left( \frac{\partial^2 \psi^{(1)}(\xi)}{\partial \xi^2} + \psi^{(2)}(\xi) \right) \right] \frac{\partial \phi^{(1)}(\xi)}{\partial \xi} \frac{\partial \phi^{(2)}}{\partial \xi} \right\} d\xi \, dt $$

It can be easily seen that $\langle (\psi^{(1)}, \phi^{(1)}), (\psi^{(1)}, \phi^{(1)}) \rangle \geq 0$ with equality if and only if $(\psi^{(1)}, \phi^{(1)}) = 0$. By means of the integration by parts and by taking into account the boundary conditions we deduce that

$$ \langle (\psi^{(1)}, \phi^{(1)}), (\psi, \phi) \rangle = \langle (\psi, \phi), (\psi^{(1)}, \phi^{(1)}) \rangle $$

and hence the Schwarz inequality furnishes

$$ \langle (\psi, \phi), (\psi, \phi) \rangle = \langle (\psi^{(1)}, \phi^{(1)}), (\psi^{(1)}, \phi^{(1)}) \rangle $$

Consequently, we have

$$ \epsilon(0) \leq \langle (\psi^{(1)}, \phi^{(1)}), (\psi^{(1)}, \phi^{(1)}) \rangle $$

and so we have obtained an above bound of $\epsilon(0)$ in terms of the given data.

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### References