Reciprocal and variational principles in linear thermoelasticity without energy dissipation

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A B S T R A C T
In the present paper we consider the equations which govern the behavior of an anisotropic and inhomogeneous centro-symmetric material within the framework of the linear theory of thermoelasticity without energy dissipation. We establish a reciprocal relation which is based on a characterization of the boundary-initial value problem in which the initial conditions are incorporated into the field equations. Further, a variational principle is presented too.

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1. Introduction

In the literature concerning thermal effects in continuum mechanics there are developed several parabolic and hyperbolic theories for describing the heat conduction. The hyperbolic theories are also called theories of second sound and there the flow of heat is modelled with finite propagation speed, in contrast to the classical model based on Fourier’s law leading to infinite propagation speed of heat signals. A review of these theories is presented in the articles by Chandrasekharaiah (1998) and Hetnarski and Ignaczak (1999, 2000).

A new thermoelastic theory without energy dissipation has been proposed by Green and Naghdi (1993). This thermomechanical theory of deformable media introduces the so-called thermal displacement relating the common temperature and uses a general entropy balance as postulated in Green and Naghdi (1977). By the procedure of Green and Naghdi (1995), the reduced energy equation is regarded as an identity for all thermodynamical processes and places some restrictions on the functional forms of the dependent constitutive variables. The theory is illustrated in detail in the context of flow of heat in a rigid solid, with particular reference to the propagation of thermal waves at finite speed. The linearized formulation allows the transmission of heat flow as thermal waves at finite speed and the evolution equations are fully hyperbolic.

The linear theory of thermoelasticity without energy dissipation for homogeneous and isotropic materials was employed by Nappa (1998) and Quintanilla (1999) in order to obtain spatial energy bounds and decay estimates for the transient solutions in connection with the problem in which a thermoelastic body is subject to boundary and initial data and body supplies having a compact support, provided positive definiteness assumptions are supposed upon the constitutive coefficients. Moreover, we have to mention that Chandrasekharaiah (1996) proves uniqueness of solutions, Iesan (1998) establishes continuous dependence results, while Quintanilla (2002) studies the question of existence. Further results of structural stability and decay type are given by Quintanilla (2001, 2003) and Quintanilla and Straughan (2000) used logarithmic convexity and Lagrange identity arguments to yield uniqueness and growth without requiring sign definiteness of the constitutive coefficients.

In this paper we establish a reciprocal theorem of Graffi type within the framework of the linear theory of thermoelasticity without energy dissipation for an inhomogeneous and anisotropic thermoelastic material with a center of symmetry at each point. The reciprocity relation is derived for a body of volume region \( B \) and surface \( \partial B \) and represents an integral relation over \( B \) and \( \partial B \) between body supplies, surface tractions and surface fluxes, displacements and thermal displacements of two solutions of the mixed problem of the thermoelasticity without energy dissipation, namely, a solution of an actual problem and a solution of an auxiliary or virtual problem. To this end we first give an alternative characterization of the solution to the mixed boundary-initial value problem in which the initial conditions are incorporated into the field equations. In the classical elasticity this characterization was established by Ignaczak (1963) and Gurtin (1964).

A reciprocal principle of Betti–Rayleigh type was established in the context of the linear thermoelasticity without energy dissipation for a homogeneous and isotropic material by Chandrasekharaiah (1998).
The first reciprocal theorem in the classical thermoelastodynamics is due to Ionescu-Cazimir (1964). The proof is based on the assumption of null initial data and systematic use of the Laplace transform. Leşan (1974) has established a reciprocal theorem without using the Laplace transform. The method of proof is based on a characterization of the boundary–initial value problem in which the initial conditions are incorporated into the basic equations of motion. Later, Leşan (1989) has established a new reciprocal theorem where the proof avoids both the use of the Laplace transform and the incorporation of the initial conditions into the basic equations of motion.

Despite its long existence, the reciprocal theorem was, until recently, not used extensively to actually solve problems. A recent book by Achenbach (2003) presents, however, novel uses of reciprocity relations for the actual determination of elastodynamic fields. Various other applications of the reciprocal theorem have been presented in Ionescu-Cazimir (1964) and Nowacki (1966).

We will use the alternative formulation of the boundary–initial value problem in order to establish a variational characterization of the solution within the framework of linear theory of thermoelasticity without energy dissipation for anisotropic and inhomogeneous materials. A functional is indicated whose variation vanishes at a solution of the boundary–initial value problem.

In the context of the classical theory of thermoelasticity, variational principles have been presented in various works (see, e.g., Carlson, 1972; Lebon, 1980; Leşan, 1966, 1998).

We have to outline that Chandrasekharaiah (1998) has established a new reciprocal principle of the field type and a variational principle of Hamilton type for thermoelasticity without energy dissipation for homogenous and isotropic materials. Such type of variational principles does not characterize completely the boundary–initial value problem since it fails to take into account the initial velocity distribution and presupposes the knowledge of the displacements at a later time.

2. The mixed boundary–initial value problem

Throughout this section we assume that the properly regular region \( B \subset \mathbb{R}^2 \) is occupied by an inhomogeneous and anisotropic thermoelastic material with a center of symmetry at each point. We denote by \( \partial B \) the boundary surface of \( B \).

In what follows we will consider the dynamic theory of thermoelasticity without energy dissipation as described in Green and Nagdhi (1993, 1995). The governing equations of the linear theory of anisotropic and inhomogeneous thermoelasticity without energy dissipation are given by the evolution equations (Green and Nagdhi, 1993, 1995)

\[
S_{ij} + \rho_b \dot{b}_i = \rho \ddot{u}_i, \\
\rho \ddot{\eta} = \frac{\partial}{\partial_0} \ddot{r} - q_i,\tag{2.1}
\]

in \( B \times (0, \infty) \), the constitutive equations

\[
S_y = C_{ijkl} \varepsilon_{kl} - M_{ij} \ddot{r}, \\
\rho \ddot{\eta} = \frac{\partial}{\partial_0} \ddot{r} - q_i, \tag{2.2}
\]

\[
q_i = -\frac{1}{\partial_0} K_{q_i}, \tag{2.3}
\]

in \( \mathbb{R} \times (0, \infty) \), the geometrical equations

\[
\varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \\
\beta_i = \tau, \tag{2.4}
\]

in \( \mathbb{R} \times (0, \infty) \). Here \( u_i \) are the components of the displacement vector, \( \tau \) is the thermal displacement, \( \dot{\theta} \) represents the temperature variation from the uniform reference temperature \( \theta_0 \), \( e_i \) are the components of the strain tensor, \( \beta_i \) are the components of the thermal displacement gradient vector, \( S_y \) are the components of the stress tensor, \( q_i \) are the components of the entropy-heat flux vector, \( \eta \) is the entropy density per unit mass and \( b_i \) represents the components of the external body force vector and \( r \) is the external rate of supply of heat per unit mass. Furthermore, \( \rho \) is the density mass, \( C_{ijkl}, \ M_{ij}, \ c \) and \( K_q \) are the constitutive coefficients satisfying the following symmetries:

\[
C_{ijkl} = C_{lijk} = C_{ijlk}, \\
M_{ij} = M_{ji}, \quad K_q = K_q, \tag{2.6}
\]

and the subscripts \( i, j, k \), \( l \) take values 1, 2, 3 and summation is implied by index repetition. Moreover, a supersposed dot denotes differentiation with respect to time and a subscript comma indicates partial differentiation. The specific Helmholtz free energy \( \psi \) is given by

\[
\psi = \frac{1}{2} C_{ijkl} \dot{e}_{ij} - M_{ij} \dot{e}_i + \frac{c}{\partial_0} \dot{r}^2 + \frac{1}{\partial_0} K_q \beta_i \beta_j, \tag{2.7}
\]

while the specific internal energy \( e \) is given by

\[
e = \frac{1}{2} C_{ijkl} \dot{e}_{ij} + \frac{c}{\partial_0} \dot{r}^2 + \frac{1}{\partial_0} K_q \beta_i \beta_j. \tag{2.8}
\]

The components of the surface traction and the heat flux at regular points of \( \partial B \) can be expressed in the form

\[
s_i = S_i n_j, \\
q = q_i n_i, \tag{2.9}
\]

where \( n_i \) are the components of the unit outward normal vector to \( \partial B \).

We assume that \( b_i, r, q_i, \theta_0 \) and the constitutive coefficients are prescribed functions with the following properties: (i) \( b_i \) and \( r \) are continuous on \( \mathbb{R} \times [0, \infty) \); (ii) \( C_{ijkl}, \ M_{ij}, \ c \) and \( K_q \) are smooth on \( \mathbb{R} \); while \( c \) is continuous on \( \mathbb{R} \times \mathbb{R} \). (iii) \( C_{ijkl}, \ M_{ij}, \ c \) and \( K_q \) satisfy the relation \( \psi, b \) is a strictly constant.

We say that \( (u, \tau) \) is a dynamically admissible state on \( \mathbb{R} \times (0, \infty) \) provided: (a) \( u_i \) and \( \tau \) are of class \( C^2 \) on \( \mathbb{R} \times [0, \infty) \); (b) \( 0 \leq \beta_i \leq \beta \) and (c) \( q_i \) are continuous on \( \mathbb{R} \times (0, \infty) \).

By an admissible system of stresses and heat fluxes on \( \mathbb{R} \times (0, \infty) \) we mean an ordered array \( (S_y, q_i) \) with properties: (a) \( S_y \) and \( q_i \) are of class \( C^1 \) on \( \mathbb{R} \times (0, \infty) \); (b) \( S_y, q_i, \) and \( q_i \) are continuous on \( \mathbb{R} \times (0, \infty) \).

By an admissible process on \( \mathbb{R} \times (0, \infty) \) we mean an ordered array of functions \( p = (u, \tau, e_i, \beta_i, S_y, q_i, \eta) \) with the following properties: (1) \( u_i, \tau \) is a dynamically admissible state on \( \mathbb{R} \times (0, \infty) \); (2) \( e_i \) and \( \beta_i \) are continuous fields on \( \mathbb{R} \times (0, \infty) \); (3) \( S_y, q_i, \) and \( q_i \) are an admissible system of stresses and heat fluxes on \( \mathbb{R} \times (0, \infty) \). The set of all admissible processes on \( \mathbb{R} \times (0, \infty) \) can be organized as a vector space provided addition and scalar multiplication are defined in an appropriate manner.

We say that \( p = (u, \tau, e_i, \beta_i, S_y, q_i, \eta) \) is a thermoelastic process corresponding to the supply terms \( (b_i, \eta) \) if \( p \) is an admissible process that satisfies the fundamental system of field Eqs. (2.1)–(2.5) on \( \mathbb{R} \times (0, \infty) \).

To the field Eqs. (2.1)–(2.5) we adjoint initial conditions and boundary conditions. In what follows we consider the initial conditions in the form

\[
u_i(0, x) = u_i^0(x), \quad \dot{u}_i(0, x) = \dot{u}_i^0(x), \quad \tau_i(0, x) = 0, \quad u_0(x) = 0, \quad x \in \mathbb{R}, \tag{2.10}
\]

where \( u_i^0, \dot{r}_i^0 \) and \( \dot{\theta}_0 \) are prescribed continuous functions on \( \mathbb{R} \) so that \( u_i^0 \) is continuous on \( \mathbb{R} \). Without loss of generality, here we have considered zero initial condition for the thermal displacement. Our
analysis in the present paper works also in the case when non-zero initial condition is considered for the thermal displacement.

Throughout this paper we will consider the mixed boundary-initial value problem. To this end we set $\Sigma_m$ ($m = 1, 2, 3, 4$) the subsets of $\partial B$ such that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$ and consider the following boundary conditions:

$$
\begin{align*}
\begin{align*}
\mathbf{u}_i &= \mathbf{u}_i \text{ on } \Sigma_1 \times (0, \infty), \\
\mathbf{s}_i &= \mathbf{s}_i \text{ on } \Sigma_2 \times (0, \infty), \\
\tau &= \tau \text{ on } \Sigma_3 \times (0, \infty), \\
q &= q \text{ on } \Sigma_4 \times (0, \infty), \\
\end{align*}
\end{align*}
\tag{2.11}
$$

where $\mathbf{u}_i$, $\tau$, $\mathbf{s}_i$, and $q$ are prescribed functions. We assume that $\mathbf{u}_i$ and $\tau$ are continuous on $\Sigma_1 \times (0, \infty)$ and $\Sigma_3 \times (0, \infty)$, while $\mathbf{s}_i$ and $q$ are continuous in time and piecewise regular on $\Sigma_2 \times (0, \infty)$ and $\Sigma_4 \times (0, \infty)$, respectively. We also assume that these data are compatible with the initial conditions.

The mixed boundary-initial value problem of the linear thermoelasticity without energy dissipation consists in finding a thermoelastic process $p$ corresponding to the supply term $(b, r)$ that satisfies the initial conditions (2.10) and the boundary conditions (2.11). We call such a thermoelastic process a solution of the mixed boundary-initial value problem.

3. An alternative characterization of the mixed problem

Throughout this section we will give a characterization of the mixed boundary–initial value problem in which the initial conditions are incorporated into the field of basic equations. To this end we follow Ignaczak (1963), Curtin (1964), Carlson (1972) and Lebon (1980) and introduce the generalized supplies

$$
\begin{align*}
F_i &= \mathcal{G}(\mathbf{g} \ast \mathbf{b}_i + \tau \mathbf{b}_i^0 + \mathbf{u}_i^0), \\
R &= \mathcal{G}\left(\frac{1}{\theta_0} r + \eta^0 \tau\right),
\end{align*}
\tag{3.1}
$$

where

$$
g(t) = t, \quad t \in [0, \infty),
$$

$$
\mathcal{G}
\begin{align*}
\mathcal{G} &= \mathcal{G}(\mathbf{g} \ast \mathbf{b}_i + \tau \mathbf{b}_i^0 + \mathbf{u}_i^0),
\end{align*}
\tag{3.2}
$$

and the sign $\ast$ denotes the convolution product, that is

$$
f(t) = \int_0^t f(s) h(t - s) ds.
$$

We also introduce the function

$$
\mathcal{G}(t) = 1, \quad t \in [0, \infty).
$$

4. Reciprocal theorem

In this section we present a reciprocal theorem of the Betti type. To this end we consider two external data systems $L^{(x)} = \left\{\mathbf{b}_i^{(x)}; r^{(x)}; \mathbf{u}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{p}_i^{(x)}; \mathbf{p}_i^{(x)}\right\}$, $(\alpha = 1, 2)$ and consider $p^{(x)} = \left\{\mathbf{u}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{p}_i^{(x)}; \mathbf{p}_i^{(x)}\right\}$, $(\alpha = 1, 2)$, the corresponding solutions of the mixed boundary–initial value problem. We introduce the notations

$$
\begin{align*}
\mathcal{G} &= \mathcal{G};
\end{align*}
\tag{4.1}
$$

and use

$$
\begin{align*}
\mathcal{G} &= \mathcal{G}\left(\mathbf{g} \ast \mathbf{b}_i + \tau \mathbf{b}_i^0 + \mathbf{u}_i^0\right),
\end{align*}
\tag{4.2}
$$

and

$$
\begin{align*}
\mathcal{G} &= \mathcal{G}\left(\mathbf{g} \ast \mathbf{b}_i + \tau \mathbf{b}_i^0 + \mathbf{u}_i^0\right),
\end{align*}
\tag{4.3}
$$

Thus, we can establish a reciprocal relation in the form of the following theorem.

**Theorem 2.** (Reciprocal theorem). Assume that the thermoelastic coefficients satisfy the symmetry relation (2.6). Let $p^{(x)} = \left\{\mathbf{u}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{p}_i^{(x)}; \mathbf{p}_i^{(x)}\right\}$ be a solution of the mixed boundary–initial value problem corresponding to the external system $L^{(x)} = \left\{\mathbf{b}_i^{(x)}; r^{(x)}; \mathbf{u}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{q}_i^{(x)}; \mathbf{p}_i^{(x)}; \mathbf{p}_i^{(x)}\right\}$, $(\alpha = 1, 2)$. Then we have

$$
\begin{align*}
\mathcal{G} &= \mathcal{G}\left(\mathbf{g} \ast \mathbf{b}_i + \tau \mathbf{b}_i^0 + \mathbf{u}_i^0\right),
\end{align*}
\tag{4.4}
$$

**Proof.** We introduce the notation

$$
\begin{align*}
\mathcal{G} &= \mathcal{G}\left(\mathbf{g} \ast \mathbf{b}_i^{(x)} + \mathbf{q}_i^{(x)} - \frac{1}{\theta_0} \mathbf{g} \ast \mathbf{K}_i \mathbf{q}_i^{(x)} \ast \mathbf{q}_i^{(x)} \ast \mathbf{q}_i^{(x)} \ast \mathbf{p}_i^{(x)}
\end{align*}
\tag{4.5}
$$

...
and note that relation (2.6) implies that

$$N_{\alpha\beta} = N_{\beta\alpha}, \quad (\alpha, \beta = 1, 2).$$

(4.6)

Further, by means of relations (3.8), (3.9) and (2.5), we have

$$N_{\alpha\beta} = g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} = \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta},$$

(4.7)

In view of the basic Eqs. (3.6) and (3.7) and by using (3.9), from (4.7) we obtain

$$N_{\alpha\beta} = \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} - \ell M_{g_{\alpha\beta}} \tau^{\alpha\beta}.$$

(4.8)

Further, we introduce

$$L_{\alpha\beta} = N_{\alpha\beta} + g u_{\alpha} u_{\beta} - \frac{c}{\partial t} \tau^{\alpha\beta} - \tau^{\alpha\beta},$$

(4.9)

and note that

$$L_{\alpha\beta} = L_{\beta\alpha}.$$  

(4.10)

On the other hand, from (4.8) we have

$$L_{\alpha\beta} = \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} + F^{\alpha\beta} + u_{\alpha} u_{\beta} - \tau^{\alpha\beta}.$$

(4.11)

If we integrate the relation (4.11) on $B$ and use the divergence theorem and the symmetry relation (4.10), then we obtain (4.4) and the proof is complete.

Other reciprocal relation was established by Chandrasekharaiah (1998). □

5. Variational principle

Let us denote by $\mathcal{M}$ the linear space of all admissible processes endowed with natural addition and scalar multiplication. For each $t \in (0, \infty)$ we define the functional on $\mathcal{M}$ by

$$A_t(p) = \int_B \left[ \frac{1}{2} \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} - \ell M_{g_{\alpha\beta}} \tau^{\alpha\beta} \right] + \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} + F^{\alpha\beta} + u_{\alpha} u_{\beta} - \tau^{\alpha\beta}.$$

(5.1)

for every $p = (u, \tau, e, q, b, S, q, \eta) \in \mathcal{M}$.

Theorem 3. Assume that the thermoelastic coefficients satisfy the symmetry relation (2.6) and assume $c \neq 0$ on $B$. Then

$$\delta A_t(p) = 0, \quad t \in (0, \infty).$$

(5.2)

at an admissible process $p$ if and only if $p$ is a solution of the mixed boundary–initial value problem.

Proof. Let $p' = (u', \tau', e', q', S', q', \eta') \in \mathcal{M}$ and $p = (u, \tau, e, q, b, S, q, \eta) \in \mathcal{M}$. Then $p + Atp' \in \mathcal{M}$ for every scalar $\lambda$. A straightforward calculation yields

$$\delta\lambda A_t(p) = \int_B \left[ \frac{1}{2} \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} - \ell M_{g_{\alpha\beta}} \tau^{\alpha\beta} \right] + \left( g + S^g_{\alpha\beta} + q^g_{\alpha\beta} + \ell M_{g_{\alpha\beta}} \right) \tau^{\alpha\beta} + F^{\alpha\beta} + u_{\alpha} u_{\beta} - \tau^{\alpha\beta}.$$

(5.3)

for all $t \in (0, \infty)$. If $p$ is a solution of the mixed boundary–initial value problem, then on the basis of relations (2.4)–(3.9) and the boundary conditions (2.11), we obtain

$$\delta\lambda A_t(p) = 0, \quad t \in (0, \infty).$$

(5.4)

for every $p' = (u', \tau', e', q', b, S', q', \eta') \in \mathcal{M}$, and therefore

$$\delta A_t(p) = 0, \quad t \in (0, \infty).$$

(5.5)

Conversely, suppose that (5.2) holds true and hence (5.4) holds. On making suitable choices of $p'$ into (5.3) and appealing Lemmas 64.1–64.3 established by Gurtin (1972), p. 224, from (5.3) and (5.4) we see that $p$ is a solution of the mixed boundary–initial value problem and the proof is complete. □

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