Strong ellipticity and progressive waves in elastic materials with voids

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In the present paper, we investigate a model for propagating progressive waves associated with the voids within the framework of a linear theory of porous media. Owing to the use of lighter materials in modern buildings and noise concerns in the environment, such models for progressive waves are of much interest to the building industry. The analysis of such waves is also of interest in acoustic microscopy where the identification of material defects is of paramount importance to the industry and medicine. Our analysis is based on the strong ellipticity of the poroelastic materials. We illustrate the model of progressive wave propagation for isotropic and transversely isotropic porous materials. We also study the propagation of harmonic plane waves in porous materials including the thermal effect.

Keywords: strong ellipticity; progressive waves; elastic materials with voids; thermoelastic materials with voids

1. Introduction

Wave motion in an elastic material containing voids is an area with immense potential for practical applications. Many theories have been developed which describe the behaviour of materials with voids and which have proved to be useful in practice. In seismic zones, buildings are constructed with much lighter porous materials and typically have thinner walls. As a consequence, there is a great need to study the acoustic properties of porous materials including the nature of the solid elastic matrix and the gas filling the pores, and the influence of temperature on these quantities. But seismology represents only one of the many fields where the theory of materials with voids is applied. Medicine, various branches of biology and the oil exploration industry are other important fields of application.

In this paper, we consider the theory of materials with voids introduced by Cowin & Nunziato (1983). The theory of elastic materials with voids is the simplest extension of the classical theory of elasticity. The basic idea of this theory is to suppose that there is a distribution of voids throughout the elastic body. Consequently, the bulk density is written as the product of two fields: the matrix material density field and the volume fraction field. This representation introduces

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an additional degree of kinematic freedom and it was employed previously by Goodman & Cowin (1972) to develop a continuum theory of granular materials. The first investigations in the theory of thermoelastic materials with voids are due to Nunziato & Cowin (1979) and Ieșan (1986). The Nunziato–Cowin theory has enjoyed much success in predicting various effects especially within the remit of linear theory, e.g. Ciarletta & Ieșan (1993), Ieșan (2004) and the recent work of Quintanilla (2001), Casas & Quintanilla (2005), Magaña & Quintanilla (2006a,b, 2007), Ghiba (2008a) and the references therein. The intended applications of the theory concern the manufacture of building materials, e.g. plasterboard, brick and concrete.

The first aim of the present study is to establish the necessary and sufficient conditions characterizing the strong ellipticity of materials with voids. Such conditions are important in discussing the uniqueness, wave propagation (e.g. Gurtin 1972, p. 86), loss of ellipticity in the context of the nonlinear elasticity of fibre-reinforced materials (Merodio & Ogden 2003a) and also in the study of the spatial behaviour of the constrained anisotropic cylinders (Chiriţă & Ciarletta 2006) or of constrained rectangular plates (Ghiba 2008b).

Simpson & Spector (1983) treated the strong ellipticity condition for isotropic nonlinearly elastic materials. They established necessary and sufficient conditions for the strong ellipticity of the equations governing an isotropic (compressible) nonlinearly elastic material at equilibrium. The main tool in their analysis consists of a representation theorem for copositive matrices. A review on the strong ellipticity for isotropic nonlinearly elastic materials is given by Dacorogna (2001).

Notable contribution concerning the strong ellipticity for the class of transverse isotropic linearly elastic materials is the work by Payton (1983) and the recent results by Padovani (2002), Merodio & Ogden (2003b) and Chiriţă (2006). For other symmetries of elastic materials, the strong ellipticity is studied by Chiriţă et al. (2007), Chiriţă & Danescu (2008) and Han et al. (2009).

In this paper, the general necessary and sufficient conditions are established for anisotropic centro-symmetric materials with voids but two types of materials are discussed in detail: isotropic and transversely isotropic elastic materials with voids. The theory of isotropic materials is a good approach for many problems in engineering and biological sciences. Ding et al. (2006) have presented methods for studying the different types of problems which arise in the theory of transversely isotropic elastic materials. Besides the well-known applications of this type of material in the mechanics of rocks (Wyllie 1999; Jaeger et al. 2007), transversely isotropic materials are very useful in many branches of biology (Humphrey 2002; Dong & Guo, 2004; Goldmann et al. 2005). The recent studies of fibre-reinforced composites (Spencer 1992; Bunsell & Renard 2005) and the modern technologies also encourage the study of transversely isotropic materials.

In the second part of this paper, we study the propagation of progressive waves in elastic materials with voids. We also consider the propagation of waves in thermoelastic isotropic materials with voids. We outline that the problem of wave propagation in materials with voids was studied in various works (Nunziato & Walsh 1977, 1978; Nunziato et al. 1978; Cowin & Nunziato 1983; Ciarletta & Straughan 2006, 2007a,b; Ciarletta et al. 2007) and a comprehensive review of the results can be found in the book by Straughan (2008). The use of acceleration waves and related analysis has proved extremely useful in the recent investigations of wave motion in various dispersive and random media.


The present mathematical results can be used in laboratories for studying the specific properties of materials with voids.

2. Basic equations of elastic model with voids

Throughout this section, $B$ is a bounded regular region of three-dimensional Euclidean space. We let $\bar{B}$ denote the closure of $B$, call $\partial B$ the boundary of $B$, and designate by $\mathbf{n}$ the outward unit normal of $\partial B$. We assume that the body occupying $\bar{B}$ is a linearly elastic solid with voids. The body is referred to a fixed system of rectangular Cartesian axes $Ox_i$ $(i = 1, 2, 3)$. Throughout this paper, Latin indices have the range 1–3, Greek indices have the range 1, 2 and the usual summation convention is employed. Moreover, subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate, while a superposed dot means the time differentiation.

Let $\mathbf{u}$ be the displacement vector field over $B$, and let $\varphi$ be the volume fraction field over $B$. Then the components $e_{rs}$ of the linear strain tensor are given by

$$ e_{rs} = \frac{1}{2} (u_{r,s} + u_{s,r}) \quad (2.1) $$

and the constitutive equations for a homogeneous and anisotropic elastic solid with voids are (cf. Cowin & Nunziato 1983)

$$
\begin{align*}
t_{rs} & = C_{rsmn} e_{mn} + B_{rs} \varphi + D_{rsk} \varphi, \\
h_r & = A_{rs} \varphi, + D_{mn} e_{mn} + d_r \varphi, \\
g & = -B_{rs} e_{rs} - \xi \varphi - d_r \varphi, \quad (2.2)
\end{align*}
$$

Here, we have used the notations: $t_{rs}$ are the components of the stress tensor, $h_r$ are the components of the equilibrated stress vector, $g$ is the intrinsic equilibrated body force and $A_{rs}, B_{rs}, C_{rsmn}, D_{rsk}, d_r$ and $\xi$ are constant constitutive coefficients satisfying the following symmetries:

$$
A_{rs} = A_{sr}, \quad B_{rs} = B_{sr}, \quad D_{rsk} = D_{skr}, \quad C_{mnr} = C_{rsmn} = C_{nmrs}. \quad (2.3)
$$

The specific internal energy $W$ is given by

$$
W(\mathbf{e}, \varphi, \text{grad} \varphi) = \frac{1}{2} C_{mnr} e_{mn} e_{rs} + \frac{1}{2} \xi \varphi^2 + \frac{1}{2} A_{rs} \varphi, e_{rs} + B_{rs} e_{rs} \varphi + D_{rsk} e_{rs} \varphi, + d_r \varphi, \varphi, \quad (2.4)
$$

The specific internal energy $W_0$ for an isotropic and homogeneous elastic material with voids is defined by

$$
W_0(\mathbf{e}, \varphi, \text{grad} \varphi) = \mu e_{rs} e_{rs} + \frac{1}{2} \lambda \varepsilon_{rr} e_{ss} + \frac{1}{2} \xi \varphi^2 + \frac{1}{2} \alpha \varphi, \varphi, + \beta \varphi, \varphi, \quad (2.5)
$$

where $\lambda, \mu, \alpha, \beta$ and $\xi$ are constant constitutive coefficients.
The transverse isotropy is characterized by the symmetry group consisting of unit tensor 1 and the rotations \( R_\theta^\phi (0 < \theta < 2\pi) \), that is, the orthogonal tensor corresponding to a right-handed rotation through the angle \( \theta, 0 < \theta < 2\pi \), about the axis in the direction of the unit vector \( e_3 \). For a transversely isotropic elastic material with voids, the specific internal energy \( W_1 \) is defined by

\[
W_1(e, \phi, \text{grad } \phi) = \frac{1}{2} c_{11} (e_1^2 + e_2^2) + \frac{1}{2} c_{33} e_3^2 + 2 c_{55} (e_{13}^2 + e_{23}^2) + 2 c_{66} e_1^2 \\
+ \frac{1}{2} \xi \phi^2 + \frac{1}{2} a_{11} (\phi_{11}^2 + \phi_{22}^2) + \frac{1}{2} a_{33} \phi_{33}^2 + c_{12} e_{11} e_{22} \\
+ c_{13} (e_{11} + e_{22}) \phi_{33} + b_{11} (e_{11} + e_{22}) \phi + b_{33} \phi_{33} \phi, 
\]

where

\[
c_{11} = C_{1111}, \quad c_{33} = C_{3333}, \quad c_{55} = C_{1313}, \quad c_{13} = C_{1133}, \quad c_{12} = C_{1122}, \\
c_{66} = \frac{1}{2} (c_{11} - c_{12}), \quad a_{11} = A_{11}, \quad a_{33} = A_{33}, \quad b_{11} = B_{11}, \quad b_{33} = B_{33}. 
\]

The other constitutive coefficients are zero.

In the absence of the body force and the extrinsic equilibrated body force, the equations of motion of the elastic material with voids are given by (cf. Cowin & Nunziato 1983)

\[
t_{rs,r} = \varrho \ddot{u}_s \]
\[h_{r,r} + g = \varrho \dot{\phi} \]

(2.8)

If we substitute equations (2.1) and (2.2) into equation (2.8), we obtain the basic equations in terms of the displacement and volume fraction fields in the form

\[
(C_{mnrs} u_{r,s} + B_{mn} \phi + D_{mnk} \phi_{,k},_{,n} = \varrho \ddot{u}_n \]
\[(A_{mn} \phi_{,n} + D_{rsm} u_{r,s} + d_{mn} \phi_{,m} - B_{rs} u_{r,s} - \xi \phi - d_{r} \phi_{,r} = \varrho \dot{\phi} \]

(2.9)

which for isotropic materials with voids becomes

\[
\mu \Delta u_r + (\lambda + \mu) u_{sr,r} + \beta \phi_{,r} = \varrho \ddot{u}_r \\
\alpha \Delta \phi - \xi \phi - \beta u_{r,r} = \varrho \dot{\phi}
\]

(2.10)

where \( \Delta \) is the Laplace operator.

We have to note that the specific internal energy \( W_0 \) is a positive definite quadratic form in terms of the variables \( (e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{31}, \phi, \phi_{,1}, \phi_{,2}, \phi_{,3}) \) if and only if

\[
\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad \left( \lambda + \frac{2}{3} \mu \right) \xi > \beta^2, 
\]

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while $W_1$ is a positive definite quadratic form in terms of the variables $(e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{31}, \varphi, \varphi_1, \varphi_2, \varphi_3)$ if and only if

$$
c_{11} > 0, \quad c_{33} > 0, \quad c_{55} > 0, \quad \xi > 0, \quad a_{11} > 0, \quad a_{33} > 0,
$$

$$
|c_{12}| < c_{11}, \quad |c_{13}| < \sqrt{\frac{1}{2}(c_{11} + c_{12})c_{33}},
$$

and

$$
\left( c_{13} - \frac{b_{11}}{\xi} \right)^2 < \left( c_{33} - \frac{b_{33}^2}{\xi} \right) \left[ \frac{1}{2}(c_{11} + c_{12}) - \frac{b_{11}^2}{\xi} \right].
$$

(2.12)

3. Strongly elliptic elastic materials with voids

We will say that the elastic material with voids is strongly elliptic if

$$
W(A, \eta, \zeta) > 0 \quad \text{for all} \ (A, \eta, \zeta) \neq 0,
$$

(3.1)

with $A = m \otimes n$, $m$, $n$ and $\zeta$ arbitrary vectors and $\eta$ an arbitrary scalar. The strongly elliptic condition (3.1) can be written in an explicit way in the following form:

$$
\frac{1}{2} C_{mnrs} m_m n_r n_s n_s + \frac{1}{2} \xi \eta^2 + \frac{1}{2} A_{rs} \zeta_r \zeta_s + B_{rs} m_r n_s \eta
$$

$$
+ D_{rsk} m_r n_s \zeta_k + d_r \eta \zeta_r > 0 \quad \text{for all} \ (m \otimes n, \eta, \zeta) \neq 0,
$$

(3.2)

where $m = (m_1, m_2, m_3)$, $n = (n_1, n_2, n_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$.

We can observe that the strong ellipticity condition (3.1) assures the strong ellipticity of the system of partial differential equations (2.9) (Fichera 1965). We have to note that the strong ellipticity condition (3.2) implies that

$$
C_{mnrs} m_m n_r n_s n_s > 0 \quad \text{for all unit vectors} \ n, m,
$$

(3.3)

$$
\xi > 0
$$

(3.4)

and

$$
A_{rs} \zeta_r \zeta_s > 0 \quad \text{for all unit vectors} \ \zeta.
$$

(3.5)

The inequality (3.2) is the general strong ellipticity condition for an arbitrary anisotropic poroelastic body. Our objective here consists of finding explicit necessary and sufficient conditions characterizing the strong ellipticity of elastic materials with voids. However, we believe that the results one obtains when the porous body has a centre of symmetry are revealing. When the poroelastic body possesses a centre of symmetry, the terms $D_{rsk}$ and $d_r$ are zero. Thus, for the centro-symmetric elastic materials with voids, the strong ellipticity condition (3.2) reduces to

$$
\frac{1}{2} C_{mnrs} m_m n_r n_s n_s + \frac{1}{2} \xi \eta^2 + \frac{1}{2} A_{rs} \zeta_r \zeta_s + B_{rs} m_r n_s \eta > 0 \quad \text{for all} \ (m \otimes n, \eta, \zeta) \neq 0.
$$

(3.6)

This condition is equivalent to the following inequalities:

$$A_{rs} \xi_r \xi_s > 0$$ for all unit vectors $\xi$ \hspace{1cm} (3.7)

and

$$\xi \eta^2 + 2B_{rs} m_r n_s \eta + C_{mns} m_m m_r n_r n_s > 0$$

for all real scalar $\eta$ and unit vectors $m, n$. \hspace{1cm} (3.8)

Further, regarding equation (3.8) as a quadratic in $\eta \in \mathbb{R}$, we deduce that it is equivalent to the following inequalities:

$$\xi > 0$$ \hspace{1cm} (3.9)

and

$$(B_{rs} m_r n_s)^2 < \xi C_{mns} m_m m_r n_r n_s$$ for all unit vectors $m, n$. \hspace{1cm} (3.10)

This last condition can be written as

$$(\xi C_{mns} - B_{mn} B_{rs}) m_m m_r n_r n_s > 0$$ for all unit vectors $m, n$, \hspace{1cm} (3.11)

that is, the tensor

$$C_{mns} = \xi C_{mns} - B_{mn} B_{rs}$$ \hspace{1cm} (3.12)

is strongly elliptic.

Concluding, we deduce that a centro-symmetric elastic material with voids is strongly elliptic if and only if $A_{rs}$ is a positive definite tensor, $\xi > 0$ and $C_{mns}$ is strongly elliptic tensor.

In what follows, we will establish the explicit necessary and sufficient conditions characterizing the strong ellipticity of the isotropic and transversely isotropic elastic materials with voids.

(a) Isotropic elastic materials with voids

In this subsection, we characterize the strong ellipticity for the class of isotropic elastic materials with voids. To this end, we note that the strong ellipticity condition (3.6) becomes

$$\mu m_r m_r n_s n_s + \frac{1}{2} \lambda m_r n_s m_r n_s + \beta \eta m_r n_r n_r + \frac{1}{2} \xi \eta^2 + \frac{1}{2} \alpha \xi_r \xi_r > 0,$$ \hspace{1cm} (3.13)

and it is equivalent to

$$\alpha > 0, \hspace{0.5cm} \xi > 0$$ \hspace{1cm} (3.14)

and

$$\beta^2 m_r n_r m_s n_s < \xi [2\mu m_r m_r n_s n_s + \lambda m_r n_r m_s n_s], \hspace{0.5cm} \text{for all unit vectors} \hspace{0.1cm} m, n. \hspace{1cm} (3.15)$$

Further, we recall that we can write equation (3.15) in the form

$$2\xi \mu \left[(m_1 n_2 - m_2 n_1)^2 + (m_2 n_3 - m_3 n_2)^2 + (m_3 n_1 - m_1 n_3)^2\right]$$

$$+ \{\xi (\lambda + 2\mu) - \beta^2 \} (m_1 n_1 + m_2 n_2 + m_3 n_3)^2 > 0,$$ \hspace{1cm} (3.16)

and note that it is equivalent to the following inequalities:

$$\xi \mu > 0$$

and

$$\xi (\lambda + 2\mu) - \beta^2 > 0.$$ \hspace{1cm} (3.17)

Concluding, we can say that an isotropic elastic material with voids is strongly elliptic if and only if the elastic coefficients satisfy the following inequalities:

\[
\begin{align*}
\mu &> 0, \\
\xi &> 0, \\
\alpha &> 0
\end{align*}
\]  

(3.18)

and

\[\xi(\lambda + 2\mu) > \beta^2.\]

It appears clear from the relations (2.11) and (3.18) that the positive definiteness assumption of the specific internal energy \(W_0\) implies the strong ellipticity of the elastic material with voids.

(b) Transversely isotropic elastic materials with voids

Many natural and man-made materials are classified as transversely isotropic. Such materials are characterized by the fact that one can find a line, which allows a rotation of the material about it without changing its elastic properties. The plane, which is perpendicular to this line (the axis of rotational symmetry), is called a plane of elastic symmetry or a plane of isotropy. Modern examples for such materials are laminates made of randomly oriented chopped fibres that are in general placed in a certain plane. The effective elastic material properties for a bundled structure have no preferred direction in that plane, which then becomes a plane of elastic symmetry. Hence, each plane that contains the axis of rotation is a plane of symmetry, and therefore transversely isotropic elastic material admits an infinite number of elastic symmetries. Thus, the transverse isotropy is appropriate to real materials having a laminate or a bundled structure.

The strong ellipticity condition (3.6) reduces to

\[
c_{66}(n_1 m_2 - n_2 m_1)^2 + c_{11}(n_1 m_1 + n_2 m_2)^2 + c_{13}n_3^2 m_3^2 + c_{55}(n_3^2 m_1^2 + n_1^2 m_3^2)
+ n_2^2 m_2^2 + n_2^2 m_2^2) + \xi \eta^2 + a_{11}(\xi_1^2 + \xi_2^2) + a_{33}\xi_3^2 + 2(c_{13} + c_{55})(n_1 m_1
+ n_2 m_2)n_3 m_3 + 2b_{11}(n_1 m_1 + n_2 m_2)\eta + 2b_{33}n_3 m_3 \eta > 0,
\]  

(3.19)

and it is equivalent to the inequalities

\[a_{11} > 0, \quad a_{33} > 0, \quad \xi > 0,\]

(3.20)

and the strong ellipticity of the elasticity tensor given by

\[
\begin{pmatrix}
\xi c_{11} - b_{11}^2 & \xi c_{12} - b_{11}^2 & \xi c_{13} - b_{11} b_{33} & 0 & 0 & 0 \\
\xi c_{12} - b_{11}^2 & \xi c_{11} - b_{11}^2 & \xi c_{13} - b_{11} b_{33} & 0 & 0 & 0 \\
\xi c_{13} - b_{11} b_{33} & \xi c_{13} - b_{11} b_{33} & \xi c_{33} - b_{33}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \xi c_{55} & 0 & 0 \\
0 & 0 & 0 & 0 & \xi c_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \xi c_{66}
\end{pmatrix}
\]  

(3.21)

In view of the results established by Merodio & Ogden (2003b), Chiriță (2006) and Chiriță et al. (2007), the elasticity tensor defined in equation (3.21) is strongly
elliptic if and only if

\[ \xi c_{11} - b_{11}^2 > 0, \quad \xi c_{33} - b_{33}^2 > 0, \quad (3.22) \]

\[ \xi c_{55} > 0, \quad \xi (c_{11} - c_{12}) > 0 \quad (3.23) \]

and

\[ \left| c_{13} + c_{55} - \frac{b_{11} b_{33}}{\xi} \right| < c_{55} + \sqrt{\left( c_{11} - \frac{b_{11}^2}{\xi} \right) \left( c_{33} - \frac{b_{33}^2}{\xi} \right)}. \quad (3.24) \]

Concluding, we deduce that a transversely isotropic elastic material with voids is strongly elliptic if and only if the constitutive coefficients satisfy the relations (3.20), (3.22), (3.23) and (3.24).

4. Progressive waves in centro-symmetric elastic materials with voids

Throughout this section, we will consider plane progressive waves in elastic materials with voids having a centre of symmetry, that is, materials for which we have

\[ D_{jk} = 0, \quad d_r = 0. \quad (4.1) \]

By a plane progressive wave in a centro-symmetric elastic material with voids, we mean a couple of functions \((u_r, \varphi)\) on \(\mathbb{R}^3 \times (-\infty, \infty)\) of the following form:

\[
\begin{align*}
\begin{aligned}
u_r(x, t) & = U_r \Lambda (n_s x_s - ct) \\
\varphi(x, t) & = \Phi \Gamma (n_s x_s - ct),
\end{aligned}
\end{align*}
\] (4.2)

where

(i) \(\Lambda\) and \(\Gamma\) are the real-valued functions of class \(C^2\) on \((-\infty, \infty)\) with

\[ \frac{d^2 \Gamma}{ds^2} = -\Gamma \neq 0, \quad \frac{d \Lambda}{ds} = \Gamma; \quad (4.3) \]

(ii) \(U = (U_1, U_2, U_3)\) is the displacement amplitude vector, \(n = (n_1, n_2, n_3)\) is a unit vector which represents the direction of propagation and \((U_1, U_2, U_3, \Phi) \neq 0;\)

(iii) \(c\) is the speed of propagation.

We say that the progressive wave is longitudinal if \(U\) and \(n\) are linearly dependent, that is, \(U \times n = 0\). It is transverse when \(U\) and \(n\) are perpendicular, that is, \(U \cdot n = 0\). For any given constant \(\gamma\) and for any given time \(t\), the displacement and volume fraction fields (4.2) are constants on the plane \(n_s x_s - ct = \gamma\). This last plane is perpendicular to \(n\) and it is moving with the speed \(c\) in the direction \(n\).

These plane waves are important in that they furnish valuable information concerning the propagation characteristics of elastic materials with voids.

If we substitute the plane progressive wave described by equation (4.2) into the basic equations (2.9) and then we take into consideration the assumptions

described into equations (4.1) and (4.3), we see that such waves are possible in an elastic material with voids if and only if
\[
\begin{align*}
(C_{mnr}n_m - \varrho c^2 \delta_{rm}) U_r + B_{mr} n_m \Phi &= 0, \\
B_{rs} n_s U_r + (\xi + A_{rs} n_r n_s - \varrho \kappa c^2) \Phi &= 0.
\end{align*}
\] (4.4)

Such a linear algebraic system admits a non-zero solution \((U_1, U_2, U_3, \Phi)\) if and only if \(c^2\) satisfies the following algebraic equation:
\[
\det \begin{pmatrix}
Q_{11}(\mathbf{n}) & Q_{12}(\mathbf{n}) & Q_{13}(\mathbf{n}) & Q_{14}(\mathbf{n}) \\
Q_{21}(\mathbf{n}) & Q_{22}(\mathbf{n}) & Q_{23}(\mathbf{n}) & Q_{24}(\mathbf{n}) \\
Q_{31}(\mathbf{n}) & Q_{32}(\mathbf{n}) & Q_{33}(\mathbf{n}) & Q_{34}(\mathbf{n}) \\
Q_{41}(\mathbf{n}) & Q_{42}(\mathbf{n}) & Q_{43}(\mathbf{n}) & Q_{44}(\mathbf{n})
\end{pmatrix} = 0, \tag{4.5}
\]
where
\[
\begin{align*}
Q_{rs}(\mathbf{n}) &= C_{mnr} n_m n_r - \varrho c^2 \delta_{rs}, \\
Q_{r4}(\mathbf{n}) &= Q_{4r}(\mathbf{n}) = B_{mr} n_m \quad \text{for } r, s = 1, 2, 3.
\end{align*}
\] (4.6)

It is easy to see that all the solutions of equation (4.5) are real numbers. In fact, from equation (4.4), we have
\[
\varrho c^2 (U_1^2 + U_2^2 + U_3^2 + \kappa \Phi^2) = C_{mnr} U_r U_n n_r n_m \\
+ (\xi + A_{rs} n_r n_s - \varrho \kappa c^2) \Phi^2 + 2B_{rs} U_r n_s \Phi. \tag{4.7}
\]

In view of the strong ellipticity condition (3.6), we can see that the right-hand side of equation (4.7) is a positive definite term and the corresponding roots \(c^2\) are positive. Thus, all solutions \(c_1, c_2, c_3\) and \(c_4\) of equation (4.5) are real numbers.

Therefore, for every direction of propagation \(\mathbf{n} = (n_1, n_2, n_3)\) in a centro-symmetric elastic material with voids, there exist three orthogonal directions of motion and four progressive waves propagating with the associated speeds of propagation \(c_1, c_2, c_3\) and \(c_4\).

(a) Isotropic poroelastic materials

For an isotropic poroelastic material, the progressive wave (4.2) can propagate with the speeds \(c_1 = c_2, c_3\) and \(c_4\), where
\[
c_1^2 = c_2^2 = \frac{\mu}{\varrho}, \tag{4.8}
\]
\[
c_3^2 = \frac{1}{2\varrho \kappa} \left\{ \xi + \alpha + \kappa (\lambda + 2\mu) + \sqrt{[\xi + \alpha - \kappa (\lambda + 2\mu)]^2 + 4\kappa \beta^2} \right\} \tag{4.9}
\]
and
\[
c_4^2 = \frac{1}{2\varrho \kappa} \left\{ \xi + \alpha + \kappa (\lambda + 2\mu) - \sqrt{[\xi + \alpha - \kappa (\lambda + 2\mu)]^2 + 4\kappa \beta^2} \right\}. \tag{4.10}
\]
Moreover, for \( c = c_1 = c_2 = \sqrt{\mu/\rho} \), we have the corresponding linear independent solutions \((U_1, U_2, U_3, \Phi)^{(1)}\) and \((U_1, U_2, U_3, \Phi)^{(2)}\) of the algebraic system (4.4)

\[
\begin{align*}
(U_1, U_2, U_3, \Phi)^{(1)} &= (-n_3, 0, n_1, 0), \\
(U_1, U_2, U_3, \Phi)^{(2)} &= (-n_2, n_1, 0, 0),
\end{align*}
\]

for the direction of propagation \( \mathbf{n} = (n_1, n_2, n_3) \). Thus, the characteristic space for \( c^2 = c^1 = c^2 \) is generated by the linear independent vectors \( U^{(1)} \) and \( U^{(2)} \). It is easy to see that we have

\[
U^{(1)} \cdot \mathbf{n} = 0, \quad U^{(2)} \cdot \mathbf{n} = 0,
\]

where \( U^{(1)} = (-n_3, 0, n_1) \) and \( U^{(2)} = (-n_2, n_1, 0) \). Therefore, the progressive waves propagating with speeds \( c = c_1 = c_2 = \sqrt{\mu/\rho} \) are transverse waves.

For \( c = c_3 \), we have, up to a scalar factor, the corresponding solution \((U_1, U_2, U_3, \Phi)^{(3)}\) of the algebraic system (4.4) given by

\[
(U_1, U_2, U_3, \Phi)^{(3)} = (n_1 c, n_2 c, n_3, \beta),
\]

where

\[
c = \rho c_3^2 - \xi - \alpha,
\]

while for \( c = c_4 \), we have the corresponding solution \((U_1, U_2, U_3, \Phi)^{(4)}\) of the algebraic system (4.4) given by

\[
(U_1, U_2, U_3, \Phi)^{(4)} = (n_1 a, n_2 a, n_3 a, \beta),
\]

with

\[
a = \rho c_4^2 - \xi - \alpha.
\]

Thus, it is easy to see that we have

\[
U^{(3)} \times \mathbf{n} = 0, \quad U^{(4)} \times \mathbf{n} = 0,
\]

where \( U^{(3)} = a \mathbf{n} \) and \( U^{(4)} = a \mathbf{n} \) and hence the progressive waves propagating with speeds \( c = c_3 \) and \( c = c_4 \) are longitudinal waves. The lines spanned by \( U^{(3)} \) and \( U^{(4)} \), respectively, are the characteristic spaces for \( c_3^2 \) and \( c_4^2 \).

Concluding, we can formulate the following result:

**Proposition 4.1.** For a strongly elliptic isotropic elastic material with voids, the progressive wave defined in equation (4,2) is either one transverse propagating with the speed \( c = c_1 = c_2 \) or it is one longitudinal propagating with the speeds \( c = c_3 \) or \( c = c_4 \).

Thus, for strongly elliptic isotropic elastic materials with voids, there are only two types of progressive waves: longitudinal and transverse.
(b) Transverse isotropic poroelastic materials

In this section, we consider the case of strongly elliptic transverse isotropic poroelastic materials. For these types of materials, the matrix $Q$ defined by equation (4.5) is given by

\[
\begin{align*}
Q_{11} &= c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 - \varphi c^2, \\
Q_{22} &= c_{66}n_1^2 + c_{11}n_2^2 + c_{55}n_3^2 - \varphi c^2, \\
Q_{33} &= c_{55}(n_1^2 + n_2^2) + c_{33}n_3^2 - \varphi c^2, \\
Q_{44} &= \xi + a_{11}(n_1^2 + n_2^2) + a_{33}n_3^2 - \varphi x c^2, \\
Q_{12} &= Q_{21} = (c_{12} + c_{66})n_1n_2, \\
Q_{13} &= Q_{31} = (c_{13} + c_{55})n_1n_3, \\
Q_{14} &= Q_{41} = b_{11}n_1, \\
Q_{23} &= Q_{32} = (c_{13} + c_{55})n_2n_3, \\
Q_{24} &= Q_{42} = b_{11}n_2, \\
Q_{34} &= Q_{43} = b_{33}n_3.
\end{align*}
\]

Then equation (4.5) reduces to

\[
[c_{66}(n_1^2 + n_2^2) + c_{55}n_3^2 - \varphi c^2]D = 0,
\]

where

\[
D = \det(q_{ij}),
\]

with

\[
\begin{align*}
q_{11} &= c_{11}(n_1^2 + n_2^2) + c_{55}n_3^2 - \varphi c^2, \\
q_{22} &= c_{55}(n_1^2 + n_2^2) + c_{33}n_3^2 - \varphi c^2, \\
q_{33} &= \xi + a_{11}(n_1^2 + n_2^2) + a_{33}n_3^2 - \varphi x c^2, \\
q_{12} &= (c_{13} + c_{55})n_3, \\
q_{21} &= (c_{13} + c_{55})(n_1^2 + n_2^2)n_3, \\
q_{13} &= b_{11}, \\
q_{31} &= b_{11}(n_1^2 + n_2^2), \\
q_{23} &= q_{32} = b_{33}n_3.
\end{align*}
\]

Thus, we can observe that the progressive wave can propagate in an arbitrary direction $\mathbf{n}$ with the first speed

\[
c_1^2 = \frac{1}{\varphi} [c_{66} + (c_{55} - c_{66})n_3^2].
\]

The other speeds $c$ are the solutions of the equation

\[
c^6 + \Lambda_1 c^4 + \Lambda_2 c^2 + \Lambda_3 = 0,
\]

where

\[
A_1 = -\frac{1}{q^2} \left[ \frac{\xi}{x} + \left( \frac{a_{11}}{x} + c_{11} + c_{55} \right) + \left( \frac{a_{33} - a_{11}}{x} + c_{33} - c_{11} \right) n_3^2 \right],
\]

\[
A_2 = \frac{1}{q^2} \left\{ [(c_{11} + c_{55}) + (c_{33} - c_{11}) n_3^2] \left( \frac{\xi}{x} + \frac{a_{11}}{x} + \frac{a_{33} - a_{11}}{x} n_3^2 \right) + [c_{11} + (c_{55} - c_{11}) n_3^2][c_{55} + (c_{33} - c_{55}) n_3^2] - (c_{13} + c_{55})^2 (1 - n_3^2) n_3^2 - \frac{b_{11}^2}{x} - \frac{b_{33}^2 - b_{11}^2}{x} n_3^2 \right\},
\]

\[
A_3 = -\frac{1}{q^3} \left( \frac{\xi}{x} + \frac{a_{11}}{x} + \frac{a_{33} - a_{11}}{x} n_3^2 \right) \left\{ [(c_{11} + (c_{55} - c_{11}) n_3^2)[c_{55} + (c_{33} - c_{55}) n_3^2] - (c_{13} + c_{55})^2 (1 - n_3^2) n_3^2 + \frac{1}{q^3 x} b_{11}^2 [c_{55} + (c_{33} - c_{55}) n_3^2] \times (1 - n_3^2) + b_{33}^2 [c_{11} + (c_{55} - c_{11}) n_3^2] n_3^2 - 2 b_{11} b_{33} (c_{13} + c_{55}) (1 - n_3^2) n_3^2 \right\},
\]

In what follows, we will discuss the following two directions of propagation: (i) the progressive wave propagates in the plane of isotropy and (ii) the progressive wave propagates along the axis of rotational symmetry Ox3.

**Case (i): n_3 = 0**

In this case, the speeds of propagation are given by

\[
\begin{align*}
\tilde{c}_1^2 &= \frac{1}{q} c_{66}, \\
\tilde{c}_2^2 &= \frac{1}{q} c_{55}, \\
\tilde{c}_3^2 &= \frac{1}{2 q x} \left[ \xi + a_{11} + x c_{11} - \sqrt{\left( \xi + a_{11} - x c_{11} \right)^2 + 4 x b_{11}^2} \right], \\
\text{and} \\
\tilde{c}_4^2 &= \frac{1}{2 q x} \left[ \xi + a_{11} + x c_{11} + \sqrt{\left( \xi + a_{11} - x c_{11} \right)^2 + 4 x b_{11}^2} \right].
\end{align*}
\]

The solutions of the system (4.4) corresponding to the speeds \( \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \) and \( \tilde{c}_4 \) are, respectively, given by

\[
(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{\Phi})^{(1)} = (- n_2, n_1, 0, 0),
\]

\[
(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{\Phi})^{(2)} = (0, 0, 1, 0),
\]

\[
(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{\Phi})^{(3)} = ((q x c_3^2 - \xi - a_{11}) n_1, (q x c_3^2 - \xi - a_{11}) n_2, 0, b_{11}),
\]

and \( (\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{\Phi})^{(4)} = ((q x c_3^2 - \xi - a_{11}) n_1, (q x c_3^2 - \xi - a_{11}) n_2, 0, b_{11}). \)
In conclusion, we have the following result:

**Proposition 4.2.** For a strongly elliptic transversely isotropic elastic material with voids, the progressive waves which propagate in the plane of isotropy are two transverse propagating with the speeds \( \hat{c}_1 \) or \( \hat{c}_2 \) and two longitudinal propagating with the speeds \( \hat{c}_3 \) or \( \hat{c}_4 \).

**Case (ii):** \( n_1 = 0, \ n_2 = 0 \)

If the direction of propagation is along the axis of rotational symmetry, then the speeds of propagation are

\[
\hat{c}_1^2 = \hat{c}_2^2 = \frac{1}{\rho} c_{55},
\]

\[
\hat{c}_3^2 = \frac{1}{2Qx} \left[ \xi + a_{33} + \sqrt{(\xi + a_{33})^2 - 4x b_{33}^2} \right]
\]

and

\[
\hat{c}_4^2 = \frac{1}{2Qx} \left[ \xi + a_{33} + \sqrt{(\xi + a_{33})^2 - 4x b_{33}^2} \right]
\]

For the double solution of equation (4.5), the corresponding solutions of the algebraic system (4.4) are given by

\[
(\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{\Phi})^{(1)} = (1, 0, 0, 0)
\]

and

\[
(\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{\Phi})^{(2)} = (0, 1, 0, 0),
\]

while the corresponding solutions for the speeds \( \hat{c}_3 \) and \( \hat{c}_4 \) are

\[
(U_1, U_2, U_3, \Phi)^{(3)} = (0, 0, \rho x c_3^2 - \xi - a_{33}, b_{33})
\]

and

\[
(U_1, U_2, U_3, \Phi)^{(4)} = (0, 0, \rho x c_4^2 - \xi - a_{33}, b_{33}).
\]

Thus, we can give the following result:

**Proposition 4.3.** For a strongly elliptic transversely isotropic elastic material with voids, the progressive waves which propagate along the axis of isotropy are two transverse propagating with the speed \( \hat{c}_1 = \hat{c}_2 \) and two longitudinal propagating with the speeds \( \hat{c}_3 \) or \( \hat{c}_4 \).

### 5. Thermal effects

In this section, we study the plane progressive waves in thermoelastic isotropic materials with voids. In the absence of the body force, the extrinsic equilibrated body force and the heat source, the equations which describe the behaviour of the thermoelastic material with voids are given by (cf. Ieuan 2004)

\[
\mu \Delta u_r + (\lambda + \mu) u_{j,r} + \beta \varphi_r - b \theta_r = \varrho \ddot{u}_r,
\]

\[
\alpha \Delta \varphi - \beta u_{j,j} - \xi \varphi + m \theta = \varrho \ddot{\varphi}
\]

and

\[
k \Delta \theta - T_0 b u_{j,j} - T_0 m \phi = T_0 \dot{a} \theta.
\]
where \(\lambda, \mu, \alpha, \beta, \xi, a, b, m\) and \(k\) are constitutive coefficients which describe the thermal properties of material and \(T_0\) is the ambient absolute temperature. We suppose the material to be strongly elliptic. As a consequence of the entropy inequality, we have that \(k\) is a positive constant.

By a plane harmonic wave in a thermoelastic material with voids, we mean a set of functions \((u_r, \varphi, \Theta)\) on \(\mathbb{R}^3 \times (-\infty, \infty)\) of the following form:

\[
\begin{align*}
    u_r(x, t) &= \text{Re}\{U_r \exp[i(n_r x_r - ct)]\}, \\
    \varphi(x, t) &= \text{Re}\{\Phi \exp[i(n_\varphi x_\varphi - ct)]\}, \\
    \Theta(x, t) &= \text{Re}\{\Theta \exp[i(n_\Theta x_\Theta - ct)]\},
\end{align*}
\]  

(5.2)

and

\[
\begin{align*}
    \n_r &= \frac{1}{2} (\mu - \varphi c^2) \delta_{ij} + (\lambda + \mu) n_r n_j U_j - i \beta n_r \Phi + i b n_r \Theta = 0, \\
    \n_j &= \frac{1}{2} (\lambda + \varphi c^2) \delta_{ij} + (\mu - \varphi c^2) \Phi - m \Theta = 0, \\
    \i_r &= \frac{1}{2} (\lambda + \varphi c^2) \delta_{ij} + (\mu - \varphi c^2) \Theta = 0, \\
    T_0 b c n_j U_j - i T_0 m c \Phi + (k - i T_0 a c) \Theta &= 0.
\end{align*}
\]  

(5.3)

This linear algebraic system admits a non-zero solution \((U_1, U_2, U_3, \Phi, \Theta)\) if and only if \(c\) is a solution of the following equation:

\[
\begin{align*}
    (\varphi c^2 - \mu)^2 \begin{vmatrix}
        \lambda + 2\mu - \varphi c^2 & -i\beta & ib \\
        i\beta & \xi + \alpha - \varphi c^2 & -m \\
        T_0 b c & -i T_0 m c & k - i T_0 a c
    \end{vmatrix} = 0,
\end{align*}
\]  

(5.4)

that is,

\[
\begin{align*}
    (\varphi c^2 - \mu)^2 (i \gamma_5 c^5 + i \gamma_4 c^4 + i \gamma_3 c^3 + i \gamma_2 c^2 + i \gamma_1 c + \gamma_0) &= 0,
\end{align*}
\]  

(5.5)

where

\[
\begin{align*}
    \gamma_5 &= -a T_0 c \xi^2, \\
    \gamma_4 &= k \xi q^2, \\
    \gamma_3 &= T_0 q \{m^2 + a \xi + b^2 \xi + a[\alpha + \xi(\lambda + 2\mu)]\}, \\
    \gamma_2 &= -k q \{\xi + \alpha + \xi(\lambda + 2\mu)\}, \\
    \gamma_1 &= -T_0 \{-2 bm \beta + m^2 (\lambda + 2\mu) + b^2 \xi + a[-\beta^2 + (\lambda + 2\mu) \xi] \\
    &\quad + \alpha[b^2 + a(\lambda + 2\mu)]\}, \\
    \gamma_0 &= k[-\beta^2 + (\lambda + 2\mu) (\xi + \alpha)].
\end{align*}
\]  

(5.6)

and

\[
\begin{align*}
    T_0 &= k[-\beta^2 + (\lambda + 2\mu) (\xi + \alpha)].
\end{align*}
\]  

(5.7)

Obviously, the above equation has the double root

\[
c = c_1 = c_2 = \sqrt{\frac{\lambda}{\varphi}}.
\]  

(5.8)

For these values of c, we have the following solution of system (5.3):

\[
(U_1, U_2, U_3, \Phi, \Theta) = (- (n_2 + n_3), n_1, n_1, 0, 0) \quad \text{if } n_1 \neq 0,
\]

\[
(U_1, U_2, U_3, \Phi, \Theta) = (n_2, -(n_1 + n_3), n_2, 0, 0) \quad \text{if } n_2 \neq 0
\]

and

\[
(U_1, U_2, U_3, \Phi, \Theta) = (n_3, n_3, -(n_1 + n_2), 0, 0) \quad \text{if } n_3 \neq 0.
\]  

(5.8)

It is easy to see that these progressive waves are transverse waves. If we take into consideration the case of transverse waves \((U \cdot \mathbf{n} = 0)\), then from equations (5.3) we find that the speeds defined by equation (5.7) are the only admissible speeds for this type of waves.

In the case of longitudinal waves, we have \(U \cdot \mathbf{n} = U\), where \(U = |U|\). For the case of longitudinal waves the system (5.3) becomes

\[
(\lambda + 2\mu - \varrho c^2)U - i\beta \Phi + ib\Theta = 0,
\]

\[
i\beta U + (\alpha + \xi - \varrho \chi c^2)\Phi - m\Theta = 0
\]

and

\[
T_0bcU - iT_0mc\Phi + (k - iT_0ac)\Theta = 0.
\]

(5.9)

This system has a non-trivial solution if and only if \(c\) is a solution of the equation

\[
i\gamma_5 c^5 + \gamma_4 c^4 + i\gamma_3 c^3 + \gamma_2 c^2 + i\gamma_1 c + \gamma_0 = 0.
\]  

(5.10)

If \(b \neq 0\) or \(m \neq 0\), then the solutions of this equation are not real numbers. In fact, if we suppose that there is a real solution of the above equation, then it has to satisfy both equations below

\[
c(\gamma_5 c^4 + \gamma_3 c^2 + \gamma_1) = 0
\]

and

\[
\gamma_4 c^4 + \gamma_2 c^2 + \gamma_0 = 0.
\]  

(5.11)

But the poroelastic material is strongly elliptic and thus we have that \(c = 0\) cannot be the solution of equation (5.11). Actually, for \(b \neq 0\) or \(m \neq 0\), the equation (5.11) do not have any common solution. In consequence, in this case, equation (5.10) has only the solutions in \(\mathbb{C} \setminus \mathbb{R}\).

If \(b \to 0\) and \(m \to 0\), then equations (5.11) have two common solutions. This case will be discussed as a particular case.

In view of the above discussion, we can conclude as follows:

**Proposition 5.1.** For a thermoelastic isotropic material with voids for which \(b \neq 0\) or \(m \neq 0\), the progressive wave defined in equation (5.2) is either one undamped transverse wave propagating with the speeds \(c = c_1 = c_2\) or there are three damped longitudinal waves propagating with the speeds \(c\) which are the solutions of the equation (5.10).

In the linear theory of thermoelastic materials with voids, there are three coupling coefficients: \(\beta, b\) and \(m\). The pair \((b, m)\) couples equation (5.1) describing the poroelastic behaviour of the material with the heat equation (5.1), while the pair \((\beta, b)\) couples equation (5.1) which describes purely the elastic behaviour of material with the porosity and heat equation (5.1).
In what follows, we use the convergence results established by Chiriță & Ciarletta (2008), in order to discuss how the thermoelastic progressive wave behaves as the coupling coefficients $\beta$, $b$ and $m$ tend to zero. This discussion helps us to compare the results obtained in thermoporoelastic theory with those established in the poroelastic theory and also with the results from the classical elasticity.

Case (i): $b \to 0$, $m \to 0$ and $\beta \neq 0$

In this case, the possible propagating speeds are

\[
\begin{align*}
&c_1^2 = c_2^2 = \frac{\mu}{\rho}, \\
&c_3^2 = \frac{1}{2Q\kappa} \left\{ \xi + \alpha + \chi(\lambda + 2\mu) + \sqrt{[\xi + \alpha - \chi(\lambda + 2\mu)]^2 + 4\chi^2} \right\}, \\
&c_4^2 = \frac{1}{2Q\kappa} \left\{ \xi + \alpha + \chi(\lambda + 2\mu) - \sqrt{[\xi + \alpha - \chi(\lambda + 2\mu)]^2 + 4\chi^2} \right\}, \\
&\text{and } c_5 = -\frac{ik}{aT_0}.
\end{align*}
\]

For the first four speeds, we retrieve the poroelastic case discussed in the §4a. The thermal wave is missing for these speeds and the poroelastic waves are not affected by the thermal field.

For the fifth speed, we can observe that we cannot have a poroelastic wave but we have a damped thermal wave which is not affected by the poroelastic field.

This behaviour is possible because the equations are uncoupled. In the coupled case, the thermal field affects the poroelastic waves and the poroelastic field affects the thermal waves.

Case (ii): $\beta \to 0$, $b \to 0$ and $m \neq 0$

In this uncoupled case, the possible propagating speeds are

\[
\begin{align*}
&c_1^2 = c_2^2 = \frac{\mu}{\rho}, \\
&c_3^2 = \frac{\lambda + 2\mu}{\rho}, \\
&c_4 = \frac{i}{3aT_0} \left[ -k + \frac{1}{xQ} \left( \frac{\sqrt{2}p}{q} - \frac{q}{\sqrt{2}} \right) \right], \\
&c_5 = \frac{i}{3aT_0} \left[ -k - \frac{1}{2xQ} \left( \frac{\sqrt{2}(1 + i\sqrt{3})p}{q} - \left( 1 - i\sqrt{3} \right) \frac{q}{\sqrt{2}} \right) \right], \\
&\text{and } c_6 = \frac{i}{3aT_0} \left[ -k - \frac{1}{2xQ} \left( \frac{\sqrt{2}(1 - i\sqrt{3})p}{q} - \left( 1 + i\sqrt{3} \right) \frac{q}{\sqrt{2}} \right) \right].
\end{align*}
\]

The waves with the propagating speed $c_1^2$ are discussed in the general case. Let us now consider the progressive wave corresponding to the speed $c_3^2$. For this speed, up to a scalar factor, we find

$$ (U_1, U_2, U_3, \Phi, \Theta) = (n_1, n_2, n_3, 0, 0). $$

(5.15)

We can observe that we retrieve the longitudinal progressive wave from the classical elasticity. This fact is a consequence of the decoupling of equations which describe the mechanical behaviour by the porosity equation and heat equation. Regarding the progressive waves which propagate with the speeds $c_{4,5,6}$, we obtain

$$ (U_1, U_2, U_3, \Phi^{(4,5,6)}, \Theta) = (0, 0, 0, k - i T_0 a c_{4,5,6}, T_0 m c_{4,5,6}). $$

(5.16)

These are porothermal waves which are not affected by the classical mechanical field but are affected by the presence of voids.

**Case (iii):** $\beta \to 0$, $b \to 0$ and $m \to 0$

This is a fully uncoupled case. In this case, the waves are purely mechanical waves with the speeds

$$ c_1^2 = c_2^2 = \frac{\mu}{\rho} $$

(5.17)

and

$$ c_3^2 = \frac{\lambda + 2\mu}{\rho} $$

or purely porous waves with the speed

$$ c_4 = \frac{\alpha + \xi}{\rho \kappa} $$

(5.18)

or purely thermal waves with the speed

$$ c_5 = -\frac{i k}{a T_0}. $$

(5.19)

Moreover, each field (mechanical, porosity, thermal field) do not affect the waves from the other fields.

### 6. Concluding remarks

The main purpose of this paper was to study the strong ellipticity condition in the theory of materials with voids. To this end, the strong ellipticity condition was formulated for general anisotropic elastic materials with voids and relevant
results were reported concerning the explicit necessary and sufficient conditions characterizing the strong ellipticity of the isotropic and transversely isotropic elastic materials with voids. We have to outline that similar results can be obtained for other elastic materials with voids with special symmetries as, for example, those with rhombic symmetry.

Supposing the material to be strongly elliptic, we studied the propagation of progressive waves in isotropic and transversely isotropic poroelastic bodies. It is shown that there are only transverse and longitudinal undamped waves and the explicit expressions are presented for the speeds of propagation. In the context of strongly elliptic poroelastic materials, we can also give a good description of the surface waves. The analysis of the behaviour of surface waves in poroelastic materials will be treated in a forthcoming paper.

In the last section, we studied the propagation of harmonic plane waves in isotropic thermoelastic materials with voids. Then, there are two undamped transverse harmonic waves and three damped longitudinal harmonic waves. When the thermal effect is taken into consideration, there are three coefficients coupling the equations that describe purely the elastic behaviour of material with the porosity and heat equations. In the fully coupled case, we give a characterization result and then we show how the coupling coefficients affect the propagation of harmonic plane waves. This discussion proves that the present results are in concordance with the results from the classical elasticity and also with the results established for poroelastic materials.

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