Saint-Venant decay rates for an inhomogeneous isotropic linear thermoelastic strip

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\begin{abstract}
In this paper we consider the state of plane strain in an isotropic and inhomogeneous thermoelastic material occupying a rectangular strip. Such a strip is maintained in equilibrium under self-equilibrated traction applied on one of the heated edges, while the other three edges are thermally insulated and traction-free. Our aim is to derive some explicit spatial estimates describing how certain appropriate measures of the Airy stress function and temperature evolve with respect to the distance from the loaded and heated edge, provided specific assumptions are made upon the derivatives of the thermoelastic coefficients. The results of the present paper prove how the spatial decay rate varies with the inhomogeneous constitutive profile.
\end{abstract}

\section{Introduction}

There is an extensive literature on the question of spatial behavior of transient and steady-state solutions within the context of various linear theories of thermoelasticity. Much of the work in this area is referenced in the survey articles of Horgan [1,2].

It is well known that (see, for example, Carlson [3]), within the context of the linear thermoelastostatics, the basic system of differential equations decouples. That means we have a second-order partial differential equation for the temperature only and a partial differential system for the displacement vector with a body force depending on the temperature field which can be assumed to be known.

On the other hand, the areas where the biharmonic equation has proved most useful is that of two-dimensional elasticity: essentially, the determination of the stress components in the context of plane strain and of generalized plane stress for a homogeneous or inhomogeneous isotropic elastic material is reducible to the solution of this equation under suitable boundary conditions (in the case of a simply connected region). The biharmonic equation is also important for slow Stokes, viscous plane flow, anti-plane shear deformations and elastic plates.

Spatial decay estimates in two-dimensional homogeneous elasticity date back to the seminal papers of Knowles [4,5]. Many studies are dedicated to Saint-Venant type decay estimates for solutions of the biharmonic equation in a finite or semi-infinite strip in the two-dimensional Euclidean space (see, for example, Flavin [6], Oleinik and Yosifian [7,8]). Additional results are obtained by Payne and Schaeffer [9] for unbounded regions by establishing Phragmén–Lindelöf type growth decay estimates.

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The corresponding issues for (smoothly varying) inhomogeneous isotropic elastic material lead to a generalized biharmonic equation (see, for example, Flavin [10]). It should be noted that the smoothly varying inhomogeneous elastic materials considered there provide a model for technologically important FGMs – functionally graded materials.

This paper considers a rectangular strip consisting of smoothly varying inhomogeneous, isotropic thermoelastic material in an equilibrium state of plane strain, three of its edges being thermally insulated and traction-free and the remaining one – corresponding to \( x_1 = 0 \) – being heated and subjected to a (necessarily) self-equilibrated (in plane) load. The formulation of the problem is given in terms of the Airy stress function and of the temperature. That means we have a second-order partial differential equation for the temperature and a partial differential equation of fourth-order for the Airy stress function depending on the temperature which can be assumed to be known. To these equations we have to associate the boundary conditions stated above. Our aim is to derive some explicit spatial estimates describing how certain measures of the Airy stress function and the temperature evolve with respect to the distance from the heated and loaded edge, provided the derivatives of the thermoelastic coefficients satisfy the stated assumptions.

For the Laplace’s equation (corresponding to a homogeneous and isotropic conductor), the energetic methods furnish an optimal estimate for the decay rate (see, for example, Horgan and Knowles [11]) in the sense that this coincides with that given by the exact solution. In this connection the estimated decay rate for the steady-state heat conduction for inhomogeneous materials is not a best possible estimate of the decay rate. Exponential decay results, with optimal decay rate, for the analogous of biharmonic equation for an inhomogeneous rectangular strip have not obtained. In this connection, due to the complex mathematical problems arising in the field of functionally graded materials, numerical methods have been carried out. So, it appears necessary to develop analytical methods for studying the properties of solutions for such problems. Our results in the present paper provide some lower bounds for the actual decay rates for the plane steady-state heat conduction and for the plane strain of thermoelastic states in inhomogeneous materials. The results are illustrated with definite choices for the constitutive coefficients. The examples also serve to clarify assumptions satisfied by the thermoelastic coefficients.

Lupoli [12] has derived Phragmén–Lindelöf type results for a non-prismatic cylinder consisting of an anisotropic, compressible, inhomogeneous thermoelastic material, subject to either null tractions or null displacements on the lateral boundary and loaded by a self-equilibrated force system at one end. However, there was used the theory developed by Knops [13] for three-dimensional problems.

### 2. State of thermoelastic plane strain

Throughout this section, \( \Sigma \) is a bounded regular region of two-dimensional Euclidean space. We let \( \partial \Sigma \) denote the closure of \( \Sigma \), call \( \partial \Sigma \) the boundary of \( \Sigma \), and designate by \( \mathbf{n} \) the outward unit normal on \( \partial \Sigma \). We assume that the body occupying \( \Sigma \) is an isotropic and inhomogeneous linearly thermoelastic solid. The body is referred to a fixed system of rectangular Cartesian axes \( O x_\alpha \ (\alpha = 1, 2) \). Throughout this paper, Greek indices have the range 1, 2 and the usual summation convention is employed. We use subscripts preceded by a comma to denote partial differentiation with respect to the corresponding coordinate.

We consider the plane strain, parallel to the \( x_1, x_2 \)-plane, characterized by

\[
\begin{align*}
   u_1 &= u_1(x_1, x_2), & u_2 &= 0, & \theta &= \theta(x_1, x_2), & (x_1, x_2) &\in \Sigma,
\end{align*}
\]

where \( u_1, u_2 \) and \( u_3 \) are the components of the displacement vector and \( \theta \) is the temperature.

The fundamental system of equations for the plane linear theory of thermoelastostatics with zero supply terms consists of the equilibrium equations

\[
S_{\beta \alpha, \beta} = 0,
\]

the heat equation

\[
q_{\alpha, \alpha} = 0,
\]

the constitutive equations

\[
S_{\alpha \beta} = \lambda \epsilon_{\rho \delta \delta \alpha \beta} + 2\mu \epsilon_{\alpha \beta} + m \theta \delta_{\alpha \beta},
\]

\[
q_{\alpha} = k \theta_{, \alpha},
\]

and the geometric equations

\[
e_{\alpha \beta} = \frac{1}{2} (u_{\alpha, \beta} + u_{\beta, \alpha}).
\]

Here, we have used the notation: \( S_{\alpha \beta} \) are the components of the symmetric stress tensor, \( q_{\alpha} \) are the components of the heat flux vector, \( \lambda, \mu, m \) and \( k \) are thermoelastic coefficients depending on the spatial variables \( x_1 \) and \( x_2 \). Throughout this paper we shall assume that the thermoelastic coefficients \( \lambda \) and \( \mu \) are of class \( C^2(\Sigma) \), and \( m \) and \( k \) are of class \( C^4(\Sigma) \) and, moreover, we suppose
\[ \mu > 0, \quad \lambda + \mu > 0, \]
\[ (2.7) \]

and
\[ k > 0. \]
\[ (2.8) \]

The boundary tractions acting at a point \( x \) on the curve \( \partial \Sigma \) are given by
\[ s_\alpha = S_\beta \alpha n_\beta, \]
\[ (2.9) \]

while the heat flux at \( x \) is
\[ h = q_\alpha n_\alpha, \]
\[ (2.10) \]

where \( n_\alpha = \cos(\mathbf{n}, x_\alpha) \) and \( \mathbf{n} \) is the unit vector of the outward normal to \( \partial \Sigma \).

For later convenience we note that the state of plane strain satisfies the following compatibility condition
\[ e_{11,22} + e_{22,11} = 2e_{12,12}, \]
\[ (2.11) \]

Moreover, on the basis of the assumption (2.7), we can write the constitutive equation (2.4) in the following convenient form
\[ e_{11} = \varepsilon S_{11} - \nu S_{22} - M \theta, \]
\[ e_{22} = -\nu S_{11} + \varepsilon S_{22} - M \theta, \]
\[ e_{12} = (\varepsilon + \nu) S_{12}, \]
\[ (2.12) \]

where the thermoelastic moduli \( \varepsilon \) and \( \nu \) are related to the Lamé moduli by means of [10]
\[ \varepsilon = \frac{\lambda + 2\mu}{4\mu(\lambda + \mu)}, \quad \nu = \frac{\lambda}{4\mu(\lambda + \mu)}, \]
\[ (2.13) \]

and
\[ M = \frac{m}{2(\lambda + \mu)}. \]
\[ (2.14) \]

Obviously, in view of the assumption (2.7), we have
\[ \varepsilon > 0. \]
\[ (2.15) \]

Moreover, we can see that \( \varepsilon \) and \( \nu \) are of class \( C^4(\Sigma) \), while \( M \) and \( k \) are of class \( C^2(\Sigma) \).

### 3. Formulation of problem

Throughout this paper, we consider \( \Sigma \) to be the rectangular region \( R : 0 < x_1 < L, \ 0 < x_2 < \ell \ (\ell < \infty) \) occupied by an isotropic inhomogeneous thermoelastic material in an equilibrium state of plane strain subject to zero source terms. The edges \( x_1 = L, x_2 = 0, \ell \) are thermally insulated and traction-free, while the remaining edge is heated and subject to a (necessarily) self-equilibrated load. Sometimes a semi-infinite rectangular strip is admitted \( (L \rightarrow \infty) \). We adopt the following notation: \( R_{x_1} \) denotes the sub-rectangle between abscissae \( (x_1, L) \), \( R_0 \) denoting the entire rectangle.

Then the heat equation (2.3) and the constitutive equation (2.5) give
\[ (k \theta)_\alpha = 0 \quad \text{in } R, \]
\[ (3.1) \]

while the thermal boundary conditions can be written as
\[ \theta(x_1, 0) = 0, \quad \theta(x_1, \ell) = 0, \quad x_1 \in [0, L], \]
\[ \theta(L, x_2) = 0, \quad x_2 \in [0, \ell], \]
\[ (3.2, 3.3) \]

in the case of a finite strip. In the limiting case when \( L \rightarrow \infty \) the condition (3.3) is substituted by the hypothesis that the thermal energy of the infinite strip to be finite. Moreover, on the edge \( x_1 = 0 \), we assume
\[ \theta(0, x_2) = g(x_2), \quad x_2 \in [0, \ell], \]
\[ (3.4) \]

where \( g \) is a prescribed function on \( [0, \ell] \).

In terms of Airy's stress function, the (relevant) stress components \( S_{11}, S_{12} \) and \( S_{22} \) are given by
\[ S_{11} = A_{22}, \quad S_{22} = A_{11}, \quad S_{12} = -A_{12}. \]
\[ (3.5) \]
Then, the relations (2.2), (2.11), (2.12) and (3.5) lead to the following differential equation in terms of the Airy stress function \( A(x_1, x_2) \) and temperature \( \theta(x_1, x_2) \),

\[
(\varepsilon A_{11}, A_{11})_{11} + 2(\varepsilon A_{12}, A_{12})_{12} + (\varepsilon A_{22}, A_{22})_{22} - v_{12} A_{11} - v_{11} A_{22} + 2v_{12} A_{12} - \Delta(M \theta) = 0,
\]

in \( R \), where \( \Delta \) is the Laplace operator in two dimensions. Moreover, the stated boundary conditions can be expressed in terms of the Airy’s stress function as (see, for example, Gurtin [14], p. 156)

\[
A(x_1, 0) = 0, \quad A_2(x_1, 0) = 0,
\]

\[
A(x_1, \ell) = 0, \quad A_2(x_1, \ell) = 0, \quad x_1 \in [0, L],
\]

and

\[
A(L, x_2) = 0, \quad A_1(L, x_2) = 0, \quad x_2 \in [0, \ell],
\]

in the case of a finite strip. In the limiting case when \( L \to \infty \) condition (3.8) is substituted by the hypothesis that a certain energetic integral to be finite. On the edge \( x_1 = 0 \) we assume

\[
A(0, x_2) = f_1(x_2), \quad A_1(0, x_2) = f_2(x_2), \quad x_2 \in [0, \ell].
\]

where \( f_1 \) and \( f_2 \) are prescribed functions on \([0, \ell]\) chosen to ensure the loads applied on the edge \( x_1 = 0 \) to be equilibrated.

Throughout this paper we assume that \( A \) and \( \theta \) are sufficiently smooth functions (e.g. \( A \in C^4(R), \theta \in C^2(R) \)).

4. Spatial behavior for the temperature

We first establish a spatial estimate for the temperature. We form the identity

\[
\theta(k\theta, \varepsilon)_\varepsilon = 0
\]

which can be written as

\[
k\theta, \varepsilon\theta, \varepsilon = (k\theta \theta, \varepsilon)_\varepsilon.
\]

By using the boundary condition (3.2), we obtain the following identity

\[
\int_0^\ell k(\theta_1^2 + \theta_2^2) \, dx_2 = \frac{d}{dx_1} \int_0^\ell k\theta_1 \, dx_2.
\]

So we are led to introduce the function

\[
I(x_1) = -\int_0^\ell k\theta_1 \, dx_2, \quad x_1 \in [0, L],
\]

and note that

\[
\frac{dI}{dx_1}(x_1) = -\int_0^\ell k(\theta_1^2 + \theta_2^2) \, dx_2 \leq 0 \quad \text{for all} \ x_1 \in [0, L]
\]

and hence \( I(x_1) \) is a non-increasing function with respect to \( x_1 \) on the interval \([0, L]\).

In view of the end boundary condition (3.3), it follows that \( I(L) = 0 \) and hence we have \( I(x_1) \geq 0 \) for all \( x_1 \in [0, L] \). Moreover, by using relation \( I(L) = 0 \) and (4.5), we obtain

\[
I(x_1) = \int_{Kx_1} k(\theta_1^2 + \theta_2^2) \, da.
\]

Thus, \( I(x_1) \) appears as a measure of the temperature, that is \( I(x_1) \geq 0 \) and \( I(x_1) = 0 \) implies \( \theta(x_1, x_2) = 0 \) in \( R \).

On the other hand, in view of the boundary condition (3.2), we have the Wirtinger type inequality [15]

\[
\int_0^\ell \theta_2^2 \, dx_2 \geq \frac{\pi^2}{L^2} \int_0^\ell \theta_2^2 \, dx_2.
\]
and hence
\[
\int_0^\ell k\theta_2^2 \, dx_2 \geq \frac{k_m(x_1)}{k_M(x_1)} \frac{\pi^2}{\ell^2} \int_0^\ell k\theta^2 \, dx_2, \tag{4.8}
\]
where
\[
k_m(x_1) = \min_{x_2 \in [0, \ell]} k(x_1, x_2), \quad k_M(x_1) = \max_{x_2 \in [0, \ell]} k(x_1, x_2).
\tag{4.9}
\]
Therefore, from (4.4), by means of the arithmetic–geometric mean inequality and (4.8), we obtain
\[
l(x_1) \leq \frac{\ell}{2\pi} \sqrt{\frac{k_m(x_1)}{k_M(x_1)}} \int_0^\ell k(\theta_1^2 + \theta_2^2) \, dx_2 \quad \text{for all } x_1 \in [0, L]. \tag{4.10}
\]
Thus, we are led to the first-order differential inequality
\[
\frac{dl}{dx_1}(x_1) + \frac{2\pi}{\ell} \sqrt{\frac{k_m(x_1)}{k_M(x_1)}} l(x_1) \leq 0 \quad \text{for all } x_1 \in [0, L], \tag{4.11}
\]
which, after integration, furnishes the following spatial estimate
\[
0 \leq l(x_1) \leq l(0) \exp \left( -\frac{2\pi}{\ell} \int_0^{x_1} \sqrt{\frac{k_m(\tau)}{k_M(\tau)}} \, d\tau \right) \quad \text{for all } x_1 \in [0, L]. \tag{4.12}
\]
Note that when the conductivity coefficient \(k\) is constant (homogeneous conductor) or it depends only on \(x_1\), then the estimated decay rate is independent of \(k\) and is \(\frac{2\pi}{\ell}\).

5. Spatial behavior for the Airy stress function

Throughout this section we will study the spatial behavior of the Airy stress function that satisfies (3.6) subject to boundary conditions (3.7)–(3.9). To this end, we will establish restrictions upon the thermoelastic coefficients that allow us to establish such spatial behavior. We first use Eq. (3.1) into (3.6) to obtain
\[
(\epsilon \cdot A, _{11})_{11} + 2(\epsilon \cdot A, _{12})_{12} + (\epsilon \cdot A, _{22})_{22} - \nu_{12} A, _{11} - \nu_{11} A, _{22} + 2\nu_{12} A, _{12} + \left( \frac{M}{k} k_{, \alpha} - 2M, _{, \alpha} \right) \theta, _{, \alpha} - \theta \Delta M = 0, \quad \text{in } \mathbb{R}. \tag{5.1}
\]
Further, we multiply (5.1) by \(A\) to obtain the following identity
\[
\epsilon (A, _{11}^2 + 2A, _{12} + A, _{22}) + \nu_{12} A, _{11}^2 + \nu_{11} A, _{22}^2 - 2\nu_{12} A, _{11} A, _{22} + \left( \frac{M}{k} k_{, \alpha} - 2M, _{, \alpha} \right) A\theta, _{, \alpha} - A\theta \Delta M
\]
\[
= -2\epsilon A, _{12} + \nu_{12} A, _{12}^2 + \left( -\epsilon A, _{11} + \epsilon A, _{12} + \epsilon A, _{22} + \frac{1}{2} \nu_{12} A, _{12}^2 \right)_{, 11} - \left[ \epsilon, _{, 1} (A, _{11}^2 + A, _{12}^2) \right]_{, 1}
\]
\[
+ \left( -\epsilon A, _{22} + \epsilon A, _{22} + \epsilon A, _{22} + \frac{1}{2} \nu_{12} A, _{22}^2 \right)_{, 22} - \left[ \epsilon, _{, 2} (A, _{11}^2 + A, _{12}^2) \right]_{, 2}. \tag{5.2}
\]
Furthermore, we integrate the identity (5.2) with respect to \(x_2\) over the interval \([0, \ell]\) and then we take into account the boundary condition (3.7) in order to obtain
\[
\int_0^\ell \left[ \epsilon (A, _{11}^2 + 2A, _{12} + A, _{22}) + \nu_{12} A, _{11}^2 + \nu_{11} A, _{22}^2 - 2\nu_{12} A, _{11} A, _{22} + \left( \frac{M}{k} k_{, \alpha} - 2M, _{, \alpha} \right) A\theta, _{, \alpha} - A\theta \Delta M \right] \, dx_2
\]
\[
= \frac{d^2}{dx_1^2} \int_0^\ell \left( -\epsilon A, _{11} + \epsilon A, _{12} + \epsilon A, _{22} + \frac{1}{2} \nu_{12} A, _{12}^2 \right) \, dx_2 - \frac{d}{dx_1} \int_0^\ell \epsilon, _{, 1} (A, _{11}^2 + A, _{12}^2) \, dx_2 \quad \text{for all } x_1 \in [0, L]. \tag{5.3}
\]
At this instant we combine the identities (4.3) and (5.3) and so we can give
\[
\int_0^\ell \left[ \varepsilon (A_{11}^2 + 2A_{12}^2 + A_{22}^2) + v_{12}A_{11}^2 + u_{11}A_{12}^2 - 2v_{12}A_{1}A_{2} + \left( \frac{M}{k}k_{,\alpha} - 2M_{,\alpha} \right) A_{\theta,\alpha} - A_{\theta} A_{\Delta M} \right. \\
+ \omega k(\theta_1^2 + \theta_2^2) \left. \right] dx_2 \\
= \frac{d^2}{dx_1^2} \int_0^\ell \left( -\varepsilon AA_{11} + \varepsilon A_{11}^2 + \varepsilon A_{22}^2 + \frac{1}{2} v_{22}A^2 \right)dx_2 - \frac{d}{dx_1} \int_0^\ell \left[ \varepsilon_{11}(A_{11}^2 + A_{22}^2) - \omega k\theta_{11} \right] dx_2
\]
for all \( x_1 \in [0, L] \), (5.4)

where \( \omega \) is a positive parameter at our disposal. This identity allows us to introduce the function
\[
J(x_1) = \int_0^\ell \left( -\varepsilon AA_{11} + \varepsilon A_{11}^2 + \varepsilon A_{22}^2 + \frac{1}{2} v_{22}A^2 \right)dx_2 + \int_{R_{x_1}} \left[ \varepsilon_{11}(A_{11}^2 + A_{22}^2) - \omega k\theta_{11} \right] da
\]
for all \( x_1 \in [0, L] \), (5.5)

By direct derivation in (5.5), we obtain
\[
\frac{dJ}{dx_1}(x_1) = \int_0^\ell \left( -\varepsilon AA_{11} + \varepsilon A_{11}^2 + 2\varepsilon A_{12}A_{12} + \omega k\theta_{11} \right) dx_2 + \int_0^\ell \left( -\varepsilon AA_{11} + \frac{1}{2} v_{22}A^2 + v_{22}AA_{11} \right) dx_2
\]
for all \( x_1 \in [0, L] \), (5.6)

while the identity (5.4) implies
\[
\frac{d^2J}{dx_1^2}(x_1) = \int_0^\ell \left[ \varepsilon (A_{11}^2 + 2A_{12}^2 + A_{22}^2) + v_{12}A_{11}^2 + u_{11}A_{12}^2 - 2v_{12}A_{1}A_{2} + \left( \frac{M}{k}k_{,\alpha} - 2M_{,\alpha} \right) A_{\theta,\alpha} \\
- A_{\theta} A_{\Delta M} + \omega k(\theta_1^2 + \theta_2^2) \right] dx_2 \quad \text{for all } x_1 \in [0, L].
\]

Now, we take into account that \( A \in C^2(R) \) and \( A = A_1 = A_2 = 0 \) on the edges \( x_2 = 0, \ell \), in order to write the following three Wirtinger type inequalities (see, e.g. Horgan [15])
\[
\int_0^\ell A_{11}^2 dx_2 \geq \frac{\pi^2}{\ell^2} \int_0^\ell A^2 dx_2, \quad \int_0^\ell A_{12}^2 dx_2 \geq \frac{\pi^2}{\ell^2} \int_0^\ell A_{11}^2 dx_2, \quad \int_0^\ell A_{22}^2 dx_2 \geq \frac{4\pi^2}{\ell^2} \int_0^\ell A_{12}^2 dx_2.
\]

Moreover, we have (see, e.g. Knowles [5], Flavin [6], Horgan and Knowles [11])
\[
\int_0^\ell (A_{11} + 2A_{12} + A_{22}) dx_2 \geq \frac{2\pi^2}{\ell^2} \int_0^\ell |A_{11} + A_{22}| dx_2.
\]
By using the Schwarz and arithmetic–geometric mean inequalities and the above inequalities, we get

\[
\left| \int_0^\ell \left( v_{22}A_{11}^2 + v_{11}A_{22}^2 - 2v_{12}A_{11}A_{22} \right) dx_2 \right|
\]

\[
\leq \int_0^\ell \left| \left[ (|v_{22}| + |v_{12}|)A_{11}^2 + (|v_{11}| + |v_{12}|)A_{22}^2 \right] \right| dx_2
\]

\[
\leq (|v_{22}| + |v_{12}|)_M \int_0^\ell A_{11}^2 dx_2 + (|v_{11}| + |v_{12}|)_M \int_0^\ell A_{22}^2 dx_2
\]

\[
\leq (|v_{22}| + |v_{12}|)_M \frac{\ell^2}{\pi^2} \int_0^\ell A_{12}^2 dx_2 + (|v_{11}| + |v_{12}|)_M \frac{\ell^2}{4\pi^2} \int_0^\ell A_{22}^2 dx_2,
\]

(5.12)

\[
\left| \int_0^\ell \left( \frac{M_k}{k} k_{\alpha} - 2M_{\alpha} \right) A_{\theta,\theta} dx_2 \right|
\]

\[
\leq \frac{M_{km}}{2\sqrt{\varepsilon_m k_m}} \int_0^\ell (A_{\theta,\theta}^2 + k_m\theta,\theta) dx_2
\]

\[
\leq \frac{M_{km} \ell^4}{8\pi^4 \sqrt{\varepsilon_m k_m}} \int_0^\ell A_{22}^2 dx_2 + \frac{M_{km}}{2\sqrt{\varepsilon_m k_m}} \int_0^\ell k\theta,\theta dx_2,
\]

(5.13)

and

\[
\left| \int_0^\ell A_{\theta} \Delta M dx_2 \right|
\]

\[
\leq \frac{|\Delta M|_M}{2\sqrt{\varepsilon_m k_m}} \left( \int_0^\ell (A_{\theta,\theta}^2 + k_m\theta,\theta) dx_2 \right) \leq \frac{|\Delta M|_M \ell^4}{8\pi^4 \sqrt{\varepsilon_m k_m}} \int_0^\ell A_{22}^2 dx_2 + \frac{|\Delta M|_M \ell^2}{2\pi^2 \sqrt{\varepsilon_m k_m}} \int_0^\ell k\theta,\theta dx_2,
\]

(5.14)

where

\[
(|v_{22}| + |v_{12}|)_M(x_1) = \max_{x_2 \in [0, \ell]} \left( |v_{22}| + |v_{12}| \right),
\]

\[
(|v_{11}| + |v_{12}|)_M(x_1) = \max_{x_2 \in [0, \ell]} \left( |v_{11}| + |v_{12}| \right),
\]

(5.15)

\[
M_{km}(x_1) = \max_{x_2 \in [0, \ell]} \left[ \left( \frac{M_k}{k} k_{\alpha} - 2M_{\alpha} \right) \left( \frac{M_k}{k} k_{\alpha} - 2M_{\alpha} \right) \right]^{1/2},
\]

\[
|\Delta M|_M(x_1) = \max_{x_2 \in [0, \ell]} \left| \Delta M(x_1, x_2) \right|,
\]

(5.16)

\[
\varepsilon_m(x_1) = \min_{x_2 \in [0, \ell]} \varepsilon(x_1, x_2), \quad \varepsilon_M(x_1) = \max_{x_2 \in [0, \ell]} \varepsilon(x_1, x_2).
\]

(5.17)

We insert the above estimates into relation (5.7) to obtain

\[
\frac{d^2}{dx_1^2} \left| f(x_1) \right| \geq \int_0^\ell \left\{ \varepsilon_{M}A_{11}^2 + \left[ 1 - \frac{\ell^2}{2\varepsilon_m \pi^2} (|v_{22}| + |v_{12}|)^2 \right] 2\varepsilon_{A_{11}}^2 \right. \]

\[
+ \left[ 1 - \frac{\ell^2}{4\varepsilon_m \pi^2} (|v_{11}| + |v_{12}|)^2 \right] M_{km} + \frac{\ell^4}{8\pi^4 \sqrt{\varepsilon_m k_m}} \left| \Delta M \right|_M \right] \varepsilon_{A_{22}}^2 dx_2
\]

\[
\left. + \left[ \omega + \frac{\ell^2}{2\sqrt{\varepsilon_m k_m}} \right] \left( M_{km} + \frac{\ell^2}{\pi^2} \left| \Delta M \right|_M \right) k_{\theta,\theta} \right\} dx_2 \quad \text{for all } x_1 \in [0, \ell].
\]

(5.18)
Let us assume henceforward that the constitutive functions \( \nu, M \) and \( k \) are such that
\[
\left( |\nu_{122}| + |\nu_{121}| \right)_{M}(x_1) < \frac{2\pi^2}{\ell^2} \epsilon_m(x_1) \quad \text{for all} \quad x_1 \in [0, L],
\]
and
\[
\frac{\ell^4}{4\pi^2 \epsilon_m(x_1)} \left( |\nu_{111}| + |\nu_{121}| \right)_{M}(x_1) + \frac{\ell^4}{8\pi^4 \sqrt{\epsilon_m(x_1)} k_m(x_1)} (M_{km}(x_1) + |\Delta M|_{M}(x_1)) < 1 \quad \text{for all} \quad x_1 \in [0, L],
\]
for all \( x \in [0, L] \), and, furthermore, we set
\[
\xi_1(x_1) = 1 - \frac{\ell^2}{2\pi^2 \epsilon_m(x_1)} (|\nu_{122}| + |\nu_{121}|)_{M}(x_1) > 0,
\]
\[
\xi_2(x_1) = 1 - \frac{\ell^2}{4\pi^2 \epsilon_m(x_1)} (|\nu_{111}| + |\nu_{121}|)_{M}(x_1) + \frac{\ell^4}{8\pi^4 \sqrt{\epsilon_m(x_1)} k_m(x_1)} (M_{km}(x_1) + |\Delta M|_{M}(x_1)) > 0,
\]
\[
\xi_3(x_1) = \omega - \frac{1}{2\sqrt{\epsilon_m(x_1)} k_m(x_1)} \left( M_{km}(x_1) + \frac{\ell^2}{\pi^2} |\Delta M|_{M}(x_1) \right) > 0.
\]

The conditions (5.19)–(5.21) individuate the class of inhomogeneous thermoelastic materials for which our analysis works. This will become more clear in Section 7 where two classes of exponentially graded materials will be considered. If we set
\[
\zeta(x_1) = \min\{1, \xi_1(x_1), \xi_2(x_1)\},
\]
then the relation (5.18) implies
\[
\frac{d^2 J}{dx_1^2}(x_1) > \zeta(x_1) \int_0^\ell \epsilon \left( A_{111}^2 + 2A_{112}^2 + A_{222}^2 \right) dx_2 + \xi_3(x_1) \int_0^\ell \epsilon k_{\theta, \rho} \xi_3 dx_2 > 0
\]
for all \( x_1 \in [0, L] \).

By taking into account the end boundary conditions (3.3) and (3.8), from (5.5) and (5.6) we deduce that \( J(L) = 0 \) and \( (dJ/dx_1)(L) = 0 \) and hence (5.26) furnishes
\[
- \frac{d J}{dx_1}(x_1) \geq \int_{R_{x_1}} \zeta \epsilon \left( A_{111}^2 + 2A_{112}^2 + A_{222}^2 \right) \, da + \int_{R_{x_1}} \xi_3 k_{\theta, \rho} \xi_3 \, da \geq 0,
\]
\[
J(x_1) \geq \int_{x_1}^L d\eta \int_{R_{x_1}} \zeta \epsilon \left( A_{111}^2 + 2A_{112}^2 + A_{222}^2 \right) \, da + \int_{x_1}^L d\eta \int_{R_{x_1}} \xi_3 k_{\theta, \rho} \xi_3 \, da \geq 0.
\]

This last relation proves that \( J(x_1) \) represents a measure for the Airy stress function and the temperature in the sense that \( J(x_1) \geq 0 \) and \( J(x_1) = 0 \) implies that \( (A_{11}, \theta)(x_1, x_2) = 0 \) in \( R \).

We proceed now to obtain an appropriate estimate for \( J(x_1) \) in terms of \( (dJ/dx_1) \) and \( (d^2 J/dx_1^2) \). To this end we first note that the estimates (5.8)–(5.11) and relations (4.8), (5.26) and (5.27) give
\[
\left| \int_0^\ell \left( -\epsilon A_{111} + \epsilon A_{112}^2 + \epsilon A_{222}^2 + \frac{1}{2} \nu_{222} A_{222}^2 \right) dx_2 \right|
\leq \frac{|\nu_{222}|(M(x_1)) \ell^4}{8\pi^4 \epsilon_m(x_1)} \int_0^\ell \epsilon A_{222}^2 dx_2 + \frac{\epsilon M(x_1) \ell^2}{2\pi^2 \epsilon_m(x_1)} \int_0^\ell \epsilon (A_{111}^2 + 2A_{112}^2 + A_{222}^2) dx_2
\leq \sigma_1 \frac{d^2 J}{dx_1^2}(x_1),
\]
and
\[
\int_{R_{x_1}} \left[ \varepsilon,1 \left( A_{11}^2 + A_{22}^2 \right) - \omega \kappa \theta,1 \right] da \leq \frac{\ell^2}{2\pi^2} \int_{R_{x_1}} \left[ \varepsilon,1 M(x_1) \right] \varepsilon,1 (A_{11}^2 + 2A_{12}^2 + A_{22}^2) da + \frac{\omega \ell}{2\pi} \int \frac{k_M(x_1)}{k_{m}(x_1)} \kappa \theta,\mu,\nu da \leq \sigma_2 \left( -\frac{dJ}{dx_1} (x_1) \right),
\]
(5.30)

where
\[
|\varepsilon,1 M(x_1)| = \max_{x_2 \in [0,\ell]} |\varepsilon,1(x_1, x_2)|,
\]
(5.31)

and
\[
\sigma_1 = \frac{\ell^2}{2\pi^2} \max_{x_1 \in [0,\ell]} \left\{ \frac{1}{\xi(x_1) E_{m}(x_1)} \left[ E_M(x_1) + \frac{\ell^2}{4\pi^2} |\varepsilon,22| M(x_1) \right] \right\},
\]
\[
\sigma_2 = \frac{\ell}{2\pi} \max_{x_1 \in [0,\ell]} \left( \frac{\ell}{\max_{x_1 \in [0,\ell]} \xi(x_1) E_{m}(x_1)} \omega \max_{x_1 \in [0,\ell]} \frac{1}{\kappa M(x_1)} \sqrt{k_M(x_1) k_m(x_1)} \right).
\]
(5.32)

For a homogeneous thermoelastic body we see that \( \xi_1 = 1, \xi_2 = 1, \xi_3 = \omega \) and hence \( \xi = 1 \) and \( \sigma_1 = \frac{\ell^2}{2\pi^2} \) and \( \sigma_2 = \frac{\ell}{2\pi} \).

While for a genuine inhomogeneous body we have \( \xi_1 < 1, \xi_2 < 1, \frac{\omega}{\xi_3} < 1 \) and then \( \xi < 1 \). In such a case \( \xi \) and \( \frac{\omega}{\xi_3} \) can be sufficiently small and then \( \sigma_1 \) and \( \sigma_2 \) can take very large values.

In view of relations (5.5), (5.29) and (5.30) we obtain the following second-order differential inequality
\[
\frac{d^2 J}{dx_1^2} (x_1) - \frac{\sigma_2}{\sigma_1} \frac{dJ}{dx_1} (x_1) - \frac{1}{\sigma_1} J(x_1) \geq 0 \quad \text{for all } x_1 \in [0, L] .
\]
(5.33)

By the Comparison Principle (see, e.g. Flavin and Rionero [16]), from (5.33) we get
\[
0 \leq J(x_1) \leq J(0) e^{-\kappa_2 x_1} + J(L) e^{-\kappa_1 (L-x_1)} \quad \text{for all } x_1 \in [0, L] ,
\]
(5.34)

where
\[
\kappa_1 = \frac{1}{2\sigma_1} (\sigma_2 + \sqrt{\sigma_2^2 + 4\sigma_1}), \quad \kappa_2 = \frac{1}{2\sigma_1} (-\sigma_2 + \sqrt{\sigma_2^2 + 4\sigma_1}).
\]
(5.35)

Since \( J(L) = 0 \), from (5.34) we deduce the following spatial decay estimate
\[
0 \leq J(x_1) \leq J(0) e^{-\kappa_2 x_1} \quad \text{for all } x_1 \in [0, L] .
\]
(5.36)

For inhomogeneous thermoelastic materials with sufficiently large values for \( \sigma_1 \) and \( \sigma_2 \) we can see that \( \kappa_2 = \frac{2}{\sigma_2 + \sqrt{\sigma_2^2 + 4\sigma_1}} \) is sufficiently small and our estimate (5.36) falls to give valuable information concerning the spatial decay of solution.

### 6. A semi-infinite strip

In this section we discuss the case of a semi-infinite strip, that is the strip \( R: 0 < x_1 < \infty, 0 < x_2 < \ell \ (L \rightarrow \infty) \). For such a region we have to assume that \( \varepsilon \) and \( \nu \) are of class \( C^3(R) \), while \( M \) and \( k \) are of class \( C^3(R) \) and, moreover, we assume that \( \varepsilon, \nu \) and \( k \), together \( \varepsilon,1, \nu,11, \nu,12, \nu,22, M, M, k, k \) and \( \kappa \) are bounded on \( R \). Further, we assume that there exist the following integrals
\[
I(x_1) = \int_{R_{x_1}} k(\theta,1^2 + \theta,2^2) da, \quad x_1 \in [0, \infty),
\]
(6.1)

\[
J(x_1) = \int_{R_{x_1}} \left[ \varepsilon(A_{11}^2 + 2A_{12}^2 + A_{22}^2) + \nu,22 A_{11}^2 + \nu,11 A_{22}^2 - 2
\nu,12 A_{1} A_{2} + \left( \frac{M}{k} k_{1,\alpha} - 2M_{\alpha} \right) A_{1,\alpha}
- A_{\alpha} \Delta M + \omega k(\theta,1^2 + \theta,2^2) \right] da, \quad x_1 \in [0, \infty),
\]
(6.2)
\[ J^*(x_1) = \int_{x_1}^{\infty} J(\eta) \, d\eta, \quad x_1 \in [0, \infty), \]  

(6.3)

where now \( R_2 \equiv [x_1, \infty) \times [0, \ell]. \)

Then \( I(x_1) \) tends to zero as \( x_1 \to \infty \), that is \( I(\infty) = 0 \) and then the analysis of Section 4 implies

\[
0 \leq I(x_1) \leq I(0) \exp \left( -\frac{2\pi}{\ell} \int_{0}^{x_1} \sqrt{\frac{k_M(\tau)}{k_m(\tau)}} \, d\tau \right) \quad \text{for all} \quad x_1 \in [0, \infty). \tag{6.4}
\]

Moreover, \( J^*(x_1) \) is bounded above for all \( x_1 \in [0, \infty) \) and hence \( J(x_1) \) is bounded above and, in view of the assumptions (5.19)-(5.24), it follows that \( J(\infty) = 0 \) and \( (dJ/dx_1)(\infty) = 0 \). Then the analysis of Section 5 proves that

\[
0 \leq J^*(x_1) \leq J(0)e^{-\kappa x_1} \quad \text{for all} \quad x_1 \in [0, \infty). \tag{6.5}
\]

7. Application to some classes of exponentially graded materials

In this section we illustrate our above analysis for an isotropic inhomogeneous thermoelastic material with exponentially variable properties along to the directions parallel with the edges of a finite strip.

In the first example we consider a thermoelastic constitutive profile depending only on the \( x_1 \) variable characterized by

\[
\varepsilon(x_1) = E_0 e^{-px_1}, \quad \nu(x_1) = e_0 e^{-px_1}, \quad M(x_1) = m_0 e^{-px_1}, \quad k(x_1) = k_0 e^{-px_1}, \tag{7.1}
\]

where \( E_0 > 0, e_0, m_0, k_0 > 0 \) and \( p > 0 \) are prescribed parameters at our disposal. Then the decay rate of the thermal effects described in relation (4.12) is

\[
v_0 = \frac{2\pi}{\ell}. \tag{7.2}
\]

Furthermore, the hypotheses (5.19)-(5.21) assumed for the constitutive functions imply the following restrictions upon the parameters entering in (7.1)

\[
\frac{\ell^2}{4\pi^2} p \left[ \frac{|e_0|}{E_0} p + \frac{\ell^2|m_0|}{2\pi^2\sqrt{E_0 k_0}} (1 + p) \right] < 1,
\]

\[
\frac{p|m_0|}{2\sqrt{E_0 k_0}} \left( 1 + \frac{\ell^2}{\pi^2 p} \right) < \omega. \tag{7.3}
\]

Moreover, relations (5.32) furnishes

\[
\sigma_1 = \frac{\ell^2}{2\pi^2\xi_2}, \quad \sigma_2 = \frac{\ell}{2\pi} \max \left( \frac{\ell p}{\xi_2}, \frac{\omega}{\xi_3} \right), \tag{7.4}
\]

where, now we have

\[
\xi_2 = 1 - \frac{\ell^2}{4\pi^2} p \left[ \frac{|e_0|}{E_0} p + \frac{\ell^2|m_0|}{2\pi^2\sqrt{E_0 k_0}} (1 + p) \right],
\]

\[
\xi_3 = \omega - \frac{p|m_0|}{2\sqrt{E_0 k_0}} \left( 1 + \frac{\ell^2}{\pi^2 p} \right). \tag{7.5}
\]

Thus, the decay rate for such materials is \( \kappa_2 \) as calculated with the formula (5.35) in which \( \sigma_1 \) and \( \sigma_2 \) take values described in (7.4).

In the second example we consider a thermoelastic material essentially characterized by

\[
\varepsilon(x_2) = E_1 e^{-rx_2}, \quad \nu(x_2) = e_1 e^{-rx_2}, \quad M(x_2) = m_1 e^{-rx_2}, \quad k(x_2) = k_1 e^{-rx_2}, \tag{7.6}
\]

where \( E_1 > 0, e_1, m_1, k_1 > 0 \) and \( r > 0 \) are prescribed parameters at our disposal. Then the decay rate of the thermal effects described in relation (4.12) is

\[
v_0 = \frac{2\pi}{\ell} e^{-\frac{\ell}{\kappa}}. \tag{7.7}
\]
Furthermore, the hypotheses (5.19)–(5.21) assumed for the constitutive functions imply the following restrictions upon the parameters entering in (7.6)

\[
\frac{|e_1|^2}{E_1} e^{r \ell} < \frac{2\pi^2}{\ell^2},
\]

\[
|m_1| r (1 + r) < \frac{8\pi^4}{\ell^4} \sqrt{E_1 k_1} e^{-r \ell},
\]

\[
|m_1| r \left(1 + \frac{\ell^2}{\pi^2} r\right) < 2\omega \sqrt{E_1 k_1} e^{-r \ell}.
\]

(7.8)

Assuming restrictions described in (7.8), it follows that

\[
\sigma_1 = \frac{\ell^2}{2\pi^2 \zeta} \left(1 + \frac{\ell^2 |e_1|^2}{4\pi^2 E_1}\right) e^{r \ell},
\]

\[
\sigma_2 = \frac{\ell \omega}{2\pi \xi_3}.
\]

(7.9)

where now we have

\[
\zeta = \min\left(1 - \frac{\ell^2 |e_1|^2}{2\pi^2 E_1} e^{r \ell}, 1 - \frac{\ell^4 |m_1| r (1 + r)}{8\pi^4 \sqrt{E_1 k_1}} e^{-r \ell}\right),
\]

\[
\xi_3 = \omega - \frac{|m_1| r}{2\sqrt{E_1 k_1}} \left(1 + \frac{\ell^2}{\pi^2} r\right) e^{r \ell}.
\]

(7.10)

8. Concluding remarks

This paper considers a rectangular strip consisting of a smoothly varying inhomogeneous isotropic thermoelastic material in an equilibrium state of plane strain, three of whose edges are thermally insulated and traction-free, the fourth is subjected to a prescribed temperature and a self-equilibrated plane load. Inequalities are derived that estimate the spatial behavior of the solution with respect to the distance from the heated and loaded edge, and show that for a semi-infinite strip the total (mechanical and thermal) energy in a partial area decays at least exponentially with distance from the heated and loaded edge of the strip, provided this energy is assumed bounded. The decay rate depends explicitly on the material constitutive coefficients and the strip width.

It should be noted that the smoothly varying inhomogeneous thermoelastic materials considered in the present paper provide a model for the technologically important functionally graded materials (see, for example, [17] and the papers cited there). Our analysis concerning these classes of thermoelastic materials establishes spatial decay rates, under convenient hypotheses upon the thermoelastic constitutive coefficients. However, the dependence of the decay rates on the constitutive profile of the material is complicated and the class of inhomogeneous thermoelastic materials addressed in the analysis is not easily characterized (at least in the generality considered in Section 5). For particular circumstances, our analysis leads to explicit estimates concerning the decay rates, under explicit restrictions upon the constitutive profile.

The spatial estimates presented in this paper can be made fully explicit by establishing, in terms of the given data, some appropriate bounds for the “amplitude” terms. The derivation of the appropriate bounds for \(I(0)\) in (4.12) and \(J(0)\) in (5.36) in terms of the given data is more or less standard, but in the interest of completeness we include here the idea developed by Flavin [10]. Let us bound first \(I(0)\) in terms of the function \(g(x_2)\) given in (3.4). To this end we note that relation (4.5), when coupled with \(I(L) = 0\), gives

\[
I(0) = \int_{R_0}^x k(\theta_1^2 + \theta_2^2) da \geq 0.
\]

(8.1)

Let \(\phi\) and \(\psi\) be any two smooth functions having the same boundary values as \(\theta\). We define the scalar product \(\langle \cdot, \cdot \rangle\) by

\[
\langle \phi, \psi \rangle = \int_{R_0}^x k(\phi_1 \psi_1 + \phi_2 \psi_2) da
\]

(8.2)

and note that \(\langle \phi, \phi \rangle \geq 0\) and \(\langle \phi, \psi \rangle = 0\) implies \(\phi = 0\). Using the integration by parts and Eq. (3.1), we can easily find that

\[
\langle \theta, \psi \rangle = \langle \psi, \psi \rangle
\]

(8.3)

and hence, by the Schwarz’s inequality, we obtain

\[
I(0) = \langle \theta, \theta \rangle \leq \langle \psi, \psi \rangle.
\]

(8.4)
where \( \psi \) is a smooth function satisfying the same boundary conditions as \( \theta \). To bound \( J(0) \) in terms of the functions \( f_1(x_2) \) and \( f_2(x_2) \) given in (3.9) we can use a similar procedure.

The relations (4.6) and (4.12) provide estimates for the components of the heat flux vector and the thermal gradient, while (5.28) and (5.36) provide estimates for the components of the stress tensor and thermal gradient (heat flux vector). Thus, we have

\[
\int_{k_{11}}^{k_{12}} \left( q_1^2 + q_2^2 \right) \, da \leq k_M I(x_1) \leq k_M I(0) \exp \left( -\frac{2\pi}{\ell} \int_0^{x_1} \frac{k_m(\tau)}{k_M(\tau)} \, d\tau \right),
\]

(8.5)

and

\[
\int_{k_{11}}^{k_{12}} \left( \theta_1^2 + \theta_2^2 \right) \, da \leq \frac{1}{k_m} I(x_1) \leq \frac{1}{k_m} I(0) \exp \left( -\frac{2\pi}{\ell} \int_0^{x_1} \frac{k_m(\tau)}{k_M(\tau)} \, d\tau \right),
\]

(8.6)

while, by means of (4.7), we have

\[
\int_{k_{11}}^{k_{12}} \theta^2 \, da \leq \frac{\ell^2}{\pi^2} \int_{k_{11}}^{k_{12}} \theta_2^2 \, da \leq \frac{\ell^2}{\pi^2} \frac{1}{k_m} I(0) \exp \left( -\frac{2\pi}{\ell} \int_0^{x_1} \frac{k_m(\tau)}{k_M(\tau)} \, d\tau \right).
\]

(8.7)

Estimates for the components of the stress tensor are furnished by relation (5.36) where we have to take into consideration (5.28). Moreover, by means of the constitutive equation (2.4), from (5.28) and (5.36) we can obtain estimates describing the spatial decay of the strain tensor.

We have to point out that for the class of isotropic and homogeneous thermoelastic strip, Eq. (3.1) reduces to the Laplace equation, while Eq. (5.1) reduces to the biharmonic equation. The decay estimate for the temperature (4.12) reduces to that given in [11] for the Laplace equation. Furthermore, now we have \( \sigma_1 = \frac{\ell^2}{2\pi^2} \), \( \sigma_2 = \frac{\ell}{\pi} \) and hence the decay rate is \( \kappa_2 = \frac{\ell}{\pi} \).

The measure \( J(x_1) \) becomes

\[
J(x_1) = \int_0^L \left( -\epsilon A_{11} - \epsilon A_{11}^2 + \epsilon A_{22}^2 \right) \, dx_2 - \int_{k_{11}}^{k_{12}} \omega k \theta_1 \, da \quad \text{for all } x_1 \in [0, L].
\]

(8.8)

or

\[
J(x_1) = \int_{k_{11}}^{k_{12}} \left[ \epsilon (A_{11}^2 + 2A_{12}^2 + A_{22}^2) + \omega k (\theta_1^2 + \theta_2^2) \right] \, da \quad \text{for all } x_1 \in [0, L],
\]

(8.9)

but now we can take \( \omega = 0 \) and our results predict the spatial decay of the Airy stress function \( A \) at the decay rate \( \kappa_2 = \frac{\ell}{\pi} \) in the measure

\[
J_0(x_1) = \int_0^L \epsilon (A_{11} - A_{11}^2 + A_{22}^2) \, dx_2 \quad \text{for all } x_1 \in [0, L],
\]

(8.10)

or

\[
J_0(x_1) = \int_{k_{11}}^{k_{12}} \epsilon (A_{11}^2 + 2A_{12}^2 + A_{22}^2) \, da \quad \text{for all } x_1 \in [0, L].
\]

(8.11)

It has to be outlined that the estimated decay rate predicted in [9] for the elastic bodies is \( \frac{\sqrt{2\pi}}{2} \), that is one better than predicted here. This is due to the coupling with thermal effects and so \( \sigma_2 = \frac{\ell}{\pi} \), a value which in [9] is zero. That is the differential inequality (5.33) differs essentially from that established in [9].

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