On the harmonic vibrations in linear theory of thermoelasticity of type III

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ABSTRACT

In the present paper we consider the linear theory of thermoelasticity of type III as developed by Green and Naghdı (1992, 1995). Within the context of the harmonic vibrations, we establish some estimates describing the spatial behavior of the corresponding amplitudes, provided the frequency is lower than a certain critical value. It is shown that such critical frequency is influenced only by the mechanical effects. Extension of results to the strongly elliptic thermoelastic materials is also discussed.

1. Introduction

The spatial behavior of harmonic vibrations has been treated by Flavin and Knops (1987) in the framework of the linearly damped wave equation and the linearly elastic damped cylinder. They proved that in both cases the existence of damping gives rise ultimately to a steady-state oscillation, whose amplitude decays exponentially from the excited end, provided the excitation frequency is less than a certain critical value. The later case has been investigated under positive definiteness assumption upon elasticity tensor. This work has been followed by further developments in Flavin et al. (1989), Flavin and Knops (1990), and Knops (1991). These results have been extended to the classical linear theory of thermoelasticity by Chirit¸a (1995, 1996). They suggest exponential decay of activity away from the excited end, provided the frequency of vibration is lower than a critical value and the elasticity tensor is a positive definite tensor.

On the other hand, Green and Naghdı (1992, 1993, 1995) proposed several thermoelastic theories for deformable continua which relies on an energy balance law rather than an entropy inequality. The thermodynamics introduces the thermal displacement $\tau$ that is related to the temperature variation $\theta$ by means of relation $\tau = \theta$. The characterization of material response for the thermal phenomena is based on three types of constitutive response functions, labeled as types I, II, and III. The nature of these three types of constitutive equations is such that when the respective theories are linearized, type I is the same as the classical theory of thermoelasticity, while the linearized versions of both types II and III theories permit propagation of thermal waves at finite speed.

The thermoelasticity of type III allows the constitutive functions for free energy, stress tensor, entropy and heat flux to depend on the strain tensor, the time derivative of the thermal displacement, the gradient of thermal displacement and the time derivative of the gradient of thermal displacement. This theory allows the dissipation energy, but the heat flux is partially determined from the Helmholtz free energy potential. A system of equations for one-dimensional linearized thermoelasticity is examined in Green and Naghdı (1992) along with some special cases that may be of interest in connection with experiments for second sound propagation. Further study on the wave propagation has been developed by Quintanilla and Straughan (2004), Puri and Jordan (2004) and Kovalev and Radayev (2010).

In the present paper we consider the general linear theory of thermoelasticity of type III for anisotropic and inhomogeneous bodies. We consider a prismatic cylinder, made of a linear thermoelastic material, which is subject to a harmonic in time vibration on its base, while the lateral boundary and the other end are clamped and thermally insulated. For the amplitude of the steady-state vibrations occurring in cylinder we associate an adequate measure. Then we infer that the decay of amplitude of vibrations is described by an exponential of distance from the excited end of the cylinder, provided that the positive definiteness of the elasticity tensor and of the heat conductivity tensor are assumed. These thermoelastic assumptions are milder than those expressing the positive definiteness of the internal energy and conductivity tensor.

Furthermore, we relax the assumption on the elasticity tensor by assuming that it is strongly elliptic. By introducing such hypothesis we are considering a very large class of materials including those new materials with extreme and unusual physical properties like...
negative Poisson’s ratio (that is, so called auxetic materials). These are materials with heterogeneous structure, including natural composites such as bone, ligament, and wood, as well as synthetic composites, biomaterials, and cellular solids with structural hierarchy (see, for example, Lakes, 1987; Jaglin’ski and Lakes, 2007). Such a material expands laterally when stretched, in contrast to ordinary materials.

2. Basic equations and harmonic in time solutions

Throughout this paper we consider a right prismatic cylinder $\Omega$ with a plane base. We choose the Cartesian system of rectangular axes such that the base is placed in the $x_1Ox_2$-plane, with the $x_3$-axis directed parallel to the axis of cylinder. We denote the generic cross-section of the cylinder by $D$, and assume that $D$ is a bounded plane two-dimensional region whose boundary $\partial D$ is sufficiently smooth to admit application of the divergence theorem. Although the cross-section is uniform for all $x_3 \geq 0$, we let $D(x_3)$ denote the cross-section at the distance $x_3$ from the base. We choose $\Omega = D \times [0, L]$, where $0 < L < \infty$ is the length of the cylinder. Moreover, we set $\Omega(x_3)$ for the portion of $\Omega$ contained between $D(x_3)$ and $D(L)$, that is $\Omega(x_3) = D \times [x_3, L]$. We assume that $\Omega$ is occupied by an inhomogeneous and anisotropic thermoelastic material. In what follows we will consider the linear dynamic theory of thermoelasticity of type III as described in Green and Nagdhi (1992, 1995). The governing equations of the linear theory of anisotropic and inhomogeneous thermoelasticity of type III are given by

\begin{equation}
\mathbf{t}_{r,s} + Q_s = Q \mathbf{u}_r, \quad (2.1)
\end{equation}

\begin{align}
\text{– the momentum balance}
\end{align}

\begin{equation}
\mathbf{Q} \dot{\mathbf{q}} = q_s \mathbf{r}_r + Q \mathbf{R}, \quad (2.2)
\end{equation}

\begin{align}
\text{– the equation of energy}
\end{align}

\begin{align}
\text{– the constitutive equations}
\end{align}

\begin{equation}
T_{rs} = C_{rsrr} \varepsilon_{rr} + C_{rsrr} \varepsilon_{ss} - M_{rr} \tau, \quad (2.3)
\end{equation}

\begin{align}
\text{for } (x, t) \in \Omega \times (0, \infty),
\end{align}

\begin{equation}
\varepsilon_{rr} = \frac{1}{2} (u_{rr} + u_{rr}), \quad \beta_r = \tau_{r,r}, \quad (2.4)
\end{equation}

\begin{align}
\text{– the geometrical relations}
\end{align}

\begin{align}
\text{for } (x, t) \in \overline{\Omega} \times (0, \infty), \text{ where } \overline{\Omega} \text{ is the closure of } \Omega. \text{ Further, the specific Helmholtz free energy is given by}
\end{align}

\begin{equation}
Q \psi = \left( \varepsilon_{rr} \varepsilon_{ss} + \frac{1}{2} \kappa_r \beta_r \beta_s - \frac{1}{2} c^2 \dot{\tau}^2 + \varepsilon_{rr} \varepsilon_{ss} \beta_m \right) - M_{rr} \varepsilon_{rr} \tau + N_r \tau \beta_r, \quad (2.5)
\end{equation}

\begin{align}
\text{while the specific internal energy is given by}
\end{align}

\begin{equation}
Q \varepsilon = \left( C_{rrrr} \varepsilon_{rr} \varepsilon_{ss} + C_{rrrr} \varepsilon_{ss} \beta_m + \frac{1}{2} \kappa_r \beta_r \beta_s + \frac{1}{2} c^2 \dot{\tau}^2 \right), \quad (2.6)
\end{equation}

\begin{align}
\text{and the internal rate of supply of heat per unit mass is defined by}
\end{align}

\begin{equation}
\dot{\theta} = B_{rs} \dot{\beta}_r \beta_s, \quad (2.7)
\end{equation}

\begin{align}
\text{and hence the heat conductivity tensor } B_{rs} \text{ satisfies dissipation inequality}
\end{align}

\begin{equation}
B_{rs} \beta_r \beta_s \geq 0. \quad (2.8)
\end{equation}

In the above relations, the thermal displacement $\tau$ is defined by

\begin{equation}
\tau(x, t) = \int_0^t \theta(x, s) \mathrm{d}s, \quad (2.9)
\end{equation}

where $\theta$ is the temperature variation from the uniform reference temperature $\theta_0$. Moreover, $\mathbf{u}_r$ are the components of the displacement vector, $\beta_r$ are the components of the strain tensor, $\eta$ are the components of the stress tensor, $\eta_S$ are the components of the entropy flux vector, $\eta$ is the heat density per unit mass and $f_r$ represent the components of the external body force vector per unit mass and $R$ is the external rate of supply of heat per unit mass. Furthermore, $\mathbf{Q}$ is the mass density, $\mathbf{C}_{rrrr}$, $\mathbf{C}_{rrss}$, $\mathbf{M}_{rr}$, $\mathbf{N}_{r}$, $c$, $\mathbf{K}_{rs}$ and $\mathbf{B}_{rs}$ are the constitutive coefficients depending on the spatial variable $x$, continuously differentiable on $\Omega$ and satisfying the following symmetries

\begin{equation}
C_{rrrr} = C_{rrrr}, \quad C_{rrss} = C_{rrss}, \quad M_{rr} = M_{rr}, \quad K_{rs} = K_{rs}, \quad B_{rs} = B_{rs}. \quad (2.10)
\end{equation}

Throughout this paper we will assume that the constitutive coefficients are bounded functions on $\overline{\Omega}$. The subscripts $r, s, m$ and $n$ take values 1, 2, 3 and summation is implied by index repetition. Moreover, a superposed dot denotes differentiation with respect to time and a subscript comma indicates partial differentiation.

The components of the surface traction and the heat flux at regular points of $\partial \Omega$ can be expressed in the form

\begin{equation}
t_v = t_m n_m, \quad q = q_r n_r, \quad (2.11)
\end{equation}

where $n_r$ are the components of the unit outward normal vector to $\partial \Omega$.

The cylinder is subject to zero body supplies and a prescribed harmonic in time vibration on its base and the lateral boundary and the other end are maintained at zero thermal displacement and zero displacement. Then, within cylinder we have harmonic in time solutions of the form

\begin{equation}
u_r(x, t) = \nu_r(x) e^{i\omega t}, \quad \theta(x, t) = T(x) e^{i\omega t}, \quad (2.12)
\end{equation}

where $\omega > 0$ is a prescribed constant and $i = \sqrt{-1}$ is the complex unit. The amplitude $(\nu_r, T)$ of the considered vibration is solution of the boundary value problem (P) defined by the basic equations

\begin{align}
S_{rr} + \alpha \nu_r = 0, \quad Q_r + \alpha \nu_r = 0, \quad \text{in } \Omega,
\end{align}

\begin{align}
S_{rr} = C_{rrrr} E_{rr} + C_{rrss} E_s - \kappa_r \beta_r, \quad (2.13)
\end{align}

\begin{align}
Q_r = C_{rrrr} E_{rr} + C_{rrss} E_s + \kappa_r \beta_r, \quad (2.14)
\end{align}

\begin{align}
E_{nn} = \frac{1}{2} (\nu_{ss} + \nu_{rr}), \quad Br = T_r, \quad \text{in } \overline{\Omega},
\end{align}

\begin{align}
\text{the lateral boundary conditions}
\end{align}

\begin{align}
\nu_r(x) = 0, \quad T(x) = 0, \quad \text{on } \pi = \partial D \times [0, L],
\end{align}

\begin{align}
\text{the end boundary conditions}
\end{align}

\begin{align}
\nu_r(x) = 0, \quad T(x) = 0, \quad \text{on } D(L),
\end{align}

\begin{align}
\text{and the base boundary conditions}
\end{align}

\begin{align}
\nu_r(x) = \tilde{v}_r(x), \quad T(x) = \tilde{T}(x), \quad \text{on } D(0),
\end{align}

where \(\tilde{v}_r(x)\) and \(\tilde{T}(x)\) are prescribed fields.

From here onwards a superscripted bar over a quantity refers to the complex conjugate of that quantity. In this connection, we note that relations (2.13)–(2.15) give the following identities
\[(\overline{T}_s \overline{S}_t + \overline{T}_Q e)_{r} = C_{rsm}(\overline{E}_m \overline{E}_m) + C_{rsm}(\overline{E}_m B_m + E_s \overline{B}_m) + K_{rs} B_r \overline{B}_s - \omega^2 (qV_p + cT) + \text{Im} M(\overline{E}_m T - \overline{E}_s T) + \text{ Im}_N(T_{T \tau} - T_{T \tau}) + \text{ Im}_B B_r B_s, \quad (2.19)\]

and

\[(\overline{v}_s \overline{S}_t + \overline{T}_Q e)_{r} = C_{rsm}(\overline{E}_m \overline{E}_m) + C_{rsm}(\overline{E}_m B_m + E_s \overline{B}_m) + K_{rs} B_r \overline{B}_s - \omega^2 (qV_p + cT) + \text{Im} M(\overline{E}_m T - \overline{E}_s T) + \text{ Im}_N(T_{T \tau} - T_{T \tau}) + \text{ Im}_B B_r B_s, \quad (2.20)\]

If we take into account the boundary condition (2.16), then from (2.19) and (2.20) we obtain

\[
\begin{align*}
\int_{D(x)} B_r B_s da = & \frac{d}{dx} \int_{D(x)} \frac{i}{2} (\overline{v}_s \overline{S}_s - \text{ Im}_S) + (\overline{T}_Q + \overline{T}_Q) da, \\
\end{align*}
\]

\[
(2.21)
\]

\[
2 \int_{D(x)} \left[ C_{rsm}(\overline{E}_m \overline{E}_m) + C_{rsm}(\overline{E}_m B_m + E_s \overline{B}_m) + K_{rs} B_r \overline{B}_s - \omega^2 (qV_p + cT) + \text{Im} M(\overline{E}_m T - \overline{E}_s T) + \text{ Im}_N(T_{T \tau} - T_{T \tau}) \right] da,
\]

\[
\int_{D(x)} \frac{d}{dx} \left[ (\overline{v}_s \overline{S}_s - \text{ Im}_S) + (\overline{T}_Q + \overline{T}_Q) \right] da. \\
\end{align*}
\]

\[
(2.22)
\]

3. Spatial behavior of harmonic vibrations

Our main aim in this section is to establish the spatial behavior of the amplitude \((v_t, T)\) of the harmonic vibration. To this end we use the following hypotheses upon the constitutive profile:

\[
\begin{align*}
\varrho > 0, \\
\mu_m \xi_m \xi_m \leq C_{rsm} \xi_m \xi_m \leq \mu_m \xi_m \xi_m \quad \text{for all } \xi_m = \xi_m, \\
v_m \zeta \xi_m \leq B_r B_s \xi_m \leq v_m \zeta \xi_m \quad \text{for all } \zeta, \\
(3.1) \\
(3.2) \\
(3.3)
\end{align*}
\]

where \(\mu_m > 0\) and \(\mu_m > 0\) are appropriate constants to the minimum and maximum eigenvalues of the positive definite elasticity tensor \(C_{rsm}\) and \(v_m\) and \(v_m\) are positive constants related to the minimum and maximum eigenvalues of the positive definite tensor \(B_m\). It is worth noting that we do not assume any positivity for the specific internal energy, which is defined by (2.6), which is a standard assumption in literature on the subject. More precisely, we do not assume any positivity for the thermoelastic coefficients \(K_{rs}\) or \(c\).

The main result of this section consists in the following spatial behavior theorem.

**Theorem 3.1.** Assume that the right prismatic cylinder \(\Omega\) is made of a thermoelastic material with the constitutive profile satisfying relations (3.1)–(3.3) and associate with the amplitude \((v_t, T)\) of the harmonic vibration the following measure

\[
\mathcal{J}(x_3) = \sigma_1 \int_{\Omega(x_3)} C_{rsm}(\overline{E}_m \overline{E}_m) dv + \sigma_2 \int_{\Omega(x_3)} B_r B_s B_s dv, \\
(3.4)
\]

with \(\sigma_1\) and \(\sigma_2\) appropriate positive parameters. For harmonic vibrations whose frequency \(\omega\) is lower than a critical value \(\omega_1\) it is possible to determine a positive constant \(v\) depending on the constitutive profile and \(\sigma_1, \sigma_2, \omega\) and \(\omega_1\), such that the measure \(\mathcal{J}(x_3)\) satisfies the decay estimate

\[
0 \leq \mathcal{J}(x_3) \leq \mathcal{J}(0)e^{-v(x_3 - h)}, \quad h \leq x_3 \leq L, \\
\end{align*}
\]

\[
\text{ for any } h \in [0, L]. \\
\end{align*}
\]

\[
(3.5)
\]

Proof. In order to prove the result, we introduce first the following function

\[
\mathcal{J}(x_3) = - \int_{D(x_3)} \left[ \frac{d}{dx} (\overline{v}_s \overline{S}_s - \text{ Im}_S) + \frac{1}{2} (\overline{v}_s \overline{S}_s + \text{ Im}_S) \right] da, \quad x_3 \geq 0, \\
(3.6)
\]

and note that, by combining the identities (2.21) and (2.22), we have

\[
\frac{d}{dx} (\overline{v}_s \overline{S}_s - \text{ Im}_S) + \frac{1}{2} (\overline{v}_s \overline{S}_s + \text{ Im}_S) + \text{ Im}_N(T_{T \tau} - T_{T \tau}) da, \quad x_3 \geq 0, \\
(3.7)
\]

where \(\delta\) is a positive parameter at our disposal.

Furthermore, the boundary condition (2.16) allows us to write

\[
\int_{D(x_3)} T_{T \tau} da \geq \lambda_0 \int_{D(x_3)} T da, \\
(3.8)
\]

where \(\lambda_0\) is the lowest positive eigenvalue in the clamped membrane problem, that is the first eigenvalue of the problem

\[
\phi, \alpha, \lambda = 0 \quad \text{ in } D, \\
\phi = 0 \quad \text{ on } \partial D. \\
(3.9)
\]

Moreover, by means of the Cauchy–Schwarz and the arithmetic–geometric mean inequalities, we obtain the following estimates

\[
\int_{D(x_3)} C_{rsm}(\overline{E}_m \overline{E}_m) da \leq \frac{G^*}{\mu_m} \int_{D(x_3)} C_{rsm}(\overline{E}_m \overline{E}_m) da, \\
+ \frac{G^*}{v_m} \int_{D(x_3)} B_r B_s B_s da, \quad \text{ for all } \xi_3 > 0, \\
(3.10)
\]

\[
\int_{D(x_3)} K_{rs} B_r B_s da \leq \frac{K^*}{v_m} \int_{D(x_3)} B_r B_s B_s da, \\
(3.11)
\]

\[
\int_{D(x_3)} c \omega^2 T T da \leq \omega^2 c^* \int_{D(x_3)} T da \leq \frac{\omega^2}{\chi_0 v_m} c^* \int_{D(x_3)} B_r B_s B_s da, \\
(3.12)
\]

\[
\int_{D(x_3)} \text{ Im}_N(T_{T \tau} - T_{T \tau}) da \leq \frac{\omega^2}{v_m \chi_0} \int_{D(x_3)} B_r B_s B_s da, \\
(3.13)
\]

\[
\int_{D(x_3)} \text{ Im}_N(T_{T \tau} - T_{T \tau}) da \leq \frac{\omega^2}{v_m \chi_0} \int_{D(x_3)} B_r B_s B_s da, \\
(3.14)
\]

where

\[
G^* = \sup \{G_{rsm} \overline{E}_m \overline{E}_m \}^{1/2}, \quad K^* = \sup \{K_{rs} B_r B_s \}^{1/2}, \quad c^* = \sup \{c \} \mu_*, \quad \mu_*, \quad \mu_* \Omega \\
= \sup \{M_3 M_\tau \}^{1/2}, \quad \mu_* = \sup \{N_3 N_\tau \}^{1/2}. \\
(3.15)
\]

Now, we use these estimates into (3.7) in order to get
\[
\frac{d\Gamma}{dx_3}(x_3) \geq \left( 1 - \frac{G^*}{\mu_m} \varepsilon_1 - \frac{om^*}{\mu_m} \varepsilon_2 \right) \int_{\Omega(x_3)} C_{sppp} E_p T_{pp} da \\
- \int_{\Omega(x_3)} Q\alpha^2 v_1 \bar{V}_1 da \\
+ \left( \delta - \frac{G^*}{\nu \varepsilon_1} - \frac{K^*}{\nu} - \frac{\alpha^2}{\xi_0 \varepsilon_1} \right) - \frac{om^*}{\nu} - \frac{2om^*}{\nu \sqrt{x_0}} \right) \\
\times \int_{\Omega(x_3)} B_{rs} B_{rs} da.
\]
(3.16)

At this instant, we follow Flavin and Knoops (1987) and suppose
\[
\omega_{1}^2 = \inf \frac{\int_B C_{ppp} v_p q V_{i3} dv}{\int_B q v_i V_{i3} dv},
\]
(3.17)

(a) in the class of all right cylinders \( B \), with uniform cross-sections \( D(x_3) \) and plane ends perpendicular to the generators, whose (axial) lengths lie between \( h \) and \( l(h < L) \), and
(b) in the class of continuously differentiable real vector fields \( V_i \) which satisfy \( V_i = 0 \) on the curved boundary.

Stated otherwise, \( \omega^* / 2\pi \) represents the minimum fundamental frequency of vibration of elastic cylinders which belong to class a) preceding, whose curved surfaces are clamped and whose plane ends are free.

Now we recall that the frequency of vibration satisfies the crucial requirement
\[
0 < \omega < \omega^*,
\]
(3.18)

Moreover, we note that the end boundary condition (2.17) and the relation (3.6) imply that \( \Omega(L) = 0 \) and hence the relations (3.16) and (3.17) prove that
\[
\Omega(x_3) \geq \sigma_1 \int_{\Omega(x_3)} C_{srrm} E_r T_{mr} dv + \sigma_2 \int_{\Omega(x_3)} B_{rs} B_{rs} dv,
\]
(3.19)

where the parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) are chosen so small, while the parameter \( \delta \) is chosen so large, to have
\[
\sigma_1 = 1 - \frac{\omega^2}{\omega_1^2} - \frac{G^*}{\mu_m} \varepsilon_1 - \frac{om^*}{\mu_m} \varepsilon_2 > 0,
\]
\[
\sigma_2 = \frac{\omega_1^2 - \frac{G^*}{\nu \varepsilon_1} - \frac{K^*}{\nu} - \frac{\alpha^2}{\xi_0 \varepsilon_1} \varepsilon_1}{\nu \varepsilon_1} - \frac{om^*}{\nu \varepsilon_1} - \frac{2om^*}{\nu \sqrt{x_0}} > 0.
\]
(3.20)

Now we introduce the function
\[
\kappa(x_3) = \frac{1}{l} \int_{x_3-h}^{x_3+h} \mathcal{J}(z) dz,
\]
(3.21)

and note that, by means of (3.4), we have successively
\[
- \frac{\partial \kappa}{\partial x_3}(x_3, h) = - \frac{1}{l} \left[ \mathcal{J}(x_3 + h) - \mathcal{J}(x_3) \right] = - \frac{1}{l} \int_{x_3-h}^{x_3+h} \frac{d}{dz} \mathcal{J}(z) dz
\]
\[
= \frac{\sigma_1}{l} \int_{\Omega(x_3, h)} C_{srrm} E_r T_{mr} dv + \frac{\sigma_2}{l} \int_{\Omega(x_3, h)} B_{rs} B_{rs} dv,
\]
(3.22)

where \( \mathcal{J}(x_3, h) = \Omega(x_3) \setminus \Omega(x_3 + h) \).

On the other hand, from (3.4), (3.6) and (3.19), we have
\[
0 \leq \mathcal{J}(x_3) \leq \int_{\Omega(x_3)} \left| v_1 S_{31} + v_7 S_{31} \right| + |T_{Q_3} + T_{Q_1}| da
\]
(3.23)

and by means of relations (2.14) and (3.17) and the Cauchy-Schwarz inequality, it follows that there are the computable positive constants \( \chi_1 \) and \( \chi_2 \), depending on the thermoelastic coefficients and \( \delta \), \( \omega \) and \( \omega^* \), so that
\[
0 \leq \kappa(x_3, h) \leq \frac{\chi_1}{h} \int_{\Omega(x_3, h)} C_{srrm} E_r T_{mr} dv \leq \frac{\chi_2}{h} \int_{\Omega(x_3, h)} B_{rs} B_{rs} dv.
\]
(3.24)

Finally, by combining relations (3.22) and (3.24), we obtain the following first-order differential inequality
\[
\nu \kappa(x_3, h) + \frac{\partial \kappa}{\partial x_3}(x_3, h) \leq 0 \quad \text{for all} \ x_3 \in [0, L - h],
\]
(3.25)

where
\[
\frac{1}{\nu} = \max \left( \frac{\chi_1}{\sigma_1}, \frac{\chi_2}{\sigma_2} \right).
\]
(3.26)

By integrating the above differential inequality (3.25), we obtain the following spatial decay estimate
\[
0 \leq \kappa(x_3, h) \leq \kappa(0, h)e^{-\nu x_3} \quad \text{for all} \ x_3 \geq 0.
\]
(3.27)

Moreover, since \( \mathcal{J}(x_3) \) is a non-increasing function on \([0, L]\), it follows that
\[
\mathcal{J}(x_3 + h) \leq \kappa(x_3, h) \leq \mathcal{J}(x_3),
\]
(3.28)

and hence (3.27) implies the spatial decay estimate (3.5) and the proof is complete. \( \Box \)

Remark 3.1. As it can be seen from the above proof the decay rate \( \nu \) depends on the constitutive profile as well as the parameters \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \delta \) chosen such that to make strictly positive \( \sigma_1 \) and \( \sigma_2 \). It is not so ready that this choice leads to a best possible value for the decay rate of the end effects. Essentially here it is the fact that it predicts an exponentially decay of these end effects.

Remark 3.2. Although we considered here only the end boundary conditions in the form expressed by (2.17) and (2.18), our analysis in this section works yet for the other common end conditions where the traction and the heat flux are assigned as harmonic in time functions. However, the zero lateral boundary conditions (2.16) are essentially for our analysis.

4. Further results

In this section we will discuss the spatial behavior of the amplitude of vibrations for the larger class of strongly elliptic thermoelastic materials, a class including important materials such as those with negative Poisson's ratio or auxetics. Such materials, when stretched, become thicker perpendicular to the applied force. This occurs due to their hinge-like structures, which flex when stretched. These materials possess important mechanical properties such as high energy absorption and fracture resistance. Auxetics may be useful in applications such as body armor, packing material, knee and elbow pads, robust shock absorbing material, and sponge mops. To this end we will use the strong ellipticity condition of the elasticity tensor, that is
\[
C_{srrm} \xi_r \xi_m \xi_n > 0 \quad \text{for all non-zero vectors}
\]
(4.1)

\[
\times (\xi_1, \xi_2, \xi_3), (\xi_1, \xi_2, \xi_3).
\]
(4.1)

a hypothesis that is weaker than the corresponding positive definite condition expressed by (3.2).

Our results will be exemplified for thermoelastic materials with rhombic symmetry. Using a standard notation for the non-zero
components of the elasticity tensor, $C_{mm}$, the strong ellipticity condition (4.1) becomes

$$c_{11}\xi_1^2\xi_1^2 + c_{22}\xi_2^2\xi_2^2 + c_{33}\xi_3^2\xi_3^2 + c_{66}(\xi_1\xi_2 + \xi_2\xi_1)^2$$
$$+ c_{55}(\xi_1\xi_3 + \xi_3\xi_1)^2 + c_{44}(\xi_2\xi_3 + \xi_3\xi_2)^2 + 2c_{12}\xi_1\xi_2\xi_2$$
$$+ 2c_{13}\xi_1\xi_3\xi_3 + 2c_{23}\xi_2\xi_3\xi_3 > 0$$
(4.2)

for all non-zero vectors $(\xi_1, \xi_2, \xi_3)$. It was shown in Chiriţă et al. (2007) that the condition (4.2) is equivalent to the following conditions

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{33} > 0, \quad c_{44} > 0, \quad c_{55} > 0, \quad c_{66} > 0, (4.3)$$
$$-2c_{66} + k_1^2\sqrt{c_{11}c_{22}} < \triangle c_1 < k_1^2\sqrt{c_{11}c_{22}},$$
$$-2c_{66} + k_2^2\sqrt{c_{11}c_{33}} < \triangle c_2 < k_2^2\sqrt{c_{11}c_{33}},$$
$$-2c_{66} + k_3^2\sqrt{c_{22}c_{33}} < \triangle c_3 < k_3^2\sqrt{c_{22}c_{33}},$$
(4.4)

where $(k_1^2, k_2^2, k_3^2)$ and $(\triangle c_1, \triangle c_2, \triangle c_3)$ are the two solutions with respect to $x$ and $y$ of the equation $x^2 + y^2 - 2x - 1 = 0$ satisfying $|x| < 1$, $y < 1$, $z < 1$, with $x$ in the set $(c_{22}/\sqrt{c_{11}c_{22}}, (c_{22} + c_{44})/\sqrt{c_{11}c_{44}})$, $y$ in the set $(c_{11}/\sqrt{c_{11}c_{33}}, (c_{11} + c_{55})/\sqrt{c_{11}c_{55}})$ and $z$ in the set $(c_{44}/\sqrt{c_{22}c_{44}}, (c_{22} + c_{44})/\sqrt{c_{22}c_{44}})$. That means all the points $(x, y, z)$ with coordinates $x \in (c_{22}/\sqrt{c_{11}c_{22}}, (c_{22} + c_{44})/\sqrt{c_{11}c_{44}})$, $y \in (c_{11}/\sqrt{c_{11}c_{33}}, (c_{11} + c_{55})/\sqrt{c_{11}c_{55}})$ and $z \in (c_{44}/\sqrt{c_{22}c_{44}}, (c_{22} + c_{44})/\sqrt{c_{22}c_{44}})$ lie inside the domain limited by the surface $S(x, y, z) = x^2 + y^2 + z^2 - 2x = 0$, $|x| < 1$, $y < 1$, $z < 1$.

On this basis we can establish the following

**Theorem 4.1.** Assume that the right prismatical prism $\Omega$ is made of a thermoelastic material with rhombic symmetry whose elastic tensor is strongly elliptic and whose heat conductivity tensor is positive definite and associate with the amplitude $(v_r, T)$ of the harmonic vibration the following measure

$$\mathcal{H}(x_3) = \int_{\mathcal{D}(x_3)} (\sigma_{11}v_{1r}\mathbf{T}_{1r} + \sigma_{22}v_{2r}\mathbf{T}_{2r} + \sigma_{33}v_{3r}\mathbf{T}_{3r} + \sigma_3^2\mathbf{T}_{3}\mathbf{T}_{3r} + \sigma_1^2\mathbf{T}_{1}\mathbf{T}_{3r} + \sigma_2^2\mathbf{T}_{2}\mathbf{T}_{3r}) \, dv,$$
(4.5)

with $\sigma_1^2, \sigma_2^2, \sigma_3^2$ appropriate positive parameters. For harmonic vibrations whose frequency $\omega$ is lower than a critical value $\omega^*$ it is possible to determine a positive constant $\bar{v}$ depending on the constitutive profile and $\sigma_1^2, \sigma_2^2, \sigma_3^2, \omega$ and $\omega^*$ such that the measure $\mathcal{H}(x_3)$ decays faster than the decaying exponential $e^{-\omega_3 x}$, for all $0 \leq x_3 \leq L$.

**Proof.** We first note that, in view of the boundary condition (2.16), we have the following identities

$$\int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{1r}) \, dv = \int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{2r}) \, dv,$$
(4.6)

$$\frac{d}{dx_3} \int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{1r}) \, dv = \int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{2r}) \, dv,$$
(4.7)

$$\frac{d}{dx_3} \int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{1r}) \, dv = \int_{\mathcal{D}(x_3)} (v_{1r}\mathbf{T}_{1r} + v_{1r}\mathbf{T}_{2r}) \, dv,$$
(4.8)

Further, we introduce the following function

$$\tilde{\omega}^2 = \sqrt{\frac{\lambda + \mu}{\rho^2}}, \quad \omega^* = \sup_{\mathcal{D}} \omega,$$
(4.15)
and assume that
\[ 0 < \omega < \omega_2. \quad (4.16) \]
Then, it becomes possible to choose the positive parameters \( \varepsilon_3 \) and \( \varepsilon_4 \) to be so small, and \( \delta \) to be so large in order to make the parameters \( \sigma_1^+, \sigma_2^+, \sigma_3^+, \sigma_4^+ \) to be strictly positive. Further, since relations (2.17) and (4.9) imply \( H(L) = 0 \), from (4.13), by an integration over \([x_3, L]\) and the use of (4.5), we obtain that
\[ H(x_3) \geq \mathcal{H}(x_3), \quad \text{for all} \quad x_3 \geq 0. \quad (4.17) \]
On the other hand, using the Cauchy–Schwarz and arithmetic–geometric mean inequalities and estimates like (3.8), it results that is possible to determine the positive constants \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) and \( \tilde{x}_4 \) depending on the constitutive coefficients and \( \lambda_0, \omega, \delta, x_1, x_2 \) and \( x_3 \), such that
\[ H(x_3) \leq \int_{(x_3)} (\tilde{x}_1 v_{r,a} v_{r,a} + \tilde{x}_2 v_{r,3} v_{r,3} + \tilde{x}_3 T_{r,a} T_{r,a} + \tilde{x}_4 T_{r,3} T_{r,3}) da. \quad (4.18) \]
Therefore, if we set
\[ \frac{1}{\mathcal{H}} = \max \left( \frac{x_1}{\sigma_1^+}, \frac{x_2}{\sigma_2^+}, \frac{x_3}{\sigma_3^+}, \frac{x_4}{\sigma_4^+} \right), \quad (4.19) \]
then the relations (4.13) and (4.18) lead to the following first-order differential inequality
\[ \frac{dH}{dx_3}(x_3) + \frac{x_3}{\mathcal{H}} H(x_3) \leq 0 \quad \text{for all} \quad x_3 \geq 0, \quad (4.20) \]
whose integration furnishes the spatial estimate
\[ 0 \leq H(x_3) \leq H(0)e^{-x_3} \quad \text{for all} \quad x_3 \geq 0. \quad (4.21) \]
Thus, the relations (4.17) and (4.21) prove that the measure \( H(x_3) \) decays exponentially with respect to \( x_3 \) and the proof is complete.

\[ \square \]

**Remark 4.1.** It can be observed that the zero end boundary condition (2.17) is essentially for the analysis of this section.

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**References**


