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On the theory of thermoelasticity with microtemperatures

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ABSTRACT

This paper studies the linear theory of thermoelastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. We discuss uniqueness results within the context of the dynamic boundary value problems under very mild assumptions upon the thermoelastic profile. We also establish the relations describing the partition of the total energy in terms of the Cesàro means of various types of component energies.

1. Introduction

This paper is concerned with the linear theory of thermoelasticity with microtemperatures as developed by Ieşan and Quintanilla [1]. Such a theory takes into consideration the microstructure of the body and, consequently, each microelement possesses a microtemperature. The theory of elastic materials with microstructures, goes back to the book of E. and F. Cosserat [2]. After that, the theory of materials with microstructures became a subject of intensive study in the literature (see, for example, the articles [3–7] and the books [8–11]).

Grot [12] was the first to take into consideration the inner structure of a body in order to develop a thermodynamic theory for thermoelastic materials where microelements, in addition to classic microdeformations, possess microtemperatures. The Clausius–Duhem inequality is modified to include microtemperatures and the first-order moment of the energy equations are added to the common balance laws of a micromorphic continuum. Riha [13] developed a further study concerning heat conduction in thermoelastic materials with inner structure. It is shown that the experimental data for the silicone rubber containing spherical aluminum particles and for human blood are conform closely to the predicted theoretical model of thermoelasticity with microtemperatures.

The theory of thermoelasticity with microtemperatures has been further investigated in various papers. Thus, Riha [14,15] studies a theory of heat conducting micropolar fluids with microtemperatures, while Ieşan [16,17] develops a linear theory for elastic materials with inner structure whose particles, in addition to the classical displacement and temperature, possess microtemperatures and can stretch and contract independently of their translations. Recently, Ieşan and Quintanilla [18] have established a linear theory of thermoelastic solids with microstructures and microtemperatures which permits the transmission of heat as thermal waves at finite speed and Ieşan [19] derives a linear theory of microstretch elastic solids with microtemperatures (see also the recent book by Straughan [20] and the papers cited therein).

In [1] Ieşan and Quintanilla consider the simplest thermomechanical theory of elastic materials that takes into account the microtemperature variables and then they establish some basic results concerning the uniqueness, existence and asymptotic

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behavior of dynamic solutions and some equilibrium specific problems. As regards the uniqueness result, it is established by assuming that the strain energy and the dissipation energy are positive semi-definite quadratic forms and the specific heat and the specific coefficient of the first moment of energy vector are strictly positive.

Since then the theory of thermoelasticity with microtemperatures has attracted much attention in connection with the study of the basic qualitative properties of solutions to the problems relating to various thermomechanical situations. Thus, Svanadze [21,22], Scalia and Svanadze [23,24] and Scalia et al. [25] study the fundamental solutions and proves some existence and uniqueness theorems for equilibrium solutions and steady state vibrations by means of the potential method, while Ieşan and Scalia [26] consider the plane strain in a homogeneous and isotropic body with microtemperatures. The behavior of shock waves and higher-order discontinuities which propagate in a thermoelastic body with inner structure and microtemperatures are studied by Ieşan [27] and the propagation of singularities of solutions to the Cauchy problem of a semilinear thermoelastic system with microtemperatures in one space variable is studied by Yang and Huang in [28]. Some basic theorems are established by Aouadi [29] and Svanadze and Tracina [30] in the linear theory of microstretch thermoelasticity for isotropic solids with microtemperatures. Finally, Ciarletta et al. [31] investigate a model for a rigid heat conductor which allows for variation of thermal properties at a microstructure level and they examine how the solution depends on changes in coupling coefficients between the macrothermal and microthermal levels.

In this paper we consider the linear theory of thermoelasticity with microtemperatures as developed in [1]. The uniqueness problem for solutions of the initial boundary value problems of the theory in concern is treated by means of the Lagrange identities method [32] without any sign-defined assumption upon the elastic energy and under very mild assumptions upon the other thermoelastic coefficients, including those describing the dissipation energy. This is possible because of the special coupling between the differential equations for the temperature variation and that for the microtemperature field. Such a treatment of the uniqueness problem is more complete with respect to that given in [1,33]. Furthermore, we show how the Lagrange identities method can be used to establish the asymptotic partition of total energy associated with the solutions of the initial–boundary value problems of the theory in concern. We materialize the idea of partition in terms of the asymptotic behavior in the Cesarò sense of various component energies. Thus, we prove that the Cesarò mean of the thermal energy due to the microtemperature fields goes to zero as time tends to infinity, while the Cesarò mean of the thermal energy due to the temperature variation tends to a constant when time tends to infinity. The results established in the present paper concerning the asymptotic partition of energy generalize those developed by Chiriţă [34].

2. Basic equations

Throughout this section B is a bounded regular region of the three-dimensional Euclidean space. We let ∂B denote the boundary of B, and designate by $\mathbf{n}$ the outward unit normal on ∂B. We assume that the body occupying B is a linearly elastic material which possesses thermal variation at a microstructure level. The body is referred to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. Throughout this paper Latin indices have the range 1, 2, 3, Greek indices have the range 1, 2 and the usual summation convention is employed. We use a subscript preceded by a comma for partial differentiation with respect to the corresponding coordinate and a superposed dot denotes partial differentiation with respect to time.

The temperature at a point $\mathbf{x}$ of the body depends on a temperature $\theta\,(\mathbf{x}, t)$, which may be thought of as an averaged temperature at $\mathbf{x}$, and three microtemperatures $w_i\,(\mathbf{x}, t)$ which contribute to the thermal microstructure of the material. The deformation of a body can be described by means of three, namely, the displacement vector field $\mathbf{u}$, the microtemperature vector field $\mathbf{w}$ and the temperature variation T, measured from the constant absolute temperature $T_0(> 0)$, over $B \times [0, \infty)$.

Within the framework of the linear theory developed by Ieşan and Quintanilla [1], the constitutive equations for a homogeneous and isotropic thermoelastic solid with microtemperatures are

$$\begin{aligned} &t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} - \beta T\delta_{ij},\\ &\varrho\eta = \beta e_{rr} + aT,\\ &\varrho\varepsilon_i = -bw_i,\\ &q_i = kT_{,i} + \kappa_1 w_i,\\ &Q_i = (\kappa_1 - \kappa_2)\, w_i + (k - \kappa_3)\, T_{,i},\\ &q_{ij} = -\kappa_4 w_{r,r}\delta_{ij} - \kappa_5 w_{i,j} - \kappa_6 w_{j,i}, \end{aligned} \tag{2.1}$$

where

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{2.2}$$

Here, t_{ij} are the components of the stress tensor, ϱ is the reference mass density, η is the entropy per unit mass, ε_i are the components of the first moment of energy vector, q_i are the components of the heat flux vector, Q_i are the components of the mean heat flux vector, q_{ij} are the components of the first heat flux moment vector, e_{ij} are the components of the strain tensor, u_i are the components of the displacement vector, w_i are the components of the microtemperature vector, T is the temperature variation, λ, μ, β, a, b, k and κ_r $(r = 1, 2, \ldots, 6)$ are constant constitutive coefficients and δ_{ij} is the Kronecker delta.

The fundamental system of field equations of the linear theory of thermoelasticity with microtemperatures consists of [1]:

– the equations of motion

$$t_{ji,j} + \varrho f_i = \varrho \ddot{u}_i, \tag{2.3}$$

– the balance energy

$$\varrho T_0 \dot{\eta} = q_{i,i} + \rho S, \tag{2.4}$$

– the first moment of energy

$$\varrho \dot{\varepsilon}_i = q_{ji,j} + q_i - Q_i + \varrho M_i, \tag{2.5}$$

where f_i are the components of the body force vector, M_i are the components of the first heat source moment vector and S is the heat supply.

The components of surface traction t_i, the heat flux q and the components of the first heat flux moment Λ_i at a regular point $\mathbf{x}$ of the boundary ∂B are given by

$$t_i = t_{ji} n_j, \qquad q = q_i n_i, \qquad \Lambda_i = q_{ji} n_j, \tag{2.6}$$

where $n_j = \cos(\mathbf{n_x}, Ox_j)$ and $\mathbf{n_x}$ is the unit vector of the outward normal to ∂B at $\mathbf{x}$.

Within the context of the linear theory of thermoelasticity considered in [1], the Clausius–Duhem inequality reduces to

$$q_i T_{,i} - T_0 q_{ji} w_{i,j} - T_0 \left(Q_i - q_i\right) w_i \geq 0, \tag{2.7}$$

which, in combination with (2.1), implies

$$\begin{aligned} &3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \qquad \kappa_6 + \kappa_5 \geq 0, \qquad \kappa_6 - \kappa_5 \geq 0, \\ &k \geq 0, \quad (\kappa_1 + T_0\kappa_3)^2 - 4T_0 k \kappa_2 \leq 0. \end{aligned} \tag{2.8}$$

To the above fundamental equations we adjoin the initial conditions

$$\begin{aligned} &u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \qquad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \\ &T(\mathbf{x}, 0) = T^0(\mathbf{x}), \qquad w_i(\mathbf{x}, 0) = w_i^0(\mathbf{x}), \quad \mathbf{x} \in \overline{B}, \end{aligned} \tag{2.9}$$

and the boundary conditions

$$\begin{aligned} &u_i(\mathbf{x}, t) = \tilde{u}_i(\mathbf{x}, t) \quad \text{on } \overline{\Sigma}_1 \times [0, \infty), \qquad t_{ji}(\mathbf{x}, t)\, n_j = \tilde{t}_i(\mathbf{x}, t) \quad \text{on } \Sigma_2 \times [0, \infty), \\ &T(\mathbf{x}, t) = \tilde{T}(\mathbf{x}, t) \quad \text{on } \overline{\Sigma}_3 \times [0, \infty), \qquad q_i(\mathbf{x}, t)\, n_i = \tilde{q}(\mathbf{x}, t) \quad \text{on } \Sigma_4 \times [0, \infty), \\ &w_i(\mathbf{x}, t) = \tilde{w}_i(\mathbf{x}, t) \quad \text{on } \overline{\Sigma}_5 \times [0, \infty), \qquad q_{ji}(\mathbf{x}, t)\, n_j = \tilde{\Lambda}_i(\mathbf{x}, t) \quad \text{on } \Sigma_6 \times [0, \infty), \end{aligned} \tag{2.10}$$

where $u_i^0(\mathbf{x}), v_i^0(\mathbf{x}), T^0(\mathbf{x}), w_i^0(\mathbf{x})$ and $\tilde{u}_i(\mathbf{x}, t), \tilde{t}_i(\mathbf{x}, t), \tilde{T}(\mathbf{x}, t), \tilde{q}(\mathbf{x}, t), \tilde{w}_i(\mathbf{x}, t), \tilde{\Lambda}_i(\mathbf{x}, t)$ are prescribed smooth functions. Moreover, Σ_r $(r = 1, 2, \ldots, 6)$ are subsets of the boundary ∂B such that $\overline{\Sigma}_1 \cup \Sigma_2 = \overline{\Sigma}_3 \cup \Sigma_4 = \overline{\Sigma}_5 \cup \Sigma_6 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \emptyset$.

The mixed problem $(\mathcal{P})$ consists of finding a solution $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$, with $u_i \in C^{2,2}(B \times [0, \infty))$, $T, w_i \in C^{2,1}(B \times [0, \infty))$ that satisfy the fundamental equations (2.1)–(2.5), the initial conditions (2.9) and the boundary conditions (2.10), provided smooth data $\mathcal{D} = \left\{\mathbf{f}, S, \mathbf{M}; \mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0; \tilde{\mathbf{u}}, \tilde{\mathbf{t}}, \tilde{T}, \tilde{q}, \tilde{\mathbf{w}}, \tilde{\mathbf{\Lambda}}\right\}$ are prescribed.

We have to outline that, by substituting the relations (2.1) and (2.2) into relations (2.3)–(2.5), we obtain the following system of linear partial differential equations for $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$

$$\begin{aligned} &\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ji} - \beta T_{,i} + \varrho f_i = \varrho \ddot{u}_i, \\ &k T_{,ii} - \beta T_0 \dot{u}_{i,i} + \kappa_1 w_{i,i} + \varrho S = a T_0 \dot{T}, \\ &\kappa_6 w_{i,jj} + (\kappa_4 + \kappa_5)\, w_{j,ji} - \kappa_3 T_{,i} - \kappa_2 w_i - \varrho M_i = b \dot{w}_i. \end{aligned} \tag{2.11}$$

It has to be observed that in the case when $\kappa_1 \kappa_3 = 0$ the first two equations in (2.11) decouple with respect to the last equation and so we have to study separately the differential system associated with the classical theory of thermoelasticity and the differential system of microtemperatures. For this reason, throughout this paper we will assume that $\kappa_1 \kappa_3 \neq 0$.

For convenience, in order to make the paper to be self-contained, we give in the Appendix a series of identities which turned out useful in our subsequent analysis.

3. Uniqueness results

In this section we will use the auxiliary lemmas of the Appendix in order to establish uniqueness results under appropriate mild constitutive hypotheses.

Theorem 1. *Suppose that*

$$\varrho > 0, \qquad k > 0, \qquad \kappa_2 > 0, \tag{3.1}$$

$$3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \qquad \kappa_6 + \kappa_5 \geq 0, \qquad \kappa_6 - \kappa_5 \geq 0, \tag{3.2}$$

$$b \geq 0, \qquad \kappa_1\kappa_3 < k\kappa_2. \tag{3.3}$$

If moreover, $a \neq 0$ *or* meas $\Sigma_3 \neq 0$, *then the mixed problem* $(\mathcal{P})$ *has at most one solution.*

Proof. Let us assume that there are two solutions of the mixed problem, $\mathcal{S}^{(\alpha)} = \left\{\mathbf{u}^{(\alpha)}, T^{(\alpha)}, \mathbf{w}^{(\alpha)}\right\}(\mathbf{x}, t)$, $(\alpha = 1, 2)$, corresponding to the same given data $\mathcal{D} = \left\{\mathbf{f}, S, \mathbf{M}; \mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0; \tilde{\mathbf{u}}, \tilde{\mathbf{t}}, \tilde{T}, \ \tilde{q}, \tilde{\mathbf{w}}, \tilde{\boldsymbol{\Lambda}}\right\}$. Then the difference $\mathcal{S} = \mathcal{S}^{(1)} - \mathcal{S}^{(2)}$, that is $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t) = \left\{\mathbf{u}^{(1)} - \mathbf{u}^{(2)}, T^{(1)} - T^{(2)}, \mathbf{w}^{(1)} - \mathbf{w}^{(2)}\right\}(\mathbf{x}, t)$, is a solution of the mixed problem $(\mathcal{P})$ corresponding to null data $\mathcal{D} = 0$. In order to prove the uniqueness result we have to prove that $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t) = 0$. For the difference solution $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$, the auxiliary identities (A.1) and (A.4) when coupled with zero body supplies and null initial and boundary data, imply

$$\begin{aligned}
&\frac{1}{2}\int_B \left[\varrho\dot{u}_i(t)\dot{u}_i(t) + \lambda e_{mm}(t)e_{nn}(t) + 2\mu e_{ij}(t)e_{ij}(t) + aT^2(t) + \sigma b w_i(t)w_i(t)\right] dv \\
&\quad + \int_0^t \int_B \left[\frac{k}{T_0}T_{,i}(s)\,T_{,i}(s) + \left(\frac{\kappa_1}{T_0} + \kappa_3\sigma\right)T_{,i}(s)w_i(s) + \kappa_2\sigma\, w_i(s)w_i(s)\right] dv ds \\
&\quad + \sigma\int_0^t \int_B \left[\kappa_4 w_{m,m}(s)\, w_{n,n}(s) + \kappa_5 w_{i,j}(s)w_{j,i}(s) + \kappa_6 w_{i,j}(s)w_{i,j}(s)\right] dv ds = 0,
\end{aligned} \tag{3.4}$$

$$\kappa_3 T_0 \int_B \left\{\varrho\dot{u}_i(t)\dot{u}_i(t) - \left[\lambda e_{mm}(t)e_{nn}(t) + 2\mu e_{ij}(t)e_{ij}(t) + aT^2(t)\right]\right\} dv - \kappa_1 \int_B b w_i(t)w_i(t) dv = 0. \tag{3.5}$$

By combining the relations (3.4) and (3.5), we obtain

$$\begin{aligned}
&\int_B \varrho\dot{u}_i(t)\dot{u}_i(t)\, dv + \frac{1}{2}\left(\sigma - \frac{\kappa_1}{\kappa_3 T_0}\right)\int_B b w_i(t)w_i(t) dv \\
&\quad + \int_0^t \int_B \left[\frac{k}{T_0}T_{,i}(s)\,T_{,i}(s) + \left(\frac{\kappa_1}{T_0} + \kappa_3\sigma\right)T_{,i}(s)w_i(s) + \kappa_2\sigma\, w_i(s)w_i(s)\right] dv ds \\
&\quad + \sigma\int_0^t \int_B \left[\kappa_4 w_{m,m}(s)\, w_{n,n}(s) + \kappa_5 w_{i,j}(s)w_{j,i}(s) + \kappa_6 w_{i,j}(s)w_{i,j}(s)\right] dv ds = 0.
\end{aligned} \tag{3.6}$$

In view of the hypothesis (3.3) we have $\kappa_1\kappa_3 < k\kappa_2$ a relation that implies that, for each σ so that

$$0 < \sigma_1 < \sigma < \sigma_2, \tag{3.7}$$

with

$$\sigma_{1,2} = \frac{1}{T_0\kappa_3^2}\left[2k\kappa_2 - \kappa_1\kappa_3 \mp 2\sqrt{k\kappa_2\left(k\kappa_2 - \kappa_1\kappa_3\right)}\right], \tag{3.8}$$

there is true

$$\left(\frac{\kappa_1}{T_0} + \kappa_3\sigma\right)^2 < \frac{4}{T_0}\sigma k\kappa_2, \tag{3.9}$$

that is

$$f(\sigma) \equiv \kappa_3^2\sigma^2 - \frac{2}{T_0}\left(2k\kappa_2 - \kappa_1\kappa_3\right)\sigma + \frac{\kappa_1^2}{T_0^2} < 0 \quad \text{for all } \sigma \in (\sigma_1, \sigma_2). \tag{3.10}$$

It follows that, for each $\sigma \in (\sigma_1, \sigma_2)$, the third integral in (3.6) is a positive term.

Let us first consider the case when $\kappa_1\kappa_3 < 0$. Then, with $\sigma \in (\sigma_1, \sigma_2)$ and by using (3.2) and (3.9), it follows that each term of the sum into relation (3.6) is positive and hence we can conclude that

$$\dot{u}_i(\mathbf{x}, t) = 0, \qquad w_i(\mathbf{x}, t) = 0, \qquad T_{,i}(\mathbf{x}, t) = 0 \quad \text{in } B \times [0, \infty). \tag{3.11}$$

Let us now consider the case when $\kappa_1\kappa_3 > 0$. Then, we observe that

$$f\left(\frac{\kappa_1}{T_0\kappa_3}\right) = -\frac{4}{T_0^2}\frac{\kappa_1}{\kappa_3}\left(k\kappa_2 - \kappa_1\kappa_3\right) < 0 \tag{3.12}$$

and hence we have $\sigma_1 < \frac{\kappa_1}{T_0\kappa_3} < \sigma_2$. Thus, if we choose σ so that

$$\sigma \in \left(\frac{\kappa_1}{T_0\kappa_3}, \sigma_2\right), \tag{3.13}$$

then each term of the sum into relation (3.6) is positive and, therefore, we can obtain the same conclusion with that expressed in (3.11).

Concluding, by taking into account the zero initial conditions, we can see that, for $a \neq 0$, the relations (2.11) and (3.11) imply that

$$u_i(\mathbf{x}, t) = 0, \qquad w_i(\mathbf{x}, t) = 0, \qquad T(\mathbf{x}, t) = 0 \quad \text{in } B \times [0, \infty) \tag{3.14}$$

and hence we have the requested uniqueness result.

Otherwise, in the case when meas $\Sigma_3 \neq 0$, we have

$$\int_B T_{,i}T_{,i}dv \geq \lambda_0 \int_B T^2 dv, \tag{3.15}$$

where λ_0 is the lowest eigenvalue of the membrane problem for B. Further, we use (3.11) and (3.15) in order to establish (3.14), that is we have again the uniqueness result. □

Remark 1. The above uniqueness result yet remains valid under the hypotheses expressed by (3.1), (3.2) and

$$b < 0, \qquad 0 < \kappa_1\kappa_3 < k\kappa_2. \tag{3.16}$$

In fact, in such a case we have to choose the arbitrary positive parameter σ so that

$$0 < \sigma_1 < \sigma < \frac{\kappa_1}{\kappa_3 T_0} \tag{3.17}$$

and so we have that each term of (3.6) is positive. Then an analysis similar to that in the proof of Theorem 1 leads to the uniqueness result. We have to mention that a uniqueness result was obtained in [33] by means of the logarithmic convexity method, under the assumption that κ_1 and κ_3 have the same sign.

Remark 2. The uniqueness result described in Theorem 1 yet remains valid when $b < 0$ and $\kappa_1\kappa_3 < 0$, but in this case we have an ill-posed problem and we have to restrict the class of solutions. More precisely, if we suppose that

$$\begin{aligned} &\varrho > 0, \qquad k > 0, \qquad \kappa_2 > 0, \\ &3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \qquad \kappa_6 + \kappa_5 \geq 0, \qquad \kappa_6 - \kappa_5 \geq 0, \\ &b < 0, \qquad \kappa_1\kappa_3 < 0 \end{aligned} \tag{3.18}$$

then there exists a strictly positive constant α so that, in the class of solutions $\{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ for which

$$\int_0^t \int_B w_i(s)w_i(s)\, dvds \leq M^2 e^{\alpha t}, \quad \text{for all } t \in [0, \infty),\ M = \text{constant}, \tag{3.19}$$

the mixed problem $(\mathcal{P})$ has at most one solution, provided $a \neq 0$ or meas $\Sigma_3 \neq 0$. In fact, under the hypotheses expressed in (3.18), we can choose $\sigma \in (\sigma_1, \sigma_2)$ so that

$$\begin{aligned} &\int_0^t \int_B \left[\frac{k}{T_0} T_{,i}(s)\, T_{,i}(s) + \left(\frac{\kappa_1}{T_0} + \kappa_3\sigma\right) T_{,i}(s)w_i(s) + \kappa_2\sigma\, w_i(s)w_i(s)\right] dvds \\ &\geq \kappa_m \int_0^t \int_B w_i(s)w_i(s)dvds, \end{aligned} \tag{3.20}$$

with κ_m a strictly positive computable constant. Then the relations (3.6) and (3.20) imply

$$\int_0^t \int_B w_i(s)w_i(s)\, dvds \leq \omega^2 \int_B w_i(t)w_i(t)dv, \tag{3.21}$$

with

$$\omega^2 = -\frac{1}{2}\frac{b}{\kappa_m}\left(\sigma - \frac{\kappa_1}{\kappa_3 T_0}\right). \tag{3.22}$$

An integration of the differential inequality (3.21) proves the result.

Remark 3. We have to observe that the assumption $\kappa_1\kappa_3 < k\kappa_2$ agrees with the thermodynamic restriction expressed by (2.8). In fact, it is easy to verify that (2.8) implies

$$4T_0\kappa_1\kappa_3 \leq (\kappa_1 + T_0\kappa_3)^2 \leq 4T_0 k\kappa_2, \tag{3.23}$$

and hence we have $\kappa_1\kappa_3 < k\kappa_2$.

Remark 4. If meas $\Sigma_6 = 0$ then we have $w_i(\mathbf{x}, t) = 0$ on $\partial B \times [0, \infty)$ in the above analysis and therefore, in the identity (3.6), we can use

$$\int_B w_{i,j}w_{j,i}dv = \int_B w_{i,i}w_{j,j}dv, \tag{3.24}$$

so that the restriction of type (3.2) can be relaxed to

$$\kappa_6 \geq 0, \quad \kappa_6 + \kappa_4 + \kappa_5 \geq 0. \tag{3.25}$$

Remark 5. The above analysis of the uniqueness of solutions is based on the auxiliary identities (A.1) and (A.4) given in the Appendix. When the auxiliary identity (A.18) is used, then, in view of relation (A.10), it results that the uniqueness Theorem 1 remains true when the assumptions expressed in (3.1)–(3.3) are substituted with the following assumption

$$\begin{aligned} &\varrho > 0, \qquad k > 0, \qquad \kappa_1\kappa_3 > 0, \qquad \kappa_1^2 < \frac{\kappa_1}{\kappa_3}k\kappa_2, \\ &3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \qquad \kappa_6 + \kappa_5 \geq 0, \qquad \kappa_6 - \kappa_5 \geq 0, \end{aligned} \tag{3.26}$$

or with

$$\begin{aligned} &\varrho > 0, \qquad k > 0, \qquad \kappa_1\kappa_3 < 0, \qquad \kappa_1^2 < \frac{\kappa_1}{\kappa_3}k\kappa_2, \\ &3\kappa_4 + \kappa_5 + \kappa_6 \leq 0, \qquad \kappa_6 + \kappa_5 \leq 0, \qquad \kappa_6 - \kappa_5 \leq 0. \end{aligned} \tag{3.27}$$

4. Asymptotic partition of energy

In this section we study the asymptotic partition of energy associated with the solutions to the mixed problem $(\mathcal{P})$. To this end we assume the following constitutive profile

$$\varrho > 0, \qquad \mu > 0, \quad 3\lambda + 2\mu > 0, \qquad a > 0, \qquad b > 0, \tag{4.1}$$

$$k > 0, \ \kappa_2 > 0, \quad \kappa_1\kappa_3 < k\kappa_2, \tag{4.2}$$

$$3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \qquad \kappa_6 + \kappa_5 \geq 0, \qquad \kappa_6 - \kappa_5 \geq 0, \tag{4.3}$$

and note that the following quadratic forms

$$W\left(e_{pq}\right) = \frac{1}{2}\lambda e_{mm}e_{nn} + \mu e_{ij}e_{ij}, \tag{4.4}$$

$$D_1\left(T, w_p, \sigma\right) = \frac{k}{T_0}T_{,i}T_{,i} + \left(\frac{\kappa_1}{T_0} + \kappa_3\sigma\right)T_{,i}w_i + \sigma\kappa_2 w_i w_i, \tag{4.5}$$

are positive definite, provided σ is fixed in the interval (σ_1, σ_2), while

$$D_2\left(w_p\right) = \kappa_4 w_{m,m}w_{n,n} + \kappa_5 w_{i,j}w_{j,i} + \kappa_6 w_{i,j}w_{i,j}, \tag{4.6}$$

is a positive semidefinite quadratic form. Moreover, we have

$$\mu_m e_{ij}e_{ij} \leq W\left(e_{pq}\right) \leq \mu_M e_{ij}e_{ij}, \tag{4.7}$$

$$D_1\left(T, w_p, \sigma\right) \geq k_m T_{,i}T_{,i} + \kappa_m w_i w_i, \tag{4.8}$$

where $2\mu_m = \min(2\mu, 3\lambda + 2\mu) > 0, 2\mu_M = \max(2\mu, 3\lambda + 2\mu) > 0$ and k_m and κ_m are strictly positive constants related to the eigenvalues of the quadratic form $D_1\left(T, w_p, \sigma\right)$.

Throughout this section we consider $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ to be the solution of the mixed problem $(\mathcal{P})$ corresponding to the given data $\mathcal{D} = \left\{\mathbf{0}, 0, \mathbf{0}; \mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0; \mathbf{0}, \mathbf{0}, 0, 0, \mathbf{0}, \mathbf{0}\right\}$, with $\mathbf{u}^0 \in \mathbf{W}_1(B)$, $\mathbf{v}^0 \in \mathbf{W}_0(B)$, $T^0 \in W_0(B)$ and $\mathbf{w}^0 \in \mathbf{W}_0(B)$. Here and in what follows we will use the notation $W_p(B)$ for the familiar Sobolev space [35] and we set $\mathbf{W}_p(B) \equiv \left[W_p(B)\right]^3$.

In view of the analysis developed in Section 3, it is easy to observe that there exists a family of rigid motions and constant temperature variation and zero microtemperature satisfying the fundamental system of Eqs. (2.11). When the boundary conditions are taken into consideration, then the rigid motion results to be zero when meas $\Sigma_1 \neq 0$, while the constant temperature variation is zero when meas $\Sigma_3 \neq 0$. Otherwise, the rigid motion and the constant temperature variation remain arbitrary. For this reason we decompose the initial data $\mathcal{J} = \left(\mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0\right)$ as follows

$$\left(\mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0\right) = \left(\mathbf{u}^0_*, \mathbf{v}^0_*, T^0_*, \mathbf{0}\right) + \left(\mathbf{U}^0, \mathbf{V}^0, \vartheta^0, \mathbf{w}^0\right), \tag{4.9}$$

where $\mathbf{u}^0_*$ and $\mathbf{v}^0_*$ are rigid displacements and T^0_* is a constant to be determined in such a way that $\mathbf{U}^0$, $\mathbf{V}^0$ and ϑ^0 satisfy the normalization restrictions

$$\int_B \mathbf{U}^0 dv = \mathbf{0}, \qquad \int_B \mathbf{x} \times \mathbf{U}^0 dv = \mathbf{0}, \qquad \int_B \mathbf{V}^0 dv = \mathbf{0}, \qquad \int_B \mathbf{x} \times \mathbf{V}^0 dv = \mathbf{0}, \tag{4.10}$$

when meas $\Sigma_1 = 0$ and

$$\int_B \boldsymbol{\vartheta}^0 dv = \mathbf{0}, \tag{4.11}$$

when meas $\Sigma_3 = 0$.

Further, we introduce the following notations

$$\hat{\mathbf{C}}(B) \equiv \left\{\mathbf{U} \in \left[C^1\left(\overline{B}\right)\right]^3 : \mathbf{U} = \mathbf{0} \text{ on } \overline{\Sigma}_1 \text{ and if meas } \Sigma_1 = 0, \text{ then } \int_B \mathbf{U} dv = \mathbf{0}, \int_B \mathbf{x} \times \mathbf{U} dv = \mathbf{0}\right\};$$

$$\hat{C}(B) \equiv \left\{\boldsymbol{\vartheta} \in C^1\left(\overline{B}\right) : \vartheta = 0 \text{ on } \overline{\Sigma}_3 \text{ and if meas } \Sigma_3 = 0, \text{ then } \int_B \boldsymbol{\vartheta} dv = \mathbf{0}\right\};$$

$$\hat{\mathbf{W}}_1(B) \equiv \text{ the completion of } \hat{\mathbf{C}}(B) \text{ by means of } \|\cdot\|_{\mathbf{W}_1(B)},$$

$$\hat{W}_1(B) \equiv \text{ the completion of } \hat{C}(B) \text{ by means of } \|\cdot\|_{W_1(B)}.$$

For the subsequent analysis it is convenient to decompose the solution $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ in the form

$$\{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t) = \left\{\mathbf{u}^0_* + t\mathbf{v}^0_*, T^0_*, \mathbf{0}\right\}(\mathbf{x}) + \{\mathbf{U}, \vartheta, \mathbf{w}\}(\mathbf{x}, t), \tag{4.12}$$

where $\{\mathbf{U}, \vartheta, \mathbf{w}\} \in \hat{\mathbf{W}}_1(B) \times \hat{W}_1(B) \times W_1(B)$ satisfies the differential system (2.11) with the corresponding zero boundary conditions and with the initial conditions

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}^0(\mathbf{x}), \qquad \dot{\mathbf{U}}(\mathbf{x}, 0) = \mathbf{V}^0(\mathbf{x}), \qquad \vartheta(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}), \qquad \mathbf{w}(\mathbf{x}, 0) = \mathbf{w}^0, \quad \mathbf{x} \in B. \tag{4.13}$$

Throughout in what follows it is convenient to introduce the following quantities associated with the solution $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ of the mixed problem $(\mathcal{P})$:

$$\begin{aligned}
\mathcal{K}(t) &\equiv \frac{1}{2}\int_B \varrho \dot{u}_i(t)\dot{u}_i(t) dv, \\
\mathcal{W}(t) &\equiv \int_B W\left(e_{pq}(t)\right) dv, \\
\mathcal{T}(t) &\equiv \frac{1}{2}\int_B aT^2(t) dv, \\
\Gamma(t) &= \frac{1}{2}\int_B bw_i(t)\, w_i(t) dv, \\
\Pi(t) &\equiv \int_0^t \int_B D_1\left(T(\tau), w_p(\tau), \sigma\right) dv d\tau, \\
\Theta(t) &\equiv \int_0^t \int_B D_2\left(w_p(\tau)\right) dv d\tau,
\end{aligned} \tag{4.14}$$

and further we define the total energy $\mathcal{E}(t)$ by

$$\mathcal{E}(t) \equiv \mathcal{K}(t) + \mathcal{W}(t) + \mathcal{T}(t) + \sigma\Gamma(t) + \Pi(t) + \sigma\Theta(t), \tag{4.15}$$

with $\sigma \in (\sigma_1, \sigma_2)$.

We introduce the Cesàro means of the above energies by

$$\mathcal{K}_C(t) \equiv \frac{1}{t}\int_0^t \mathcal{K}(\tau)\,d\tau,$$
$$\mathcal{W}_C(t) \equiv \frac{1}{t}\int_0^t \mathcal{W}(\tau)\,d\tau,$$
$$\mathcal{T}_C(t) \equiv \frac{1}{t}\int_0^t \mathcal{T}(\tau)\,d\tau,$$
$$\Gamma_C(t) \equiv \frac{1}{t}\int_0^t \Gamma(\tau)\,d\tau,$$
$$\Pi_C(t) \equiv \frac{1}{t}\int_0^t \Pi(\tau)\,d\tau,$$
$$\Theta_C(t) \equiv \frac{1}{t}\int_0^t \Theta(\tau)\,d\tau, \quad t \in (0,\infty). \tag{4.16}$$

In view of the relations (4.7) and (4.8), it follows that, for all $\mathbf{U} \in \hat{\mathbf{W}}_1(B)$ and $\vartheta \in \hat{W}_1(B)$, we have

$$\int_B W\left(e_{pq}(\mathbf{U}(t))\right) dv \geq m_1 \int_B U_i(t)U_i(t)\,dv, \tag{4.17}$$

$$\int_B D_1\left(\vartheta(t), w_p(t), \sigma\right) dv \geq m_2 \int_B \vartheta^2(t)\,dv + m_3 \int_B w_i(t)w_i(t)\,dv, \tag{4.18}$$

with m_1, m_2 and m_3 computable positive constants.

We have now assembled all the preliminary material needed to derive the asymptotic partition of energy in terms of the Cesàro means of energies defined by relations (4.14) and (4.16).

Theorem 2. *Suppose the thermoelastic coefficients satisfy (4.1)–(4.3) and let* $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ *be the solution of the mixed problem* $(\mathcal{P})$ *corresponding to the given data* $\mathcal{D} = \left\{\mathbf{0}, 0, \mathbf{0}; \mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0; \mathbf{0}, \mathbf{0}, 0, 0, \mathbf{0}, \mathbf{0}\right\}$, *with* $\mathbf{u}^0 \in \mathbf{W}_1(B)$, $\mathbf{v}^0 \in \mathbf{W}_0(B)$, $T^0 \in W_0(B)$ *and* $\mathbf{w}^0 \in \mathbf{W}_0(B)$. *Then for all choices of the initial data we have*

$$\lim_{t\to\infty} \Gamma_C(t) = 0, \tag{4.19}$$

$$\lim_{t\to\infty} \mathcal{T}_C(t) = \frac{1}{2}\int_B a\left(T_*^0\right)^2 dv, \tag{4.20}$$

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{W}_C(t) + \frac{1}{2}\int_B \varrho \mathbf{v}_*^0 \cdot \mathbf{v}_*^0 dv, \tag{4.21}$$

$$\begin{aligned}\lim_{t\to\infty}\left[\Pi_C(t) + \sigma\Theta_C(t)\right] &= \mathcal{E}(0) - 2\lim_{t\to\infty}\mathcal{K}_C(t) + \frac{1}{2}\int_B \varrho\mathbf{v}_*^0\cdot\mathbf{v}_*^0 dv - \frac{1}{2}\int_B a\left(T_*^0\right)^2 dv \\ &= \mathcal{E}(0) - 2\lim_{t\to\infty}\mathcal{W}_C(t) - \frac{1}{2}\int_B \varrho\mathbf{v}_*^0\cdot\mathbf{v}_*^0 dv - \frac{1}{2}\int_B a\left(T_*^0\right)^2 dv.\end{aligned} \tag{4.22}$$

In the above relations we have to observe that $\mathbf{v}_*^0 = \mathbf{0}$ *when* meas $\Sigma_1 \neq 0$ *and* $T_*^0 = 0$ *when* meas $\Sigma_3 \neq 0$.

Proof. Since the solution $\mathcal{S} = \{\mathbf{u}, T, \mathbf{w}\}(\mathbf{x}, t)$ corresponds to the given data $\mathcal{D} = \left\{\mathbf{0}, 0, \mathbf{0}; \mathbf{u}^0, \mathbf{v}^0, T^0, \mathbf{w}^0; \mathbf{0}, \mathbf{0}, 0, 0, \mathbf{0}, \mathbf{0}\right\}$ and by using the relations (4.14)–(4.16), it follows that the auxiliary identities (A.1), (A.11) and (A.18) give

$$\mathcal{E}(t) = \mathcal{E}(0), \tag{4.23}$$

$$\begin{aligned}\mathcal{K}_C(t) - \mathcal{W}_C(t) - \mathcal{T}_C(t) - \frac{\kappa_1}{\kappa_3 T_0}\Gamma_C(t) &= \frac{1}{4t}\int_B \varrho\left[u_i(0)\dot{u}_i(2t) + \dot{u}_i(0)\,u_i(2t) - 2u_i(0)\dot{u}_i(0)\right]dv \\ &+ \frac{1}{4t}\int_0^t\int_B \varrho\eta(0)\left[T(t-s) - T(t+s) - 2T(s)\right]dvds \\ &+ \frac{1}{4t}\frac{\kappa_1}{\kappa_3 T_0}\int_0^t\int_B b w_i(0)\left[w_i(t-s) - w_i(t+s) - 2w_i(s)\right]dvds, \\ & t \in (0,\infty).\end{aligned} \tag{4.24}$$

Let us now choose $\sigma \in (\sigma_1, \sigma_2)$. In view of relations (4.12), (4.14) and (4.18), we obtain

$$\Pi(t) = \int_0^t\int_B D_1\left(\vartheta(\tau), w_p(\tau), \sigma\right) dvd\tau \geq m_2\int_0^t\int_B \vartheta^2(\tau)\,dvd\tau + m_3\int_0^t\int_B w_i(\tau)\,w_i(\tau)\,dvd\tau. \tag{4.25}$$

On the other hand, in view of the hypotheses of theorem, from (4.14), (4.15) and (4.23), we deduce

$$\Pi(t) \leq \mathcal{E}(0) \quad \text{for all } t \in [0, \infty). \tag{4.26}$$

Consequently, from (4.25) and (4.26) we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \int_B w_i(\tau)\, w_i(\tau)\, dv d\tau = 0, \tag{4.27}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \int_B \vartheta^2(\tau)\, dv d\tau = 0 \tag{4.28}$$

and therefore, we obtain the relation (4.19). Moreover, in view of relations (4.12), (4.14) and (4.16), we have

$$\mathcal{T}_C(t) = \frac{1}{2} \int_B a \left(T_*^0\right)^2 dv + \frac{1}{t} \int_0^t \int_B a T_*^0 \vartheta(\tau)\, dv d\tau + \frac{1}{2t} \int_0^t \int_B a \vartheta^2(\tau)\, dv d\tau, \tag{4.29}$$

so that, by means of (4.28), we are led to the relation (4.20).

Furthermore, in view of the relations (4.12), (4.19), (4.20), (4.23), (4.27) and (4.28), from (4.24) we get

$$\lim_{t\to\infty} [\mathcal{K}_C(t) - \mathcal{W}_C(t)] = \lim_{t\to\infty} \frac{1}{4t} \int_B \varrho \dot{u}_i(0) u_i(2t)\, dv + \frac{1}{2} \int_B a \left(T_*^0\right)^2 dv - \frac{1}{2} \int_B T_*^0 \varrho \eta(0) dv, \tag{4.30}$$

and hence

$$\lim_{t\to\infty} [\mathcal{K}_C(t) - \mathcal{W}_C(t)] = \frac{1}{2} \int_B \varrho \mathbf{v}_*^0 \cdot \mathbf{v}_*^0 dv. \tag{4.31}$$

Finally, from (4.15) and (4.23) we deduce that

$$\mathcal{K}_C(t) + \mathcal{W}_C(t) + \mathcal{T}_C(t) + \sigma \Gamma_C(t) + \Pi_C(t) + \sigma \Theta_C(t) = \mathcal{E}(0) \tag{4.32}$$

and consequently, by using (4.19) and (4.20), we obtain

$$\lim_{t\to\infty} [\Pi_C(t) + \sigma \Theta_C(t)] = \mathcal{E}(0) - \lim_{t\to\infty} [\mathcal{K}_C(t) + \mathcal{W}_C(t)] - \frac{1}{2} \int_B a \left(T_*^0\right)^2 dv. \tag{4.33}$$

By combining the relations (4.31) and (4.33) we are led to the relation (4.22) and the proof is completed. □

Remark 6. The exponential stability of solutions in the classical theory of the one-dimensional thermoelasticity was first considered by Dafermos [36] and Slemrod [37]. More recently, Lebeau and Zuazua [38] have used the decoupling method in order to establish the asymptotic stability of solutions in the two-dimensional and the three-dimensional theory of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions. The exponential decay of solutions when microtemperatures are considered is established by Casas and Quintanilla [39] for the one-dimensional problem.

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Appendix. Some auxiliary identities

In this Appendix we establish some auxiliary identities concerning the solutions of the mixed problem $(\mathcal{P})$ defined in Section 2.

Lemma 1. *For every solution of the mixed problem $(\mathcal{P})$ corresponding to zero body supplies and zero boundary data and for any positive parameter σ, we have*

$$\begin{aligned} &\frac{1}{2} \int_B \left[\varrho \dot{u}_i(t) \dot{u}_i(t) + 2W\left(e_{pq}(t)\right) + aT^2(t) + \sigma b w_i(t) w_i(t)\right] dv \\ &\quad + \int_0^t \int_B \left\{D_1\left(T(s), w_p(s), \sigma\right) + \sigma D_2\left(w_p(s)\right)\right\} dv ds \\ &= \frac{1}{2} \int_B \left[\varrho \dot{u}_i(0) \dot{u}_i(0) + 2W\left(e_{pq}(0)\right) + aT^2(0) + \sigma b w_i(0) w_i(0)\right] dv. \end{aligned} \tag{A.1}$$

Proof. In view of Eqs. (2.1)–(2.4) and the divergence theorem, we get

$$\frac{1}{2}\int_B \left[\varrho\dot{u}_i(t)\dot{u}_i(t) + 2W\left(e_{pq}(t)\right) + aT^2(t)\right]dv + \int_0^t\int_B\left[\frac{k}{T_0}T_{,i}(s)\,T_{,i}(s) + \frac{\kappa_1}{T_0}T_{,i}(s)\,w_i(s)\right]dvds$$
$$= \frac{1}{2}\int_B\left[\varrho\dot{u}_i(0)\dot{u}_i(0) + 2W\left(e_{pq}(0)\right) + aT^2(0)\right]dv. \tag{A.2}$$

Moreover, by using Eqs. (2.1) and (2.5) and the divergence theorem, we obtain

$$\frac{1}{2}\int_B bw_i(t)w_i(t)dv + \int_0^t\int_B\left[\kappa_3 T_{,i}(s)\,w_i(s) + \kappa_2 w_i(s)w_i(s)\right]dvds$$
$$+\int_0^t\int_B D_2\left(w_p(s)\right)dvds = \frac{1}{2}\int_B bw_i(0)w_i(0)dv. \tag{A.3}$$

Now, a combination of the relations (A.2) and (A.3) leads to the relation (A.1) and the proof is completed. □

Lemma 2. *For every solution of the mixed problem* $(\mathcal{P})$ *corresponding to zero body supplies and zero boundary data, we have*

$$\kappa_3 T_0\int_B\left[\varrho\dot{u}_i(t)\dot{u}_i(t) - 2W\left(e_{pq}(t)\right) - aT^2(t)\right]dv - \kappa_1\int_B bw_i(t)w_i(t)dv$$
$$= -\kappa_1\int_B bw_i(0)w_i(2t)dv + \kappa_3 T_0\int_B\left\{\varrho\dot{u}_i(0)\dot{u}_i(2t)\right.$$
$$\left. - \left[\lambda e_{mm}(0)e_{nn}(2t) + 2\mu e_{ij}(0)e_{ij}(2t) + aT(0)T(2t)\right]\right\}dv. \tag{A.4}$$

Proof. We start with the identity

$$\frac{\partial}{\partial s}\left[\varrho\dot{u}_i(t-s)\,\dot{u}_i(t+s)\right] = \varrho\left[\dot{u}_i(t-s)\,\ddot{u}_i(t+s) - \dot{u}_i(t+s)\,\ddot{u}_i(t-s)\right], \tag{A.5}$$

which, when integrated with respect to s over $[0, t]$ and with respect to $\mathbf{x}$ over B and by using Eqs. (2.1)–(2.3) and (2.6), implies

$$\int_B\left[\varrho\dot{u}_i(t)\dot{u}_i(t) - 2W\left(e_{pq}(t)\right)\right]dv = \int_B\left\{\varrho\dot{u}_i(0)\dot{u}_i(2t) - \left[\lambda e_{mm}(0)e_{nn}(2t) + 2\mu e_{ij}(0)e_{ij}(2t)\right]\right\}dv$$
$$+\int_0^t\int_B\beta\left[T(t-s)\,\dot{e}_{mm}(t+s) - T(t+s)\,\dot{e}_{mm}(t-s)\right]dvds. \tag{A.6}$$

Further, we use the relations (2.1), (2.4), the integration by parts and the divergence theorem in order to obtain

$$\int_0^t\int_B\beta\left[T(t-s)\,\dot{e}_{mm}(t+s) - T(t+s)\,\dot{e}_{mm}(t-s)\right]dvds$$
$$= \int_0^t\int_B\frac{\kappa_1}{T_0}\left[T_{,i}(t+s)\,w_i(t-s) - T_{,i}(t-s)\,w_i(t+s)\right]dvds + \int_B aT^2(t)dv - \int_B aT(0)\,T(2t)dv. \tag{A.7}$$

On the other hand, by using the identity

$$\frac{\partial}{\partial s}\left[bw_i(t-s)\,w_i(t+s)\right] = bw_i(t-s)\,\dot{w}_i(t+s) - b\dot{w}_i(t-s)\,w_i(t+s) \tag{A.8}$$

and the relations (2.1) and (2.5) and the divergence theorem, we have

$$\int_0^t\int_B\kappa_3\left[T_{,i}(t+s)\,w_i(t-s) - T_{,i}(t-s)\,w_i(t+s)\right]dvds = \int_B bw_i(t)w_i(t)dv - \int_B bw_i(0)w_i(2t)dv. \tag{A.9}$$

Finally, we combine the relations (A.6), (A.7) and (A.9) and so we obtain the relation (A.4) and the proof is completed. □

For the next two lemmas it is convenient to introduce

$$\mathcal{I}(t) = \int_B\varrho u_i(t)u_i(t)dv + \int_0^t\int_B\left\{\frac{k}{T_0}\left(\int_0^s T_{,i}(\tau)\,d\tau\right)\left(\int_0^s T_{,i}(\tau)\,d\tau\right)\right.$$
$$\left. + \frac{2\kappa_1}{T_0}\left(\int_0^s T_{,i}(\tau)\,d\tau\right)\left(\int_0^s w_i(\tau)\,d\tau\right) + \frac{\kappa_1\kappa_2}{T_0\kappa_3}\left(\int_0^s w_i(\tau)\,d\tau\right)\left(\int_0^s w_i(\tau)\,d\tau\right)\right\}dvds$$

$$+\frac{\kappa_1}{T_0\kappa_3}\int_0^t\int_B\left\{\kappa_4\left(\int_0^s w_{m,m}(\tau)\,d\tau\right)\left(\int_0^s w_{n,n}(\tau)\,d\tau\right)\right.$$
$$\left.+\kappa_5\left(\int_0^s w_{i,j}(\tau)\,d\tau\right)\left(\int_0^s w_{j,i}(\tau)\,d\tau\right)+\kappa_6\left(\int_0^s w_{i,j}(\tau)\,d\tau\right)\left(\int_0^s w_{i,j}(\tau)\,d\tau\right)\right\}dvds. \tag{A.10}$$

Lemma 3. *For every solution of the mixed problem $(\mathcal{P})$ corresponding to zero body supplies and zero boundary data, we have*

$$\frac{d\mathcal{I}(t)}{dt}=2\int_B\varrho u_i(0)\dot{u}_i(0)dv+2\int_0^t\int_B\varrho\eta(0)T(s)dvds+\frac{2\kappa_1}{T_0\kappa_3}\int_0^t\int_B bw_i(0)w_i(s)dvds$$
$$+2\int_0^t\int_B\left[\varrho\dot{u}_i(s)\,\dot{u}_i(s)-2W\left(e_{pq}(s)\right)-aT^2(s)-\frac{\kappa_1}{T_0\kappa_3}bw_i(s)w_i(s)\right]dvds. \tag{A.11}$$

Proof. We start with the identity

$$\frac{\partial}{\partial s}\left[\varrho u_i(s)\dot{u}(s)\right]=\varrho\left[\dot{u}_i(s)\dot{u}_i(s)+u_i(s)\ddot{u}_i(s)\right], \tag{A.12}$$

which, by means of relations (2.1) and (2.3), gives

$$\int_B\varrho u_i(t)\dot{u}_i(t)dv=\int_B\varrho u_i(0)\dot{u}_i(0)\,dv+\int_0^t\int_B\left[\varrho\dot{u}_i(s)\,\dot{u}_i(s)-2W\left(e_{pq}(s)\right)-aT^2(s)\right]dvds$$
$$+\int_0^t\int_B T(s)\varrho\eta(s)\,dvds. \tag{A.13}$$

Now we integrate (2.4) with respect to the time variable in order to obtain

$$\varrho\eta(t)=\varrho\eta(0)+\frac{1}{T_0}\int_0^t q_{i,i}(\tau)\,d\tau, \tag{A.14}$$

which, when substituted into (A.13), leads to

$$\int_B\varrho u_i(t)\dot{u}_i(t)dv+\int_B\frac{k}{2T_0}\left(\int_0^t T_{,i}(\tau)\,d\tau\right)\left(\int_0^t T_{,i}(\tau)\,d\tau\right)dv$$
$$+\int_B\frac{\kappa_1}{T_0}\left(\int_0^t T_{,i}(\tau)\,d\tau\right)\left(\int_0^t w_i(\tau)\,d\tau\right)dv$$
$$=\int_B\varrho u_i(0)\dot{u}_i(0)dv+\int_0^t\int_B T(s)\varrho\eta(0)dvds$$
$$+\int_0^t\int_B\left[\varrho\dot{u}_i(s)\,\dot{u}_i(s)-2W\left(e_{pq}(s)\right)-aT^2(s)\right]dvds+\frac{\kappa_1}{T_0}\int_0^t\int_B w_i(s)\int_0^s T_{,i}(\tau)\,d\tau dvds. \tag{A.15}$$

On the other hand, from (2.5) we obtain

$$bw_i(t)=bw_i(0)-\int_0^t q_{ji,j}(\tau)\,d\tau-\kappa_2\int_0^t w_i(\tau)\,d\tau-\kappa_3\int_0^t T_{,i}(\tau)\,d\tau, \tag{A.16}$$

which, by means of (2.1), leads to

$$\kappa_3\int_0^t\int_B w_i(s)\int_0^s T_{,i}(\tau)\,d\tau dvds=-\int_0^t\int_B bw_i(s)w_i(s)\,dvds+\int_0^t\int_B bw_i(0)w_i(s)dvds$$
$$-\frac{1}{2}\int_B\kappa_2\left(\int_0^t w_i(\tau)\,d\tau\right)\left(\int_0^t w_i(\tau)\,d\tau\right)dv$$
$$-\frac{1}{2}\int_B\left[\kappa_4\left(\int_0^t w_{m,m}(\tau)\,d\tau\right)\left(\int_0^t w_{n,n}(\tau)\,d\tau\right)\right.$$
$$+\kappa_5\left(\int_0^t w_{i,j}(\tau)\,d\tau\right)\left(\int_0^t w_{j,i}(\tau)\,d\tau\right)$$
$$\left.+\kappa_6\left(\int_0^t w_{i,j}(\tau)\,d\tau\right)\left(\int_0^t w_{i,j}(\tau)\,d\tau\right)\right]dv. \tag{A.17}$$

By combining the relations (A.15) and (A.17) we are led to relation (A.11) and the proof is completed. □

Lemma 4. *For every solution of the mixed problem ($\mathcal{P}$) corresponding to zero body supplies and zero boundary data, we have*

$$\frac{d\mathcal{I}(t)}{dt} = \int_B \varrho \left[u_i(0)\dot{u}_i(2t) + \dot{u}_i(0)\, u_i(2t)\right] dv + \int_0^t \int_B \varrho\eta(0) \left[T(t-s) - T(t+s)\right] dv ds + \frac{\kappa_1}{T_0\kappa_3} \int_0^t \int_B b w_i(0) \left[w_i(t-s) - w_i(t+s)\right] dv ds. \tag{A.18}$$

Proof. We start with the identity

$$\frac{\partial}{\partial s} \left[\varrho u_i(t-s)\, \dot{u}_i(t+s) + \varrho \dot{u}_i(t-s)\, u_i(t+s)\right] = \varrho \left[u_i(t-s)\, \ddot{u}_i(t+s) - u_i(t+s)\, \ddot{u}_i(t-s)\right], \tag{A.19}$$

which, when used with relations (2.1), (2.3) and (A.14), gives

$$\begin{aligned}
&2\int_B \varrho u_i(t)\dot{u}_i(t) dv + \int_B \frac{k}{T_0} \left(\int_0^t T_{,i}(\tau)\, d\tau\right)\left(\int_0^t T_{,i}(\tau)\, d\tau\right) dv \\
&\quad + \int_B \frac{2\kappa_1}{T_0} \left(\int_0^t T_{,i}(\tau)\, d\tau\right)\left(\int_0^t w_i(\tau)\, d\tau\right) dv \\
&= \int_B \varrho \left[u_i(0)\dot{u}_i(2t) + \dot{u}_i(0)\, u_i(2t)\right] dv \\
&\quad + \int_0^t \int_B \varrho\eta(0) \left[T(t-s) - T(t+s)\right] dv ds \\
&\quad + \frac{\kappa_1}{T_0} \int_0^t \int_B \left[w_i(t-s) \int_0^{t+s} T_{,i}(\tau)\, d\tau - w_i(t+s) \int_0^{t-s} T_{,i}(\tau)\, d\tau\right] dv ds.
\end{aligned} \tag{A.20}$$

Further, by using (A.16) we obtain

$$\begin{aligned}
&\kappa_3 \int_0^t \int_B \left[w_i(t-s) \int_0^{t+s} T_{,i}(\tau)\, d\tau - w_i(t+s) \int_0^{t-s} T_{,i}(\tau)\, d\tau\right] dv ds \\
&= -\kappa_2 \int_B \left(\int_0^t w_i(\tau)\, d\tau\right)\left(\int_0^t w_i(\tau)\, d\tau\right) dv \\
&\quad - \int_B \left\{\kappa_4 \left(\int_0^t w_{m,m}(\tau)\, d\tau\right)\left(\int_0^t w_{n,n}(\tau)\, d\tau\right) + \kappa_5 \left(\int_0^t w_{i,j}(\tau)\, d\tau\right)\left(\int_0^t w_{j,i}(\tau)\, d\tau\right)\right. \\
&\quad \left. + \kappa_6 \left(\int_0^t w_{i,j}(\tau)\, d\tau\right)\left(\int_0^t w_{i,j}(\tau)\, d\tau\right)\right\} dv + \int_0^t \int_B b w_i(0) \left[w_i(t-s) - w_i(t+s)\right] dv ds.
\end{aligned} \tag{A.21}$$

A combination of the relations (A.20) and (A.21) leads to the result (A.18) and the proof is completed. □

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