Several Results on Uniqueness and Continuous Data Dependence in Thermoelasticity of Type III

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Published online: 27 Jun 2011.

To cite this article: Stan Chiriţă & Michele Ciarletta (2011) Several Results on Uniqueness and Continuous Data Dependence in Thermoelasticity of Type III, Journal of Thermal Stresses, 34:8, 873-889, DOI: 10.1080/01495739.2011.586277

To link to this article: http://dx.doi.org/10.1080/01495739.2011.586277

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SEVERAL RESULTS ON UNIQUENESS AND CONTINUOUS DATA DEPENDENCE IN THERMOELASTICITY OF TYPE III

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In the present paper we consider the linear theory of thermoelasticity of type III for anisotropic and inhomogeneous materials as developed by Green and Naghdi (1992, 1995). We consider the general initial-boundary value problems associated with the theory in concern and then we identify some mild constitutive hypotheses that lead to appropriate energetic measures of solutions. On this basis we are able to prove some new uniqueness and continuous data dependence theorems. When the boundary surface is maintained at zero thermal displacement, we are able to obtain uniqueness and continuous data dependence results by using only the positiveness of the density mass, the positive definiteness of the elasticity tensor and the positive definiteness of the heat conductivity tensor, without any sign defined assumption upon the specific heat. These results are established by means of new techniques which are essentially different from those previously used for the classical theory of thermoelasticity.

Keywords: Continuous dependence results; Thermoelasticity of type III; Transient solutions; Uniqueness

INTRODUCTION

Green and Naghdi [1–3] proposed several thermoelastic theories for deformable continua which rely on an energy balance law rather than an entropy inequality. They also derived a theory of thermoelastic bodies based on a new entropy balance law. The thermodynamics introduces the thermal displacement \( \tau \) that is related to the temperature variation \( \theta \) by means of relation \( \tau = \theta \). One of these theories leads to a hyperbolic equation for the thermal displacement and so it permits the transmission of heat as thermal waves at finite speed.

The thermoelasticity of type III allows the constitutive functions for free energy, stress tensor, entropy and heat flux to depend on the strain tensor, the time derivative of the thermal displacement, the gradient of thermal displacement and
the time derivative of the gradient of thermal displacement. This theory allows the
dissipation energy, but the heat flux is partially determined from the Helmholtz free
energy potential.

The linear theory of thermoelasticity of type III was a subject of great interest
in literature of last years. Thus, the wave propagation was studied by Puri and
Jordan [4] and Quintanilla and Straughan [5], growth and uniqueness of solutions
in centrosymmetric thermoelasticity of type III was established by Quintanilla and
Straughan [6], uniqueness and energy bounds for some non-standard problems have
been considered by Quintanilla [7] and Quintanilla and Straughan [8], time decay of
solutions under an absorbing boundary was studied by Lazzari and Nibbi [9] and
Messaoudi and Soufyane [10]. The plate theory based on thermoelasticity of type
III was studied by Leseduarte and Quintanilla [11] and Liu and Quintanilla [12],
while the spatial behavior for centrosymmetric bodies was addressed by Quintanilla
Structural stability and continuous data dependence for centrosymmetric bodies have
been studied by Quintanilla [15], Quintanilla and Racke [16] and Liu and Lin [17].

The essential feature of the thermoelasticity theory of type III for
centrosymmetric bodies considered in [6] consists of the fact that it leads to a
symmetric partial differential system that allows one to apply Lagrange identity
method or the logarithmic convexity method in order to establish uniqueness and
continuous dependence results, provided some mild assumptions are required on the
positive definiteness of the corresponding thermoelastic coefficients. Such methods
fail to work when the genuine general anisotropy is considered because the partial
differential system does not have the appropriate symmetries. For these situations
techniques more elaborate have to be developed.

One of the most important tasks in the study of the suitability
of a thermomechanical model is to prove that the solutions of problems depend
continuously on the supply terms, which may be subjected to error in the measured
data or perturbations in the mathematical modelling process. That means small
changes in the supply loads in the partial differential equations and changes in the
equations have to be reflected physically by small changes in the corresponding
solutions. Estimates of continuous dependence play a central role in obtaining
numerical approximations to these kinds of problems.

In the present paper we consider the general linear theory of thermoelasticity
of type III for anisotropic and inhomogeneous bodies. Such a model is described
by a partial differential system which is not symmetric and, therefore the Lagrange
and logarithmic convexity methods previously applied in the centrosymmetric case
fail to work. We proceed here to develop new techniques that allow us to establish
uniqueness and continuous data dependence results for solutions of the initial-
boundary value problems corresponding to the anisotropic bodies. To this end we
study some constitutive profiles leading to appropriate energetic measures involving
the displacement vector and the thermal displacement. The first of them is based
on the positiveness of the internal energy and the dissipation energy and leads to
uniqueness and continuous dependence results like those in the classical theory of
thermoelasticity.

While the other is based upon the positive definiteness of the elasticity tensor
and the positive definiteness of the heat conductivity tensor. This last constitutive
profile is found on the basis of two auxiliary results described in Propositions 1
and 2. On this basis we are able to infer new results on the uniqueness and continuous data dependence of solutions. These results are different from those obtained in [6].

Moreover, when the boundary surface is maintained at zero thermal displacement, we are able to obtain uniqueness and continuous data dependence results by only the positiveness of the density mass, the positive definiteness of the elasticity tensor and the positive definiteness of the heat conductivity tensor, without any sign defined assumption upon the specific heat. This aspect is not investigated in the aforementioned paper [6] on the thermoelasticity of type III. Our results are also new in this respect and are essentially different from those known in literature on subject.

BASIC EQUATIONS

Throughout this paper we assume that the properly regular region \( B \subset \mathbb{R}^3 \) is occupied by an inhomogeneous and anisotropic thermoelastic material. We denote by \( \partial B \) the boundary surface of \( B \). In what follows we will consider the linear dynamic theory of thermoelasticity of type III as described in [1] and [3]. The governing equations of the linear theory of anisotropic and inhomogeneous thermoelasticity of type III are given by

- the evolution equations
  \[ T_{rs,r} + a f_s = a \ddot{u}_s \]  
  \[ \text{(1)} \]

- the equation of energy
  \[ a \dot{\eta} = q_{r,r} + a R \]  
  \[ \text{(2)} \]

in \( B \times (0, \infty) \),

- the constitutive equations
  \[ T_{rs} = C_{rmm} e_{mn} + G_{rnm} \beta_m - M_{rs} \dot{\tau} \]
  \[ a \eta = M_{rs} e_{rs} - N_r \beta_r + c \dot{\tau} \]
  \[ q_r = G_{mn} e_{mn} + K_{rs} \beta_s + N_r \dot{\tau} + B_{rs} \dot{\beta}_s \]  
  \[ \text{(3)} \]

in \( \overline{B} \times [0, \infty) \), and

- the geometrical relations
  \[ e_{rs} = \frac{1}{2} (u_{r,s} + u_{s,r}) \]
  \[ \beta_r = \tau_{r} \]  
  \[ \text{(4)} \]

in \( \overline{B} \times [0, \infty) \).

Here \( u_r \) are the components of the displacement vector, \( \tau \) is the thermal displacement, \( \theta = \dot{\tau} \) is the temperature variation from the uniform reference
temperature $\theta_0$, $\varepsilon_{rs}$ are the components of the strain tensor, $\beta_s$ are the components of the thermal displacement gradient vector, $T_{rs}$ are the components of the stress tensor, $q_r$ are the components of the entropy flux vector, $\eta$ is the heat density per unit mass and $f_s$ represent the components of the external body force vector per unit mass and $R$ is the external rate of supply of heat per unit mass. Furthermore, $\varrho$ is the density mass, $C_{rsmn}$, $G_{rsm}$, $M_{rs}$, $c$, $K_{rs}$ and $B_{rs}$ are the constitutive coefficients depending on the spatial variable $x$, continuously differentiable on $B$ and satisfying the following symmetries

$$
C_{rsmn} = C_{msrn}, \quad G_{rsm} = G_{srnm}, \quad M_{rs} = M_{sr}
$$

The subscripts $r$, $s$, $m$, $n$ take values 1, 2, 3 and summation is implied by index repetition. Moreover, a superposed dot denotes differentiation with respect to time and a subscript comma indicates partial differentiation. The specific Helmholtz free energy is given by

$$
\varrho \psi = \frac{1}{2} C_{rsmn} \varepsilon_{rs} \varepsilon_{mn} + \frac{1}{2} K_{rs} \beta_s \beta_s + \frac{1}{2} c \dot{\varepsilon}^2 + G_{rsm} \varepsilon_{rs} \beta_m - M_{rs} \varepsilon_{rs} \dot{\varepsilon} + N_\tau \beta_s \dot{\beta}_s
$$

while the specific internal energy is given by

$$
\varrho \epsilon = \frac{1}{2} C_{rsmn} \varepsilon_{rs} \varepsilon_{mn} + G_{rsm} \varepsilon_{rs} \beta_m + \frac{1}{2} K_{rs} \beta_s \beta_s + \frac{1}{2} c \dot{\varepsilon}^2
$$

and the internal rate of supply of heat per unit mass is

$$
\theta \dot{\xi} = B_{rs} \dot{\beta}_r \dot{\beta}_s
$$

and hence the heat conductivity tensor $B_{rs}$ satisfies dissipation inequality

$$
B_{rs} \dot{\beta}_r \dot{\beta}_s \geq 0
$$

The components of the surface traction and the heat flux at regular points of $\partial B$ can be expressed in the form

$$
t_n = T_{mn} n_m, \quad q = q_r n_r
$$

where $n_r$ are the components of the unit outward normal vector to $\partial B$.

To the field equations we adjoin initial and boundary conditions. The initial conditions are

$$
u_i(x, 0) = u_i^0(x), \quad \dot{u}_i(x, 0) = \dot{u}_i^0(x), \quad \tau(x, 0) = 0, \quad \dot{\tau}(x, 0) = \theta^0(x), \quad x \in \bar{B}
$$

where $u_i^0(x)$, $\dot{u}_i^0(x)$, $\theta^0(x)$ are prescribed functions on $\bar{B}$. In relation (11) we have set

$$
\tau(x, 0) = 0
$$
because we have taken
\[ \tau(x, t) = \int_0^t \theta(x, s) ds \] (13)

The boundary conditions are
\[ u_r(x, t) = \tilde{u}_r(x, t) \quad \text{on} \quad \Sigma_1 \times [0, \infty), \quad T_{ms}(x, t)n_m = \tilde{t}_n(x, t) \quad \text{on} \quad \Sigma_2 \times [0, \infty) \]
\[ \tau(x, t) = \tilde{\tau}(x, t) \quad \text{on} \quad \Sigma_3 \times [0, \infty), \quad q_n(x, t)n_n = \tilde{q}(x, t) \quad \text{on} \quad \Sigma_4 \times [0, \infty) \] (14)

where \( \tilde{u}_r(x, t), \tilde{t}_n(x, t), \tilde{\tau}(x, t) \) and \( \tilde{q}(x, t) \) are prescribed functions and \( \Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B \) and \( \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset \).

Let \( (u_r, \tau) \) be a solution of the initial-boundary value problem \( (\mathcal{P}) \) defined by the basic equations \( (1)–(4) \) and the initial conditions \( (11) \) and the boundary conditions \( (14) \). Then we easily can establish the following identity
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \dot{u}_r \dot{u}_r + \varrho e \right) + B_{rs} \dot{\theta}_s = \varrho f \dot{u}_r + \varrho R \dot{\tau} + (T_{rs} \dot{u}_s + q_r \dot{\tau})_r \] (15)

Finally, it is worth mentioning that a substitution of relations \( (3) \) and \( (4) \) into \( (1) \) and \( (2) \) leads to the following partial differential system:
\[ \varrho \ddot{u}_s = (C_{rsm} u_{m,n} + G_{rms} \tau_{,m} - M_{rs} \dot{\tau})_s + \varrho f_s \]
\[ c \ddot{\tau} = -M_{rs} \dot{u}_{r,s} + N_r \dot{\tau}_r + (G_{mn} u_{m,n} + K_{rs} \tau_{,s} + N_r \dot{\tau} + B_{rs} \dot{\tau},_s)_s + \varrho R \] (16)

For a centro-symmetric body we have
\[ G_{rms} = 0, \quad N_r = 0 \] (17)

and the above system reduces to
\[ \varrho \ddot{u}_r = (C_{rsm} u_{m,n} - M_{rs} \dot{\tau})_s + \varrho f_s \]
\[ c \ddot{\tau} = -M_{rs} \dot{u}_{r,s} + (K_{rs} \tau_{,s} + B_{rs} \dot{\tau},_s)_s + \varrho R \] (18)
a form for which the Lagrange identity method or the logarithmic convexity method can be successfully applied to study the uniqueness and continuous dependence results (see, for example, [6]). However, such methods fail when applied to the system \( (16) \).

SOME CONSTITUTIVE ASSUMPTIONS AND RELATED MEASURES

On the basis of the identity \( (15) \) and by using the initial conditions \( (11) \), we obtain the following identity
\[ \mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{B(t)} \varrho (f_s \dot{u}_s + R \dot{\tau}) dv ds + \int_0^t \int_{\partial B(t)} (T_{rs} n_s \dot{u}_s + q_r \dot{\tau}) d a d s \] (19)
where
\[ \mathcal{E}(t) = \int_{B(t)} \left( \frac{1}{2} \dot{\rho} \ddot{u} + q \mathcal{E} \right) dv + \int_{0}^{t} \int_{B(t)} B_{mn} \dot{B}_{mn} dv ds \] (20)
and \( B(t) \) is denoting that the quantity under integral is evaluated at time \( t \).

It can be shown that \( \mathcal{E}(t) \) represents a possible candidate for a measure of the solution \( (u_{\tau}, \tau) \) under one of the following sets of constitutive profiles:

(H1) the density mass \( \rho \) is strictly positive, the specific internal energy \( \mathcal{E} \) is positive and the heat conductivity tensor \( B_{rs} \) is positive, that is
\[ \rho > 0 \]
\[ C_{rsmn} \xi_{rs} \xi_{mn} + 2G_{rsmn} \xi_{rs} \xi_{mn} + K_{rsmn} \xi_{rs} + c \phi^{2} \geq 0 \quad \text{for all } \xi_{rs} = \xi_{sr}, \xi_{r} \]
\[ B_{rs} \xi_{rs} \geq 0 \quad \text{for all } \xi_{r} \] (21)

(H2) the density mass and the specific heat are strictly positive and the elasticity tensor \( C_{rsmn} \) and the heat conductivity tensor \( B_{rs} \) are positive definite tensors, that is
\[ \rho > 0, \quad c > 0 \]
\[ C_{rsmn} \xi_{rs} \xi_{mn} \geq c_{0} \xi_{rs} \xi_{rs} \quad \text{for all } \xi_{rs} = \xi_{sr} \]
\[ B_{rs} \xi_{rs} \geq b_{0} \xi_{rs} \xi_{rs} \quad \text{for all } \xi_{r} \] (22)
where \( c_{0} > 0 \) and \( b_{0} > 0 \) are appropriate constants related to the minimum eigenvalue of the elasticity tensor \( C_{rsmn} \) and the minimum eigenvalue of the heat conductivity tensor \( B_{rs} \), respectively.

Let us denote by \( (P_{b}) \) the initial-boundary value problem \( (P) \) when the whole boundary surface is thermally insulated (that is, when \( \Sigma_{3} = \partial B) \). In treating the uniqueness question for the initial-boundary value problem \( (P_{b}) \) a variant of the hypothesis (H2) is more convenient, namely,

(H3) the density mass is strictly positive and the elasticity tensor \( C_{rsmn} \) and the heat conductivity tensor \( B_{rs} \) are positive definite tensors, that is
\[ \rho > 0 \]
\[ C_{rsmn} \xi_{rs} \xi_{mn} \geq c_{0} \xi_{rs} \xi_{rs} \quad \text{for all } \xi_{rs} = \xi_{sr} \]
\[ B_{rs} \xi_{rs} \geq b_{0} \xi_{rs} \xi_{rs} \quad \text{for all } \xi_{r} \] (23)

It is obvious that (H1) makes \( \mathcal{E}(t) \geq 0 \) for all \( t \geq 0 \) and \( \mathcal{E}(t) = 0 \) implies \( \dot{u}_{\tau} = 0 \) and hence \( \mathcal{E}(t) \) can be a possible candidate for a measure of solution \( (u_{\tau}, \tau) \). Regarding the hypotheses (H2) and (H3), we note the following results.

**Proposition 1.** Suppose that the heat conductivity tensor \( B_{mn} \) is a positive definite tensor and \( \beta_{m}(x, t) \) is continuous differentiable with respect to time variable and
\[ \beta_{m}(x, 0) = 0 \quad \text{for all } x \in \overline{B} \] (24)
Then, for any constant $C > 0$, we have

$$\int_0^t \int_{B(t)} B_{mn} \beta_m \dot{\beta}_n dvds \geq C \int_{B(t)} B_{mn} \beta_m dv, \quad \text{for all } t \geq 0 \quad (25)$$

**Proof.** In view of the initial condition (24) it is possible to have

$$\beta_m(x, t) = 0 \quad \text{for all } (x, t) \in B \times [0, t_1], \quad t_1 \geq 0 \quad (26)$$

and then (25) is identically satisfied on the time interval $[0, t_1]$. Suppose now that (25) is not true on the time interval $(t_1, t_2)$, that is

$$\int_{t_1}^t \int_{B(t)} B_{mn} \beta_m \dot{\beta}_n dvds < C \int_{B(t)} B_{mn} \beta_m dv, \quad \text{for all } t \in (t_1, t_2) \quad (27)$$

and hence, we have

$$\int_{B(t)} B_{mn} \beta_m dv > 0, \quad \text{for all } t \in (t_1, t_2) \quad (28)$$

On the other hand, from (26) and the Cauchy-Schwarz inequality, we obtain

$$\int_{B(t)} B_{rs} \beta_r \beta_s dv = \int_{B(t)} B_{rs} \beta_r \beta_s dv + \int_{t_1}^t \int_{B(t)} 2B_{mn} \beta_m \beta_n dvds$$

$$\leq 2 \left( \int_{t_1}^t \int_{B(t)} B_{mn} \beta_m \beta_n dvds \right)^{1/2} \left( \int_{t_1}^t \int_{B(t)} B_{rs} \beta_r \beta_s dvds \right)^{1/2} \quad (29)$$

and hence, by using (27), we are led to the following Gronwall inequality

$$\int_{B(t)} B_{mn} \beta_m \beta_n dv \leq 4C \int_{t_1}^t \int_{B(t)} B_{mn} \beta_m \beta_n dvds \quad \text{for all } t \in (t_1, t_2) \quad (30)$$

that is,

$$\phi(t) \leq 4C \int_{t_1}^t \phi(s) ds \quad \text{for all } t \in (t_1, t_2) \quad (31)$$

where

$$\phi(t) = \int_{B(t)} B_{mn} \beta_m \beta_n dv > 0 \quad (32)$$

If we set

$$\phi^2(t) = 4C \int_{t_1}^t \phi(s) ds \quad (33)$$

then (31) gives

$$\phi(t) \leq \phi^2(t) \quad \text{for all } t \in (t_1, t_2) \quad (34)$$
By direct differentiation into (33) and the use of relation (34), we obtain
\[ \dot{\varphi}(t) \leq 2C\varphi(t) \] (35)
which, integrated, furnishes \( \varphi(t) = 0 \) and hence
\[ \beta_m(x, t) = 0 \quad \text{for all} \quad (x, t) \in \overline{B} \times (t_1, t_2) \] (36)
The conclusion expressed by (36) is in contradiction with relation (28) and this proves that our assumption (27) cannot be true. Thus, the relation (25) is true and the proof is complete. □

**Proposition 2.** Suppose that the elasticity tensor \( C_{rsmn} \) and the heat conductivity tensor \( B_{mn} \) are positive definite tensors. Then we have
\[ \int_{B(t)} (C_{rsmn} \varepsilon_{rs} \varepsilon_{mn} + 2G_{rs} \varepsilon_{rs} \beta_m + K_{rs} \beta_r \beta_s) \, dv + 2 \int_0^t \int_{\partial B(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n \, dv \, ds \geq 0 \] (37)
for all solutions \((u_r, \tau)\) of the initial-boundary value problem \((\mathcal{P})\).

**Proof.** Let \((u_r, \tau)\) be a solution of the initial-boundary value problem \((\mathcal{P})\) and let us set
\[ \mathcal{D}(t) = \int_{B(t)} (C_{rsmn} \varepsilon_{rs} \varepsilon_{mn} + 2G_{rs} \varepsilon_{rs} \beta_m + K_{rs} \beta_r \beta_s) \, dv \]
\[ + 2 \int_0^t \int_{\partial B(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n \, dv \, ds, \quad t \geq 0 \] (38)

Furthermore, by means of the Cauchy-Schwarz and arithmetic-geometric mean inequalities, we have
\[ |G_{rm} \varepsilon_{rs} \beta_m| \leq (G_{rm} G_{rm})^{1/2} (\varepsilon_{pq} \beta_p \varepsilon_{pq} \beta_q)^{1/2} \]
\[ \leq \frac{G}{2c_0} C_{rsmn} \varepsilon_{rs} \varepsilon_{mn} + \frac{G}{2b_0} B_{mn} \beta_m \beta_n \quad \text{for all} \quad \varepsilon > 0 \] (39)
where
\[ G = \max_{x \in B} (G_{rm} G_{rm})^{1/2} \] (40)
Moreover, we obtain
\[ |K_{rs} \beta_r \beta_s| \leq (K_{rs} K_{rs})^{1/2} (\beta_m \beta_m \beta_m \beta_m)^{1/2} \leq \frac{K}{B_0} B_{mn} \beta_m \beta_n \] (41)
where
\[ K = \max_{x \in B} (K_{rs} K_{rs})^{1/2} \] (42)
If we use relations (39) and (41) into (38), then we find
\[ \mathcal{D}(t) \geq \int_{\Omega(t)} \left[ \left( 1 - \frac{G\varepsilon}{c_0} \right) C_{rnn} e_{rt} e_{rn} - \frac{1}{b_0} \left( K + \frac{G}{\varepsilon} \right) B_{rs} \beta_r \beta_s \right] dv \]
+ \[ 2 \int_0^t \int_{\Omega(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv ds \quad \text{for all } \varepsilon > 0 \quad (43) \]

Further, we choose \( \varepsilon > 0 \) so that \( \frac{G}{\varepsilon} c_0 \leq 1 \) and we set
\[ C = \frac{1}{2b_0} \left( K + \frac{G}{\varepsilon} \right), \quad D = 1 - \frac{G\varepsilon}{c_0} \quad (44) \]
and note that (43) implies
\[ \mathcal{D}(t) \geq D \int_{\Omega(t)} C_{rnn} e_{rt} e_{rn} dv + 2 \int_0^t \int_{\Omega(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv ds \]
\[ - 2C \int_{\Omega(t)} B_{mn} \beta_m \beta_n dv \quad (45) \]

Now, on the basis of the relations (4), (12) and (13), we deduce that (24) holds true. Therefore, the relation (25) holds true and (45) implies that \( \mathcal{D}(t) \geq 0 \) for all \( t \geq 0 \) and the proof is finished. \( \square \)

**Remark 3.** The analysis of the above proof proves that, for each solution \( (u_r, \tau) \) of the initial-boundary value problem \( \mathcal{P} \), there exist the positive constants \( \gamma_1 \) and \( \gamma_2 \) so that
\[ \int_{\Omega(t)} \left( C_{rnn} e_{rt} e_{rn} + 2G_{rnn} \dot{e}_{rn} \dot{\beta}_m + K_{rs} \beta_r \beta_s \right) dv + \int_0^t \int_{\Omega(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv ds \]
\[ \geq \gamma_1 \int_{\Omega(t)} C_{rnn} e_{rt} e_{rn} dv + \gamma_2 \int_{\Omega(t)} B_{mn} \beta_m \beta_n dv, \quad \text{for all } t \geq 0 \quad (46) \]

The above propositions together with the hypotheses (H2) and (H3) lead to other possible candidates for a measure of solution \( (u_r, \tau) \). They are essential for our subsequent analysis.

**UNIQUENESS RESULTS**

On the basis of the above analysis we can infer the following uniqueness results.

**Theorem 4.** Let us suppose that \((H1)\) holds true and \( c > 0 \). Then the initial-boundary value problem \( \mathcal{P} \) has at most one solution.

**Proof.** Considering \( (u_r, \tau) \) as the difference of two solutions of the initial-boundary value problem \( \mathcal{P} \) corresponding to the same given data, it follows that
\[ \mathcal{E}(t) = 0 \quad \text{for all } t \in [0, \infty) \quad (47) \]
The hypothesis (H1) and relations (20) and (47) implies that
\[ \dot{u}_r(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (48)

Then the zero initial conditions implies that
\[ u_r(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (49)

Furthermore, the relations (20), (47) and (48) give
\[ \frac{1}{2} \int_{\mathcal{B}^0} \left( c \dot{v}^2 + K_{mn} \dot{p}_m \dot{p}_n \right) dv + \int_0^t \int_{\mathcal{B}^0} B_{mn} \dot{p}_m \dot{p}_n dv ds = 0 \quad \text{for all } t \in [0, \infty) \] (50)

and, since \( c > 0 \) and \( K_{mn} \dot{p}_m \dot{p}_n \geq 0 \), we deduce that
\[ \ddot{T}(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (51)

and, by using the zero initial condition, we infer
\[ \dot{T}(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (52)

Relations (49) and (52) prove that \( (u_r, \tau)(x, t) = 0 \) for all \( (x, t) \in \overline{B} \times [0, \infty) \). Thus, the uniqueness of solution follows and the proof is complete. \( \square \)

**Remark 5.** Suppose that (H1) holds true, \( c = 0 \) and the heat conductivity tensor is positive definite. Then the initial-boundary value problem \((\mathcal{P}_0)\) has at most one solution. In fact, relation (50) implies
\[ \ddot{p}_m(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (53)

and hence, by means of relation (4) and the zero initial condition, we obtain
\[ \tau_r(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (54)

Since \( \text{meas } \Sigma_3 \neq 0 \), from (54) we can deduce relation (52) and hence uniqueness of solution follows. The same conclusion is valid if we substitute the positive definiteness of \( B_{mn} \) with the positive definiteness of \( K_{mn} \).

**Remark 6.** The result described in the above theorem represents the counterpart of one established in the classical theory of thermoelasticity.

A direct consequence of the Propositions 1 and 2 is the following uniqueness result.

**Theorem 7.** Suppose that (H2) holds true. Then the initial-boundary value problem \((\mathcal{P})\) has at most one solution.
**Proof.** Considering \((u_\tau, \tau)\) as the difference of two solutions of the initial-boundary value problem \((\mathcal{P}_0)\) corresponding to the same given data, it follows that \(\mathcal{E}(t) = 0\) for all \(t \in [0, \infty)\). Then, the relations (20), (22), (24), (25) and (37) imply \((u_\tau, \tau)(x, t) = 0\) for all \((x, t) \in \mathcal{B} \times [0, \infty)\) and so, by means of zero initial condition, we get \((u_\tau, \tau)(x, t) = 0\) for all \((x, t) \in \mathcal{B} \times [0, \infty)\), that is the uniqueness of solution. Thus, the proof is complete. \(\square\)

**Remark 8.** We have to note that, it is possible to have \(c = 0\) in the above theorem and then we can conclude that the initial-boundary value problem \((\mathcal{P}_0)\) has at most one solution.

**Remark 9.** The result described in the above theorem is new and it is greatly based upon the Propositions 1 and 2. It is characteristic for the theory of thermoelasticity of type III for genuine anisotropic bodies.

**Theorem 10.** Suppose that \((H3)\) holds true. In the class of solutions \((u_\tau, \tau)\) for which
\[
\int_0^t \int_{\mathcal{B}(t)} \dot{\tau}^2 dv ds \leq M^2 e^a, \quad \text{for all } t \in [0, \infty), \quad M = \text{const.}, \quad 0 \leq a < \frac{b_0 \lambda_0}{\max_\pi |c|} \tag{55}
\]
the initial-boundary value problem \((\mathcal{P}_0)\) has at most one solution. In the above relation \(\lambda_0\) is the smallest positive eigenvalue of the clamped membrane problem for \(B\).

**Proof.** Considering \((u_\tau, \tau)\) as the difference of two solutions of the initial-boundary value problem \((\mathcal{P}_0)\) corresponding to the same given data, it follows that \(\mathcal{E}(t) = 0\) for all \(t \in [0, \infty)\). In view of relations (7) and (20), we have
\[
\int_{\mathcal{B}(t)} (q \ddot{u}_\tau \ddot{u}_\tau + c \ddot{\tau}^2 + C_{r\alpha m} e_{r\alpha} e_{m\tau} + 2G_{r\alpha m} e_{r\alpha} \beta_m + K_{r\alpha} \beta_{r\alpha}) dv + 2 \int_0^t \int_{\mathcal{B}(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv ds = 0 \quad \text{for all } t \in [0, \infty) \tag{56}
\]
Since \(\tau(x, t) = 0\) for all \((x, t) \in \partial \mathcal{B} \times [0, \infty)\), we have
\[
\int_{\mathcal{B}(t)} \dot{\tau}^2 dv \geq \lambda_0 \int_{\mathcal{B}(t)} \ddot{\tau}^2 dv \tag{57}
\]
where \(\lambda_0\) is the smallest positive eigenvalue of the clamped membrane problem. So, in view of the hypothesis \((H3)\), we get
\[
\int_{\mathcal{B}(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv \geq b_0 \lambda_0 \int_{\mathcal{B}(t)} \ddot{\tau}^2 dv \quad \text{for all } t \in [0, \infty) \tag{58}
\]
Then the relations (56) and (58) imply
\[
0 \geq \int_0^t \int_{\mathcal{B}(t)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv ds + \int_{\mathcal{B}(t)} (C_{r\alpha m} e_{r\alpha} e_{m\tau} + 2G_{r\alpha m} e_{r\alpha} \beta_m + K_{r\alpha} \beta_{r\alpha}) dv + \int_{\mathcal{B}(t)} q \ddot{u}_\tau \ddot{u}_\tau dv + b_0 \lambda_0 \int_0^t \int_{\mathcal{B}(t)} \ddot{\tau}^2 dv ds - \max_\pi |c| \int_{\mathcal{B}(t)} \ddot{\tau}^2 dv \quad \text{for all } t \in [0, \infty) \tag{59}
\]
Furthermore, in view of relation (46), it follows that

$$\int_0^t \int_{\frac{B}{\epsilon}} B_{mn} \dot{\dot{B}}_n \dot{d}vds + \int_{\frac{B}{\epsilon}} (C_{rs} e_{rs} e_{mn} + 2G_{rs} e_{rs} \dot{\beta}_m + K_{rs} \dot{\beta}_r \dot{\beta}_s) dv \geq 0$$  \hspace{1cm} (60)$$

for all $t \in [0, \infty)$ and, consequently, relation (59) furnishes

$$b_0 \lambda_0 \int_0^t \int_{\frac{B}{\epsilon}} \dot{\dot{\tau}}^2 dvds - \max_{\epsilon} |c| \int_{\frac{B}{\epsilon}} \dot{\tau}^2 dv \leq 0 \text{ for all } t \in [0, \infty)$$  \hspace{1cm} (61)$$

Therefore, we have

$$\int_0^t \int_{\frac{B}{\epsilon}} \dot{\tau}^2 dvds \leq \omega^2 \int_{\frac{B}{\epsilon}} \dot{\tau}^2 dv \text{ for all } t \in [0, \infty)$$  \hspace{1cm} (62)$$

where

$$\omega^2 = \frac{1}{b_0 \lambda_0} \max_{\epsilon} |c|$$  \hspace{1cm} (63)$$

At this instant, we can conclude that if $\max_{\epsilon} |c| = 0$, then we have uniqueness of solution. Consequently, in what follows we shall assume that $\max_{\epsilon} |c| > 0$. Further, we set

$$\chi(t) = \int_0^t \int_{\frac{B}{\epsilon}} \dot{\dot{\tau}}^2 dvds, \quad t \in [0, \infty)$$  \hspace{1cm} (64)$$

and note that relation (62) becomes

$$\chi'(t) \leq 2\omega^2 \chi(t) \dot{\tau}(t), \quad t \in [0, \infty)$$  \hspace{1cm} (65)$$

Now, it is clear that if $\chi(t) = 0$ for all $t \in [0, \infty)$, then it results that $\dot{\tau}(t) = 0$ for all $(x, t) \in \overline{B} \times [0, \infty)$ and hence we have

$$\tau(x, t) = 0 \text{ for all } (x, t) \in \overline{B} \times [0, \infty)$$  \hspace{1cm} (66)$$

If we substitute (66) into relation (59), then we can deduce that

$$u_r(x, t) = 0 \text{ for all } (x, t) \in \overline{B} \times [0, \infty)$$  \hspace{1cm} (67)$$

Thus, the relations (66) and (67) implies the uniqueness of solution for the initial-boundary value problem ($\mathcal{P}_0$).

Let us now assume that there exists $\alpha \in [0, \infty)$ so that $\chi(\alpha) > 0$ and hence we have

$$\chi(t) > 0 \text{ for all } t \in [\alpha, \infty)$$  \hspace{1cm} (68)$$

Then the relation (65) implies that

$$\frac{d}{dt} \{\chi(t)e^{-t/(2\omega^2)}\} \geq 0 \text{ for all } t \in [\alpha, \infty)$$  \hspace{1cm} (69)$$
and hence, we deduce that
\[ \chi(x) e^{-u/(2\omega^2)} \leq \chi(t) e^{-t/(2\omega^2)} \leq \lim_{t \to \infty} \left[ \chi(t) e^{-t/(2\omega^2)} \right] \quad \text{for all } t \in [a, \infty) \] (70)

In view of the assumption (55), it follows that
\[ \chi^2(t) \leq M^2 e^{at}, \quad \text{for all } t \in [0, \infty), \quad 0 \leq a < \frac{1}{\omega^2} \] (71)

and hence
\[ \lim_{t \to \infty} \left[ \chi(t) e^{-t/(2\omega^2)} \right] = 0 \] (72)

Further, the relation (70) implies that
\[ \dot{\chi}(x, t) = 0 \quad \text{for all } (x, t) \in \overline{B} \times [0, \infty) \] (73)

and hence the relations (66) and (67) hold true, that is we have the uniqueness of solution for the initial-boundary value problem (\(\mathcal{P}_0\)). Thus, the proof is complete. □

**Remark 11.** The result described in Theorem 10 is new in literature on thermoelasticity theory.

**CONTINUOUS DATA DEPENDENCE RESULTS**

In this section we discuss how the above hypotheses can be used to study the continuous data dependence of solutions of the initial-boundary value problem (\(\mathcal{P}\)). It becomes clear from the analysis of the preceding two sections that, under one of the hypotheses (H1), with \(c > 0\), and (H2), \(\mathcal{E}(t)\) is a measure of the solution to the initial-boundary value problem (\(\mathcal{P}\)).

**Theorem 12.** Suppose \((u, \tau)\) is a solution to the initial-boundary value problem (\(\mathcal{P}\)) with vanishing boundary data. Under one of hypotheses (H1), with \(c > 0\), and (H2), \(\mathcal{E}(t)\) is a measure of the solution to the initial-boundary value problem (\(\mathcal{P}\)).

\[ \mathcal{E}(t)^{1/2} \leq \mathcal{E}(0)^{1/2} + \frac{1}{\sqrt{2}} \int_0^t g(s) ds, \quad \text{for all } t \geq 0 \] (74)

where
\[ g(t) = \left[ \int_{\mathcal{B}(t)} \rho \left( f_s f_s + \frac{\rho R^2}{c^2} \right) dv \right]^{1/2} \] (75)

**Proof.** By using the vanishing boundary conditions and the Cauchy-Schwarz inequality in (19), we obtain
\[ \mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t \left( \int_{\mathcal{B}(s)} \rho \left( f_s f_s + \frac{\rho R^2}{c^2} \right) dv \right)^{1/2} \left( \int_{\mathcal{B}(s)} (q\ddot{u} + c\dot{\tau}) dv \right)^{1/2} ds \] (76)
that is, by means of relations (24) and (20), we have the following Gronwall inequality

\[ \mathcal{E}(t) \leq \mathcal{E}(0) + \int_0^t g(s)(2\mathcal{E}(s))^{1/2} ds, \quad \text{for all } t \geq 0 \]  

(77)

By applying Gronwall lemma, from (77) we obtain the estimate (74) and the proof is complete. □

The result expressed by the continuous dependence inequality (74) is the counterpart of that of the classical linear thermoelasticity.

In what follows we will use the hypothesis (H3) to establish an estimate describing the continuous dependence of solutions of the initial-boundary value problem \( (\mathcal{P}_0) \) with respect to supply terms.

**Theorem 13.** Suppose the hypothesis \( (H3) \) holds true. Let \( (u_r, \tau) \) be a solution of the initial-boundary value problem \( (\mathcal{P}_0) \) with vanishing boundary and initial data and supply terms \( (f_r, R)(x, t) \) satisfying

\[ \int_0^\infty e^{-\frac{1}{2\omega^2}t} h(s) ds < \infty \]  

(78)

where

\[ h(t) = 2r(t) + 2\left( \int_0^t f(s) ds \right)^2 \]

\[ r(t) = \frac{1}{b_0 \lambda} \int_0^t \int_0^s \int_{B(\xi)} q^2 R^2 dv d\xi ds \]

\[ f(t) = \left( \int_0^t \int_{B(\xi)} q f_r f_r dv d\xi \right)^{1/2} \]  

(79)

Let us introduce

\[ \mathcal{F}(t) = \frac{1}{2} \int_0^t \int_{B(\xi)} (\dot{q} u_i \dot{u}_r + C_{rmm} \epsilon_{r \epsilon_m} + 2G_{rmm} \epsilon_{r \beta_m} + K_{r \beta_r \beta_s}) dv d\xi \]

\[ + \int_0^t \int_0^s \int_{B(\xi)} B_{mn} \dot{\beta}_m \dot{\beta}_n dv d\xi ds, \quad t \in [0, \infty) \]  

(80)

In the class of solutions for which

\[ \lim_{t \to \infty} e^{-\frac{1}{2\omega^2}t} \mathcal{F}(t) = 0 \]  

(81)

we have the following estimate

\[ 0 \leq \mathcal{F}(t) \leq \frac{1}{2\omega^2} \int_t^\infty e^{-\frac{1}{2\omega^2}(t-s)} h(s) ds, \quad \text{for all } t \geq 0 \]  

(82)
Proof. Under our constitutive hypothesis (H3), from (80) and (46), we can see that
\[ \mathcal{F}(t) \geq \frac{1}{2} \int_0^t \int_{B(\partial)} \varrho \dot{u}_r \dot{u}_r \, dvds + \frac{1}{2} \int_0^t \int_{B(\partial)} B_{mn} \dot{\hat{\beta}}_m \dot{\hat{\beta}}_n \, dv \, \hat{\zeta} \, ds \geq 0, \text{ for all } t \geq 0 \quad (83) \]
and \( \mathcal{F}(t) = 0 \), for all \( t \geq 0 \), implies
\[ (u_r, \tau)(x, t) = 0 \quad \text{for all } (x, t) \in \bar{B} \times [0, \infty) \quad (84) \]
Moreover, by a direct differentiation into (80) and by using the relations (46) and (58), we deduce that
\[ \mathcal{F}(t) \geq \mathcal{F}(t) + \frac{1}{2} \int_0^t \int_{B(\partial)} \hat{\zeta}^2 \, dv \, ds \quad (85) \]
Since \((u_r, \tau)\) satisfies zero boundary and initial conditions, the fundamental identity (19) becomes
\[ \mathcal{F}(t) = \frac{1}{2} \int_0^t \int_{B(\partial)} -c \hat{\zeta}^2 \, dv \, ds + \int_0^t \int_{B(\partial)} (qR \tau + q \hat{\tau}) \, dv \, \hat{\zeta} \, ds, \quad \text{for all } t \geq 0 \quad (86) \]
By using the Cauchy-Schwarz and arithmetic-geometric mean inequalities and relations (58) and (85), from (86) we obtain
\[ \mathcal{F}(t) \leq \omega^2 \mathcal{F}(t) + \frac{1}{4} \int_0^t \int_{B(\partial)} B_{mn} \ddot{\hat{\beta}}_m \ddot{\hat{\beta}}_n \, dv \, \hat{\zeta} \, ds + r(t) \]
\[ + \int_0^t f(s) \left( \int_{B(\partial)} \varrho \dot{u}_r \dot{u}_r \, dv \right)^{1/2} \, ds \quad (87) \]
and hence we have
\[ \mathcal{F}(t) \leq \omega^2 \mathcal{F}(t) + \frac{1}{4} \int_0^t \int_{B(\partial)} B_{mn} \ddot{\hat{\beta}}_m \ddot{\hat{\beta}}_n \, dv \, \hat{\zeta} \, ds + r(t) \]
\[ + \left( \int_0^t \int_{B(\partial)} \varrho \dot{u}_r \dot{u}_r \, dv \, \hat{\zeta} \right)^{1/2} \int_0^t f(s) \, ds \quad (88) \]
Further, by the arithmetic-geometric mean inequality, we have
\[ \mathcal{F}(t) \leq \omega^2 \mathcal{F}(t) + r(t) + \left( \int_0^t f(s) \, ds \right)^2 \]
\[ + \frac{1}{4} \int_0^t \int_{B(\partial)} \varrho \dot{u}_r \dot{u}_r \, dv \, \hat{\zeta} \, ds + \frac{1}{4} \int_0^t \int_{B(\partial)} B_{mn} \ddot{\hat{\beta}}_m \ddot{\hat{\beta}}_n \, dv \, \hat{\zeta} \, ds \quad (89) \]
In view of relation (83), from (89) we deduce that
\[ \mathcal{F}(t) \leq 2\omega^2 \mathcal{F}(t) + h(t), \quad \text{for all } t \geq 0 \quad (90) \]
which can be written as
\[
\frac{d}{dt} \left( e^{- \frac{1}{2 \omega^2} t} \mathcal{F}(t) + \frac{1}{2 \omega^2} \int_0^t e^{- \frac{1}{2 \omega^2} s} h(s) \, ds \right) \geq 0, \quad \text{for all } t \geq 0 \tag{91}
\]

Thus, in view of the hypothesis (81), it follows that for all \( t \in [0, \infty) \), from (91) we obtain
\[
0 \leq e^{- \frac{1}{2 \omega^2} t} \mathcal{F}(t) + \frac{1}{2 \omega^2} \int_0^t e^{- \frac{1}{2 \omega^2} s} h(s) \, ds \leq \frac{1}{2 \omega^2} \int_0^\infty e^{- \frac{1}{2 \omega^2} s} h(s) \, ds \tag{92}
\]

The conclusion (82) follows now from (92) and the proof is finished. \( \square \)

The result embodied into Theorem 13 is new and it is essentially based on the Propositions 1 and 2.

DISCUSSION AND CONCLUDING REMARKS

The present paper provides several uniqueness results and continuous dependence inequalities for the linear theory of thermoelasticity of type III, provided appropriate constitutive profiles are assumed. Note that the main tools to prove uniqueness results in thermoelasticity are Lagrange identity method and logarithmic convexity method. In this sense, it is worth recalling the contribution by Quintanilla and Straughan [6], in which they establish uniqueness results in thermoelasticity theory of type III for centrosymmetric bodies. Our arguments do not agree with those used by Quintanilla and Straughan [6]. This is because all our results are obtained by means of methods which do not use any material symmetries of the bodies. Thus, this paper contains some new ideas in the study of uniqueness and continuous data dependence results in thermoelasticity of type III.

However, our contribution in this paper represents a good complement to the uniqueness results established in [6]. In fact, the results reported here refer to general anisotropic thermoelastic materials. When any material symmetry is invoked our results predict uniqueness, but our assumed constitutive hypothesis is more restrictive than that used in [6]. In this respect the results established by means of Lagrange or logarithmic convexity methods cover a larger class of centrosymmetric thermoelastic materials.

Last but not least, the methods presented here is believed to be used successfully for other continuous dependence results as, for example, those relating the structural stability of the model in concern. In this area there are still many things to explore in thermoelasticity of type III.

REFERENCES