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PLEASE SCROLL DOWN FOR ARTICLE
SOME NON-STANDARD PROBLEMS RELATED WITH THE MATHEMATICAL MODEL OF THERMOELASTICITY WITH MICROTEMPERATURES

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This article considers the linear theory of thermoelastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. We discuss some non-standard problems within the context of the dynamic boundary value problems. In fact, we use the Lagrange-Brun type identities combined with some differential inequalities in order to show that the final boundary value problem associated with the linear thermoelasticity with microtemperatures has at most one solution in appropriate classes of displacement-temperature-microtemperature fields. Furthermore, we study the constrained motion of a prismatic cylinder made of a thermoelastic material with microtemperatures and subjected to final given data that are proportional, but not identical with, the respective values at a prescribed early time. We show that certain cross-sectional integrals of the solution spatially evolve with respect to the axial variable. Conditions are derived upon the proportionality coefficients in order to show that the cross-sectional integrals exhibit alternative behavior and, in particular for a semi-infinite cylinder, that there is either at least exponential growth or at most exponential decay.

Keywords: Constrained motions; Final boundary value problems; Growth and decay spatial estimates; Non-standard problems; Thermoelasticity with microtemperatures; Uniqueness

INTRODUCTION

In [1] Iesan and Quintanilla consider the simplest thermomechanical theory of elastic materials that takes into account the microtemperature variables and then they establish some basic results concerning the uniqueness, existence and asymptotic behavior of dynamic solutions and some equilibrium specific problems. As regards the uniqueness result, it is established by assuming that the strain energy and dissipation energy are positive semi-definite quadratic forms and the specific heat and the specific coefficient of the first moment of energy vector are strictly
positive. This result has been improved by Quintanilla [2] and by Chirîţă, Ciarletta, and D’Apice [3] by establishing uniqueness of solutions of the forward-in-time problems under very mild assumptions upon the thermoelastic profile.

The theory of thermoelasticity with microtemperatures has attracted much attention in connection with the study of the basic qualitative properties of solutions to the problems relating to various thermomechanical situations. Thus, Svanadze [4, 5] and Scalia and Svanadze [6, 7] and Scalia et al. [8] studied the fundamental solutions and proves some existence and uniqueness theorems for equilibrium solutions and steady-state vibrations by means of the potential method, while Ieşan and Scalia [9] considered the plane strain in a homogeneous and isotropic body with microtemperatures. The behavior of shock waves and higher-order discontinuities that propagate in a thermoelastic body with inner structure and microtemperatures are studied by Ieşan [10] and the propagation of singularities of solutions to the Cauchy problem of a semilinear thermoelastic system with microtemperatures in one-space variable was studied by Yang and Huang in [11]. Some basic theorems were established by Aouadi [12] and Svanadze and Tracina [13] in the linear theory of microstretch thermoelasticity for isotropic solids with microtemperatures. Finally, Ciarletta et al. [14] investigated a model for a rigid heat conductor that allows for variation of thermal properties at a microstructure level, and they examine how the solution depends on changes in coupling coefficients between the macro- and microthermal levels.

On the other hand, the backward-in-time problems related to linear models of thermoelasticity based on the Fourier law for the heat flux vector lead to so-called improperly posed (or ill-posed) problems, since they fail to have a global solution, or they fail to have a unique solution, or if the solution does not depend continuously on the data. Knowing whether or not a solution is unique is important for numerical evaluation or for completeness of constructed by semi-inverse or similar methods. The non-standard problems are extensively studied in literature on subject. In this connection we have to cite the papers concerning the study of the heat equation (see, for example, Ames and Payne, [15], Ames, Payne, and Schaefer [16, 17], Ames, Payne, and Song [18], Chirîţă [19]) and those concerning the generalized heat equation (see, for example, Payne, Schaefer and Song [20] and linear thermoelasticity of second sound (see, for example, Quintanilla and Straughan [21] and also the book by Straughan [22] and the papers cited therein).

The final boundary value problems in the classical linear thermoelasticity have been considered in many papers like those by Ames and Payne [23], Ciarletta and Chirîţă [24, 25], Ciarletta [26] and Chirîţă [27], while similar problems for thermoelasticity of materials with microstructure have been considered by Iovane and Passarella [28], Passarella and Tibullo [29] and Passarella, Tibullo, and Zampoli [30]).

This article addresses some non-standard problems within the context of the linear theory of thermoelasticity with microtemperatures as developed by Ieşan and Quintanilla [1]. We treat the uniqueness problem, for solutions of the final boundary value problems for the theory in concern, by means of an appropriate combination between the Lagrange identities method [31] and some integral inequalities, provided some sign-defined assumptions are made upon the elastic energy and under very mild assumptions upon the other thermoelastic coefficients, including those describing the dissipation energy. This is possible because of the
special coupling between the differential equations for the temperature variation and
that for the microtemperature fields. In the class of solutions having a dissipation
energy growing in time, but lower than an appropriate growing exponential, the
uniqueness result holds true under mild assumptions upon the thermoelastic profile.

Furthermore, we consider a non-standard problem associated with the linear
thermoelasticity with microtemperatures for a prismatic cylinder and establish decay
and growth exponential estimates with respect to the axial variable for certain time-
weighted integrals of the cross-sectional energy. The final values for displacement,
velocity, temperature and microtemperatures fields are not prescribed, nor are
conditions specified on the upper end for a finite cylinder, or at asymptotically
large axial distance for the semi-infinite cylinder. The time-dependent displacement,
temperature and microtemperatures prescribed over the base end constrain the
motion such that the displacement, velocity, temperature and microtemperature
fields at some given time are proportional to their unknown final values. We want
to predict the deformation at each cross-section of the cylinder in terms of the given
data on the base end. We derive conditions upon the proportionality coefficients
in order to see how certain time-weighted integrals of the cross-sectional energetic
terms evolve with respect to the axial distance to the end base.

BASIC EQUATIONS AND THE FINAL BOUNDARY VALUE PROBLEMS

Throughout this section, \( B \) is a bounded regular region of three-dimensional
Euclidean space. We let \( \partial B \) denote the boundary of \( B \), and designate by \( n \) the
outward unit normal on \( \partial B \). We assume that the body occupying \( B \) is a linearly
elastic material which possesses thermal variation at a microstructure level. The
body is referred to a fixed system of rectangular Cartesian axes \( OX_i \) (\( i = 1, 2, 3 \)).
Throughout this article Latin indices have the range 1, 2, 3; Greek indices have the
range 1, 2; and, the usual summation convention is employed. We use a subscript
preceded by a comma for partial differentiation with respect to the corresponding
coordinate and a superposed dot denotes partial differentiation with respect to time.

The temperature at a point \( x \) of the body depends on a temperature \( \theta(x, t) \), which
may be thought of as an averaged temperature at \( x \), and three microtemperatures
\( \omega_i(x, t) \), which contribute to the thermal microstructure of the material. The
deformation of a body can be described by means of three, namely, the displacement
vector field \( u \), the microtemperature vector field \( w \) and the temperature variation \( T \),
measured from the constant absolute temperature \( T_0 \) (\( > 0 \)), over \( \overline{B} \times (-\infty, 0] \).

Within the framework of the linear theory developed by Ieşan and Quintanilla
[1], the constitutive equations for a homogeneous and isotropic thermoelastic solid
with microtemperatures are

\[
\begin{align*}
t_{ij} &= \lambda \varepsilon_{ij} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta T \delta_{ij} \\
\gamma \eta &= \beta \varepsilon_{ij} + a T \\
\dot{\varepsilon}_i &= -bu_i \\
q_i &= k T_j \varepsilon_j + \kappa_1 \omega_i \\
Q_i &= (\kappa_1 - \kappa_2) \omega_i + (k - \kappa_2) T_i \\
q_{ij} &= -\kappa_4 \omega_{i, r} \delta_{ij} - \kappa_5 \omega_{i,j} - \kappa_6 \omega_{j,i}
\end{align*}
\] (1)
where
\[ e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]  

(2)

Here, \( t_{ij} \) are the components of the stress tensor, \( \varrho \) is the reference mass density, \( \eta \) is the entropy per unit mass, \( \varepsilon_i \) are the components of the first moment of energy vector, \( q_i \) are the components of the heat flux vector, \( Q_i \) are the components of the mean heat flux vector, \( e_{ij} \) are the components of the strain tensor, \( u_i \) are the components of the displacement vector, \( w_i \) are the components of the microtemperature vector, \( T \) is the temperature variation, \( \lambda, \mu, \beta, a, b, k \) and \( \kappa_r \) \((r = 1, 2, \ldots, 6)\) are constant constitutive coefficients and \( \delta_{ij} \) is the Kronecker delta.

The fundamental system of field equations of the linear theory of thermoelasticity with microtemperatures consists of [1]:

- the equations of motion
  \[ t_{ji} = \varrho f_i - \frac{\partial}{\partial x_j} \]  
  \[ (3) \]

- the balance energy
  \[ \varrho T \dot{\eta} = q_{ii} + \rho S \]  
  \[ (4) \]

- the first moment of energy
  \[ \varrho \dot{\varepsilon}_i = q_{ji,i} + q_i - Q_i + qM_i \]  
  \[ (5) \]

in \( B \times (-\infty, 0) \), where \( f_i \) are the components of the body force vector, \( M_i \) are the components of the first heat source moment vector and \( S \) is the heat supply.

The components of surface traction \( t_{i} \), the heat flux \( q \) and the components of the first heat flux moment \( \Lambda_{ij} \) at a regular point \( x \) of the boundary \( \partial B \) are given by

\[ t_i = t_{i,j}n_j \]

\[ q = q_{i}n_i \]

\[ \Lambda_{ij} = q_{i,j}n_j \]  

(6)

where \( n_j = \cos(n_i, OX_j) \) and \( n_i \) is the unit vector of the outward normal to \( \partial B \) at \( x \).

Within the context of linear theory of thermoelasticity considered in [1], the Clausius-Duhem inequality reduces to

\[ q_i T_{,i} - T_0 q_{i,j} w_{i,j} - T_0 (Q_i - q_i) w_i \geq 0 \]  

(7)

which, in combination with (1), implies

\[ 3\kappa_4 + \kappa_5 + \kappa_6 \geq 0, \quad \kappa_6 + \kappa_5 \geq 0, \quad \kappa_6 - \kappa_5 \geq 0 \]

\[ k \geq 0, \quad (\kappa_1 + T_0 \kappa_2)^2 - 4T_0 k \kappa_2 \leq 0 \]  

(8)
To the above fundamental equations we adjoin the final conditions
\[ u_i(x, 0) = u_i^0(x), \quad \dot{u}_i(x, 0) = v_i^0(x) \]
\[ T(x, 0) = T^0(x), \quad w_i(x, 0) = w_i^0(x), \quad x \in \overline{B} \]
(9)
and the boundary conditions
\[ u_i(x, t) = \tilde{u}_i(x, t) \quad \text{on} \quad \Sigma_1 \times (-\infty, 0] \]
\[ t_{\beta}(x, t) n_j = \tilde{t}_j(x, t) \quad \text{on} \quad \Sigma_2 \times (-\infty, 0] \]
\[ T(x, t) = \tilde{T}(x, t) \quad \text{on} \quad \Sigma_3 \times (-\infty, 0] \]
\[ q_i(x, t) n_j = \tilde{q}(x, t) \quad \text{on} \quad \Sigma_4 \times (-\infty, 0] \]
\[ w_i(x, t) = \tilde{w}_i(x, t) \quad \text{on} \quad \Sigma_5 \times (-\infty, 0] \]
\[ q_{\beta}(x, t) n_j = \tilde{\Lambda}_j(x, t) \quad \text{on} \quad \Sigma_6 \times (-\infty, 0] \]
(10)
where \( u_i^0(x), \ v_i^0(x), \ T^0(x), \ w_i^0(x) \) and \( \tilde{u}_i(x, t), \ \tilde{t}_j(x, t), \ \tilde{T}(x, t), \ \tilde{q}(x, t), \ \tilde{w}_i(x, t), \ \tilde{\Lambda}_j(x, t) \) are prescribed smooth functions. Moreover, \( \Sigma_r \ (r = 1, 2, \ldots, 6) \) are subsets of the boundary \( \partial \overline{B} \) such that \( \Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma_5 \cup \Sigma_6 = \partial \overline{B} \) and \( \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \emptyset \).

The mixed final boundary value problem \( \mathcal{P} \) consists of finding a solution \( \mathcal{F} = \{u, T, w\}(x, t) \), with \( u_i \in C^{3,2}(B \times (-\infty, 0]), \ T, w_j \in C^{3,1}(B \times (-\infty, 0]) \) that satisfy the fundamental equations (1)–(5), the final conditions (9) and the boundary conditions (10), provided smooth data \( \mathcal{D} = \{f, S, M; u^0, v^0, T^0, w^0; \tilde{u}, \tilde{t}, \tilde{q}, \tilde{w}, \tilde{\Lambda}\} \) are prescribed.

We have to outline that, by substituting the relations (1) and (2) into relations (3)–(5), we obtain the following system of linear partial differential equations for \( \mathcal{F} = \{u, T, w\}(x, t) \)
\[ \mu u_{rs,t} + (\lambda + \mu) u_{ss,rr} - \beta T_{r,t} + qf_r = \varrho \tilde{u}_r \]
\[ k T_{ss} - \beta T_0 \tilde{u}_{s,s} + \kappa_1 w_{ss} + qS = a T_0 \tilde{T} \]
\[ \kappa_6 w_{rs,rr} + (\kappa_4 + \kappa_5) w_{ss,rr} - \kappa_3 T_{r,r} - \kappa_2 w_r - q M_r = b \tilde{w}_r \]
(11)
in \( B \times (-\infty, 0) \). It has to be observed that, in the case when \( \beta \kappa_1 \kappa_3 = 0 \), the three equations in (11) decouple with respect to one to the other and so we have to study separately the differential system associated with the classical theory of elasticity and with the heat equation and with the differential system of microtemperatures. For this reason, throughout this article we assume that \( \beta \kappa_1 \kappa_3 \neq 0 \).

To study the solutions of the backward-in-time problem \( \mathcal{P} \) it is convenient to transform it in a forward-in-time problem \( \mathcal{P}^* \) by making \( t \to -t \). Thus, by means of some appropriate notations, the forward-in-time problem \( \mathcal{P}^* \) is defined by the equations
\[ \mu u_{rs,t} + (\lambda + \mu) u_{ss,rr} - \beta T_{r,t} + qf_r = \varrho \tilde{u}_r \]
\[ k T_{ss} - \beta T_0 \tilde{u}_{s,s} + \kappa_1 w_{ss} + qS = a T_0 \tilde{T} \]
\[ \kappa_6 w_{rs,rr} + (\kappa_4 + \kappa_5) w_{ss,rr} - \kappa_3 T_{r,r} - \kappa_2 w_r - q M_r = b \tilde{w}_r \]
(12)
(13)
(14)
Lemma 1. For every solution of the forward-in-time problem \( \mathcal{P}^* \), we have

\[
\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \dot{q} \dot{u}_i(t) + W(e_{pq}(t)) \right] dv = \int_{\Omega} \beta T(t) \dot{u}_{ir}(t) dv
\]

\[
+ \int_{\Omega} g f_i(t) \dot{u}_i(t) dv + \int_{\Omega} t_{ij}(t) n_j \dot{u}_i(t) da
\]

\[
- \int_{\Omega} \frac{\partial}{\partial T_0} S(t) T(t) dv - \int_{\partial \Omega} \frac{1}{T_0} q_i(t) n_i T(t) da
\]  \hspace{1cm} (17)

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} aT^2(t) dv = - \int_{\Omega} \beta T(t) \dot{u}_{on}(t) dv + \int_{\Omega} \frac{1}{T_0} \left[ k T_i(t) T_{ij}(t) + \kappa_1 T_i(t) w_i(t) \right] dv
\]

\[
- \int_{\partial \Omega} \frac{1}{T_0} q_i(t) n_i T(t) da
\]  \hspace{1cm} (18)

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} b w_i(t) w_i(t) dv = \int_{\Omega} D_2 \left( W(e_{pq}(t)) + \frac{1}{2} aT^2(t) + \frac{1}{2} \sigma w_i(t) w_i(t) \right) dv
\]

\[
+ \int_{\partial \Omega} q M_i(t) w_i(t) dv + \int_{\partial \Omega} w_i(t) q_i(t) n_i da
\]  \hspace{1cm} (19)

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \dot{q} \dot{u}_i(t) + W(e_{pq}(t)) + \frac{1}{2} aT^2(t) + \frac{1}{2} \sigma w_i(t) w_i(t) ] dv
\]

\[
= \int_{\Omega} \left[ D_1 \left( T(t), w_i(t), \sigma \right) + \sigma D_2 \left( w_i(t) \right) \right] dv
\]
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\[ + \int_B \left[ f_i(t) \dot{u}_i(t) - \frac{1}{T_0} S(t) T(t) + \sigma M_i(t) w_i(t) \right] dv \]

\[ + \int_{2B} \left[ t_j(t) n_j(t) - \frac{1}{T_0} q_i(t)n_i T(t) + \sigma w_i(t) q_i(t)n_i \right] da \]  \hspace{1cm} (20)

where \( \sigma \) is a real parameter at our disposal and

\[ W(e_{pq}) = \frac{1}{2} \lambda e_{mm} e_{nn} + \mu e_{ij} e_{ij} \] \hspace{1cm} (21)

\[ D_1 (T, w_p, \sigma) = \frac{k}{\bar{r}} T_i T_i + \left( \frac{\kappa_1}{T_0} + \kappa_2 \right) T_i w_i + \sigma \kappa_2 w_i w_i \] \hspace{1cm} (22)

\[ D_2 (w_p) = \kappa e w_{i,j} w_{i,j} + \kappa_2 w_{i,j} w_{i,j} + \kappa_4 w_{m,m} w_{n,n} \] \hspace{1cm} (23)

**Proof.** From Eq. (12) we obtain

\[ \int_B \varrho \dot{u}_i(t) \dot{u}_i(t) = \int_B \varrho f_i(t) \dot{u}_i(t) dv + \int_{2B} t_j(t) n_j \dot{u}_i(t) da - \int_B t_j(t) \dot{e}_j(t) dv \] \hspace{1cm} (24)

which, by means of the relations (1), and (21), leads to the identity (17). Similarly, the use of Eq. (13) and relations (1)\(_4\) lead to the identity (18), while the use of the equation (14) and the relations (1)\(_6\) and (23) leads to the identity (19). Finally, an appropriate combination of relations (17)–(19) and the use of relation (22) give identity (20), and the proof is complete. \(\square\)

**Lemma 2.** For every solution of the forward-in-time problem \((\mathcal{P}^+)*\), we have the following Lagrange identity

\[ \int_B \left\{ \varrho \dot{u}_i(t) \dot{u}_i(t) - \left[ 2W(e_{pq}(t)) + aT^2(t) + \frac{\kappa}{T_0 \kappa_3} b w_i(t) w_i(t) \right] \right\} dv \]

\[ = \int_0^t \int_B \left\{ f_i(t-s) \dot{u}_i(t+s) - f_i(t) \dot{u}_i(t-s) \right\} dv ds \]

\[ + \frac{1}{T_0} \left[ S(t-s) T(t+s) - S(t+s) T(t-s) \right] \]

\[ + \frac{\kappa_1}{T_0 \kappa_3} \left[ M_i(t+s) w_i(t-s) - M_i(t-s) w_i(t+s) \right] \] \hspace{1cm} \[ dv ds \]

\[ + \int_0^t \int_{2B} \left\{ \dot{u}_i(t+s) t_j(t-s) n_j - \dot{u}_i(t-s) t_j(t+s) n_j \right\} dv ds \]

\[ + \frac{1}{T_0} \left[ T(t+s) q_i(t-s) n_i - T(t-s) q_i(t+s) n_i \right] \]

\[ + \frac{\kappa_1}{T_0 \kappa_3} \left[ w_i(t-s) q_i(t+s) n_i - w_i(t+s) q_i(t-s) n_i \right] \] \hspace{1cm} \[ dv ds \]

\[ + \int_B \left\{ \varrho \dot{u}_i(0) \dot{u}_i(2t) - \left[ \lambda e_{mm}(0) e_{mm}(2t) + 2\mu e_{ij}(0) e_{ij}(2t) \right] \right\} dv \]

\[ + a T(0) T(2t) + \frac{\kappa}{T_0 \kappa_3} b w_i(0) w_i(2t) \] \hspace{1cm} (25)

for all \( t \geq 0. \)
Proof. We start with the Lagrange identity

\[
\frac{\hat{\epsilon}}{\mathcal{S}} [\hat{q} \hat{\epsilon} (t - s) \hat{u} (t + s)] = q [\hat{u} (t - s) \hat{u} (t + s) - \hat{\epsilon} _{\alpha}(t - s) \hat{\epsilon} _{\alpha}(t + s)]
\]  

which implies

\[
\int_B q \hat{\epsilon} (t) \hat{u}(t) \, dv = \int_B q \hat{\epsilon} (0) \hat{u}(2t) \, dv + \int_0^t \int_B q [\hat{u} (t - s) \hat{u} (t + s) - \hat{\epsilon} _{\alpha}(t - s) \hat{\epsilon} _{\alpha}(t + s)] \, dv \, ds
\]  

Next, we introduce Eq. (12) in (27) and then we use the constitutive equation (1) and the geometrical relation (2) in order to obtain

\[
\int_B q \hat{\epsilon} (t) \hat{u}(t) \, dv = \int_B q \hat{\epsilon} (0) \hat{u}(2t) \, dv + \int_0^t \int_B q [\hat{u} (t - s) \hat{u} (t + s) - \hat{\epsilon} _{\alpha}(t - s) \hat{\epsilon} _{\alpha}(t + s)] \, dv \, ds
\]

Further, we use Eq. (13) so that we can obtain

\[
\int_0^t \int_B \frac{\hat{\beta}}{T_0} [\hat{T} (t - s) \hat{\epsilon} _{mm}(t + s) - \hat{T} (t + s) \hat{\epsilon} _{mm}(t - s) + 2 \mu \hat{e}_{ij}(t) \hat{e}_{ij}(t)] \, dv \, ds
\]

Finally, we use Eq. (14) in order to obtain that

\[
\int_0^t \int_B \frac{\kappa_3}{T_0} [\hat{T} (t - s) w_i (t + s) - \hat{T} (t + s) w_i (t - s)] \, dv \, ds = \int_B b w_i (t) w_i (t) \, dv
\]
Now we combine the identities (28)–(30) to obtain relation (25), and so the proof is complete.

**UNIQUENESS OF SOLUTIONS TO THE FINAL BOUNDARY VALUE PROBLEM**

Here we present the uniqueness problem for the backward-in-time problem \( (\mathcal{P}^b) \). That is equivalent to study the uniqueness problem for the forward-in-time problem \( (\mathcal{P}^f) \). Let us assume that there are two solutions of the forward-in-time problem \( (\mathcal{P}^f) \), \( S^{(1)} = \{u^{(1)}, T^{(1)}, w^{(1)}\}(x, t), (z = 1, 2), \) corresponding to the same given data \( D = \{f, S, M; u^0, v^0, T^0, w^0; \dot{u}, \dot{v}, \dot{T}, \dot{w}, \Lambda\} \). Then the difference \( S = S^{(1)} - S^{(2)}, \) that is \( S = \{u, T, w\}(x, t) = \{u^{(1)} - u^{(2)}, T^{(1)} - T^{(2)}, w^{(1)} - w^{(2)}\}(x, t) \) is a solution of the forward-in-time problem \( (\mathcal{P}^f) \) corresponding to the null data \( D = 0 \). In order to establish the uniqueness result we have to prove that \( S = \{u, T, w\}(x, t) = 0 \) on \( B \times [0, \infty) \). For the difference solution \( S = \{u, T, w\}(x, t) \), the auxiliary identities (17)–(20) and (25) imply

\[
\int_B \left[ \frac{1}{2} \dot{u}_i(t) \dot{u}_i(t) + W(e_{pq}(t)) \right] dv = \int_0^t \int_B \beta T(s) \dot{u}_{r,r}(s) dvds
\]

\[
\int_B \frac{1}{2} \dot{T}^2(t) dv = -\int_0^t \int_B \beta T(s) \dot{u}_{r,r}(s) dvds
\]

\[
+ \int_0^t \int_B \frac{1}{T_0} \left[ kT_i(s) T_i(s) + \kappa_1 T_i(s) w_i(s) \right] dvds
\]

\[
\int_B \frac{1}{2} b w_i(t) w_i(t) dv = \int_0^t \int_B D_2(w_p(s)) dvds
\]

\[
+ \int_0^t \int_B \left[ \kappa_2 T_i(s) w_i(s) + \kappa_2 w_i(s) w_i(s) \right] dvds
\]

\[
= \int_0^t \int_B \left[ D_1(T(s), w_p(s), \sigma) + \sigma D_2(w_p(s)) \right] dvds
\]

and

\[
\int_B \left\{ \dot{q}_i(t) \dot{u}_i(t) - \left[ 2W(e_{pq}(t)) + \frac{1}{2} \dot{a} T^2(t) + \frac{1}{2} \sigma b w_i(t) w_i(t) \right] \right\} dv = 0
\]

From these identities we also obtain

\[
\int_B \frac{1}{2} \left[ \dot{a} T^2(t) + \sigma b w_i(t) w_i(t) \right] dv = -\int_0^t \int_B \beta T(s) \dot{u}_{r,r}(s) dvds
\]

\[
+ \int_0^t \int_B \left[ D_1(T(s), w_p(s), \sigma) + \sigma D_2(w_p(s)) \right] dvds
\]

(36)
and
\[
\int_B g \dot{u}_i (t) \dot{u}_i (t) \, dv = \int_B \left[ 2W (e_{pq} (t)) + aT^2 (t) + \frac{\kappa_1}{T_0k_3} bw_i (t) w_i (t) \right] \, dv
\]
\[
= \int_0^t \int_B \left[ D_1 (T (s), w_{,p} (s), \sigma) + \sigma D_2 (w_{,p} (s)) \right] \, dv \, ds
\]
\[
+ \frac{1}{2} \left( \frac{\kappa_1}{T_0k_3} - \sigma \right) \int_B bw_i (t) w_i (t) \, dv
\]
(37)

Throughout this section we assume the hypothesis (H):
\[
k > 0, \quad \kappa_2 > 0, \quad \kappa_1 \kappa_3 < k \kappa_2
\]
\[
3 \kappa_4 + \kappa_5 + \kappa_6 \geq 0, \quad \kappa_6 + \kappa_5 \geq 0, \quad \kappa_6 - \kappa_5 \geq 0
\]
(38)
which agree with the thermodynamic consequences expressed by relation (8). In view of the assumption \( \kappa_1 \kappa_3 < k \kappa_2 \) it follows that for all \( \sigma \) so that
\[
0 < \sigma_1 < \sigma < \sigma_2
\]
(39)
with
\[
\sigma_{1,2} = \frac{1}{T_0k_3} \left[ 2k \kappa_2 - \kappa_1 \kappa_3 \mp 2\sqrt{k \kappa_2 (k \kappa_2 - \kappa_1 \kappa_3)} \right]
\]
(40)
we have
\[
\left( \frac{\kappa_1}{T_0} + \kappa_3 \sigma \right)^2 < \frac{4}{T_0} \sigma k \kappa_2
\]
(41)
Therefore, we have
\[
f (\sigma) \equiv \kappa_3^2 \sigma^2 - \frac{2}{T_0} (2k \kappa_2 - \kappa_1 \kappa_3) \sigma + \frac{\kappa_1^2}{T_0} < 0 \quad \text{for all } \sigma \in (\sigma_1, \sigma_2)
\]
(42)
and hence \( D_1 (T, w_{,p}, \sigma) \) is a positive definite quadratic form and so we can write
\[
k_0 T_i T_j + \kappa_0 w_i w_j \leq D_1 (T, w_{,p}, \sigma) \leq k^0 T_i T_j + \kappa^0 w_i w_j
\]
(43)
and \( k_0, \kappa_0 \) and \( k^0, \kappa^0 \) are strictly positive constants related to the eigenvalues of the positive definite quadratic form \( D_1 (T, w_{,p}, \sigma) \). Moreover, it follows then that \( D_2 (w_{,p}) \) is a positive semi-definite quadratic form.

**Theorem 1.** Suppose that the hypothesis (H) holds true, \( q > 0 \) and \( \text{meas } \Sigma_4 = 0 \) and, moreover, we have to hold one of the following assumptions
\[
\mu \geq 0, \quad 3 \lambda + 2 \mu \geq 0, \quad \kappa_1 \kappa_3 > 0
\]
(44)
or

\[ a \leq 0, \quad b \leq 0 \quad (45) \]

Then the forward-in-time problem \((\mathcal{P}_*)\) has at most one solution.

**Proof.** Let \( [u, T, w](x, t) \) be a solution of the forward-in-time problem \((\mathcal{P}_*)\) corresponding to zero given data so that the identity \((31)\) holds true.

Let us first suppose that the assumption \((44)\) holds true and, therefore, \( W(e_{pq}) \) is a positive semi-definite quadratic form. Furthermore, we have to observe that

\[
 f \left( \frac{\kappa_1}{T_0 \kappa_3} \right) = -\frac{4}{T_0^2} \frac{\kappa_1}{\kappa_3} (k \kappa_2 - \kappa_1 \kappa_3) < 0 \quad (46)
\]

and hence we have that \( \sigma_1 < \frac{\kappa_1}{T_0 \kappa_3} < \sigma_2 \) and moreover, \( D_1(T, w_p, \frac{\kappa_1}{T_0 \kappa_3}) \) is a positive definite quadratic form satisfying \((43)\).

In view of the above considerations, it follows that \( T = 0 \) on \( \partial B \times [0, \infty) \) and hence, by means of the divergence theorem, from \((31)\) it follows that

\[
 \int_B \left[ \frac{1}{2} \dot{q}_i (t) \ddot{u}_i (t) + W(e_{pq}(t)) \right] \ dv = - \int_0^t \int_B \beta T_i (s) \ddot{u}_i (s) \ dv \ ds \quad (47)
\]

By means of the Schwarz inequality and the arithmetic-geometric mean inequality and by using the relation \((43)\) and the positive semi-definiteness of the quadratic forms \( W(e_{pq}) \) and \( D_2(w_p) \), from \((47)\) it follows that

\[
 \int_B \frac{1}{2} \dot{q}_i (t) \ddot{u}_i (t) \ dv \leq \int_0^t \int_B \left[ \frac{\epsilon}{2} k_0 T_i (s) T_i (s) + \frac{1}{2 \epsilon} \frac{\beta^2}{q_k_0} \dot{q}_i (s) \ddot{u}_i (s) \right] \ dv \ ds \\
\leq \frac{\epsilon}{2} \int_0^t \int_B \left[ D_1 \left( T(s), w_p(s), \frac{\kappa_1}{T_0 \kappa_3} \right) + \frac{\kappa_1}{T_0 \kappa_3} D_2(w_p(s)) \right] \ dv \ ds \\
+ \frac{1}{2 \epsilon} \frac{\beta^2}{q_k_0} \int_0^t \int_B \dot{q}_i (s) \ddot{u}_i (s) \ dv \ ds \quad (48)
\]

for every positive parameter \( \epsilon \) at our disposal. Now we set \( \sigma = \frac{\kappa_1}{T_0 \kappa_3} \) into \((37)\) and then we use the result in \((48)\), followed by the choice

\[
 \epsilon = \frac{1}{2} \quad (49)
\]

to obtain

\[
 \int_B \dot{q}_i (t) \ddot{u}_i (t) \ dv \leq \frac{4 \beta^2}{q_k_0} \int_0^t \int_B \dot{q}_i (s) \ddot{u}_i (s) \ dv \ ds, \quad \text{for all } t \geq 0 \quad (50)
\]

which, by applying Gronwall’s lemma, implies

\[
 \ddot{u}_i (x, t) = 0 \quad \text{in } B \times (0, \infty) \quad (51)
\]
By taking into account the zero initial conditions, from (51), we deduce that
\[ u_i(x, t) = 0 \text{ in } B \times [0, \infty) \quad (52) \]

Further we substitute (52) in (37) and then we use the relation (43) and the zero boundary conditions to obtain
\[ T(x, t) = 0, \quad w_i(x, t) = 0 \text{ in } B \times [0, \infty) \quad (53) \]

that is, we have the uniqueness result.

Let us now consider the assumption expressed by (45). We choose \( \sigma \in (\sigma_1, \sigma_2) \).

If \( \kappa_1 \kappa_3 < 0 \), then (37) implies
\[ \int_B g\dot{u}_i(t) \dot{u}_i(t) \, dv \leq \int_0^t \int_B \left[ D_1(T, w_p, \sigma) + \sigma D_2(w_p) \right] \, dv \, ds \quad (54) \]

for all \( \sigma \in (\sigma_1, \sigma_2) \). If \( \kappa_1 \kappa_3 > 0 \), then, in view of relation (46), we can choose \( \sigma \in \left( \frac{\kappa_1 \kappa_3}{\omega}, \sigma_2 \right) \) so that (37) again implies (54). Furthermore, from relations (45) and (36), we deduce that
\[ \int_0^t \int_B \left[ D_1(T, w_p, \sigma) + \sigma D_2(w_p) \right] \, dv \, ds \leq \epsilon \int_0^t \int_B g\dot{u}_i(t) \dot{u}_i(t) \, dv \, ds \quad (55) \]

and by using relation (54) and by setting \( \epsilon = 1 \) we arrive at the following Gronwall inequality
\[ \phi(t) \leq \frac{\beta^2}{2 \epsilon k_0} \int_0^t \phi(s) \, ds \quad (56) \]

where
\[ \phi(t) = \int_0^t \int_B \left[ D_1(T(s), w_p, \sigma) + \sigma D_2(w_p) \right] \, dv \, ds \quad (57) \]

By an integration we deduce that \( \phi(t) = 0 \) for all \( t \in [0, \infty) \) and so, with the help of the zero boundary conditions, we arrive to the conclusion expressed by relation (53). Further, we introduce (53) into (37) and so we obtain (51) and this leads to uniqueness by means of the zero initial conditions. Thus, the proof is complete. \( \square \)

**Theorem 2.** Suppose that the hypothesis (H) holds true, \( q > 0 \) and \( \text{meas } \Sigma_1 \neq 0 \) and \( W(\varepsilon_p) \) is a negative semi-definite quadratic form. Then there exists a strictly positive
constant $\zeta$ so that, in the class of displacement-temperature-microtemperature fields $\{u, T, w\}(x, t)$ defined on $B \times [0, \infty)$, that satisfy

$$\int_0^t \int_B \left[ D_1 (T (s) , w_p (s) , \sigma) + \sigma D_2 (w_p (s)) \right] \, dv \, ds \leq \epsilon^2 \, \epsilon^t, \quad \text{for all } t \in [0, \infty) \quad (58)$$

with $M$ a positive constant, the forward-in-time problem $(\mathcal{P}^*)$ has at most one solution.

**Proof.** Let $\{u, T, w\}(x, t)$ be a solution of the forward-in-time problem $(\mathcal{P}^*)$ corresponding to zero given data. Then, from the identity (37) we obtain

$$\int_B \left[ 2W (e_{p_2} (t)) + aT^2 (t) + \frac{1}{2} \left( \sigma - \frac{\kappa_1}{T_0 K_3} \right) bw_{i} (t) w_{j} (t) \right] \, dv$$

$$= \int_0^t \int_B \left[ D_1 (T (s) , w_p (s) , \sigma) + \sigma D_2 (w_p (s)) \right] \, dv \, ds \quad (59)$$

For further convenience we take $\sigma$ to range into $(\sigma_1, \sigma_2)$ so that $D_1 (T, w_p, \sigma)$ satisfies relation (43) and we recall that $D_2 (w_p)$ is a positive semi-definite quadratic form. Since $W (e_{p_2})$ is a negative semi-definite quadratic form, from (59) we deduce that

$$\int_0^t \int_B \left[ D_1 (T (s) , w_p (s) , \sigma) + \sigma D_2 (w_p (s)) \right] \, dv \, ds$$

$$\leq \int_B \left[ aT^2 (t) + \frac{1}{2} \left( \sigma - \frac{\kappa_1}{T_0 K_3} \right) bw_{i} (t) w_{j} (t) \right] \, dv \quad (60)$$

It is worth to see that when $a \leq 0$ and $(\sigma - \frac{\kappa_1}{T_0 K_3}) b \leq 0$ relation (60) implies the uniqueness result. Otherwise, we recall that $meas \Sigma_3 \neq 0$ and $T = 0$ on $\Sigma_3 \times [0, \infty)$ allow us to deduce that

$$\int_B T_{i} (t) T_{j} (t) \, dv \geq \lambda_0 \int_B T^2 (t) \, dv \quad (61)$$

where $\lambda_0 > 0$ is the smallest eigenvalue for the fixed membrane problem for $B$. Therefore, by using relations (43) and (61) into (60), we obtain

$$\int_0^t \int_B \left[ D_1 (T (s) , w_p (s) , \sigma) + \sigma D_2 (w_p (s)) \right] \, dv \, ds$$

$$\leq \chi \int_B \left[ D_1 (T (t) , w_p (t) , \sigma) + \sigma D_2 (w_p (t)) \right] \, dv, \quad \text{for all } t \in (0, \infty) \quad (62)$$

where

$$\chi = \max \left( \frac{a}{\lambda_0 K_0}, \frac{1}{2 \kappa_0} \left( \sigma - \frac{\kappa_1}{T_0 K_3} \right) b \right) \quad (63)$$

In terms of the function

$$\psi (t) = \left\{ \int_0^t \int_B \left[ D_1 (T (s) , w_p (s) , \sigma) + \sigma D_2 (w_p (s)) \right] \, dv \right\}^{1/2} \quad (64)$$
relation (62) becomes

$$\psi^2(t) \leq 2\gamma \psi(t) \dot{\psi}(t), \quad t \in (0, \infty)$$

(65)

If $\psi(t) = 0$, for all $t \in (0, \infty)$, then it follows that

$$T(x, t) = 0, \quad w_i(x, t) = 0 \quad \text{in } \overline{B} \times [0, \infty)$$

(66)

and, by substituting this into identity (37), we are lead to relation (52); that is, we have the expected uniqueness result.

Suppose now that there exists $t_0 \in (0, \infty)$ so that $\psi(t_0) > 0$ and hence we have

$$\psi(t) \geq \psi(t_0) > 0 \quad \text{for all } t \in (t_0, \infty)$$

(67)

In such a case, from (65) we obtain the differential inequality

$$\dot{\psi}(t) \geq \frac{1}{2\gamma} \psi(t) \quad \text{for all } t \in (t_0, \infty)$$

(68)

which, when integrated, furnishes

$$\psi(t_0) e^{-\gamma(t-t_0)/2\gamma} \leq \psi(t) e^{-\gamma(t-t_0)/2\gamma} \leq \lim_{t \to \infty} \left[ \psi(t) e^{-\gamma(t-t_0)/2\gamma} \right] \quad \text{for all } t \in [t_0, \infty)$$

(69)

If now we take into account the hypothesis expressed by (58) and then we choose

$$0 \leq \zeta < \frac{1}{\gamma}$$

(70)

the relation (69) implies that $\psi(t) = 0$ for all $t \in (t_0, \infty)$ and this is in contradiction with our initial assumption (67). Thus, the proof is complete. $\Box$

**SPATIAL BEHAVIOR FOR THE CONSTRAINED MOTION IN A THERMOELASTIC CYLINDER WITH MICROTEMPERATURES**

Here, we assume $B$ to be the prismatic cylinder whose plane bounded uniform cross-section $\Sigma$ has piecewise continuously differentiable boundary curve $\partial \Sigma$. The origin of a rectangular Cartesian coordinate system is located in the cylinder’s base and the positive $x_3$-axis is directed along that of the cylinder and, therefore, $B \equiv \Sigma \times (0, L)$, where $L$ can be infinity. The cylinder is made of an isotropic and homogeneous thermoelastic material with microtemperatures and it is subject to zero body supplies and zero lateral specific boundary conditions. Its motion is induced by time-dependent displacement, temperature variation and microtemperatures specified pointwise over the base. The final values for the displacement, velocity, temperature variation and microtemperatures fields at points in the cylinder are not prescribed and they are in given proportion to, but not identical with, their respective values at an early specified time. Therefore, in terms of the forward-in-time problem ($\beta^+$) we have, for a specified time $\tau > 0$,

$$u_i(x, 0) = \gamma u_i(x, \tau), \quad \dot{u}_i(x, 0) = \delta \dot{u}_i(x, \tau)$$

$$T(x, 0) = \xi T(x, \tau), \quad w_i(x, 0) = \nu w_i(x, \tau), \quad x \in \overline{B}$$

(71)
MODEL OF THERMOELASTICITY WITH MICROTEMPERATURES

where $\gamma$, $\delta$, $\zeta$ and $\nu$ are given parameters satisfying the conditions

$$|\gamma| > 1, \quad |\delta| > 1, \quad |\zeta| > 1, \quad |\nu| > 1 \quad (72)$$

Moreover, we have the lateral boundary conditions

$$u_j \left[ \lambda \mu_{m,m} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) - \beta T \delta_{ij} \right] n_j = 0$$

$$T (k T_x + \kappa_1 w_i) n_i = 0$$

$$w_i (\kappa_4 w_{m,m} \delta_{ij} + \kappa_3 w_{j,i} + \kappa_6 w_{i,j}) n_j = 0 \quad (73)$$

for $(x, t) \in (\partial \Sigma \times (0, L)) \times (0, \tau)$ and the base boundary conditions

$$u_i (x, t) = g_i(x_1, x_2, t), \quad T(x, t) = h(x_1, x_2, t), \quad w_i (x, t) = \ell_i(x_1, x_2, t), \quad (x, t) \in \Sigma_0 \times (0, \tau) \quad (74)$$

where $\Sigma_0$ indicates the cross-section $\Sigma$ whose distance from the origin is $x_3$ and $g_i(x_1, x_2, t)$, $h(x_1, x_2, t)$ and $\ell_i(x_1, x_2, t)$ are prescribed differentiable functions.

In what follows we are interested in the study of the spatial behavior of the solution $S = \{u, T, w\}(x, t)$ of the forward-in-time problem $(P_0^u)$ as defined by Eqs. (12)–(14) (with zero body supplies) and the initial-final conditions (71), the lateral boundary conditions (73) and the base boundary conditions (74). To treat such a problem we will assume the following constitutive profile

$$c > 0, \quad \mu > 0, \quad 3 \lambda + 2 \mu > 0, \quad a > 0, \quad b > 0$$

$$k > 0, \quad \kappa_2 > 0, \quad \kappa_1 \kappa_3 < k \kappa_2$$

$$3 \kappa_4 + \kappa_5 + \kappa_6 > 0, \quad \kappa_6 + \kappa_5 > 0, \quad \kappa_6 - \kappa_5 > 0 \quad (75)$$

To solve the problem of concern we first apply identity (20) to the cylinder

$$B_{x_3} \equiv \Sigma \times (0, x_3) \quad (76)$$

where we take into account that the body supplies are vanishing and also we have the lateral boundary conditions (73). Thus, we get

$$\frac{d}{dt} \int_{B_{x_3}} \left[ \frac{1}{2} [\dot{q} u_i(t) \dot{u}_i(t) + 2W (e_{pq}(t)) + a T^2 (t) + \sigma w_i(t) w_i(t)] dv \right.$$

$$= \int_{B_{x_3}} \left[ D_1 (T(t), w_p(t), \sigma) + \sigma D_2 (w_p(t)) \right] dv$$

$$+ \int_{\Sigma_{t_0}} \left[ t_{3i} (t) \dot{u}_i(t) - \frac{1}{T_0} q_3 (t) T(t) + \sigma w_i(t) q_{3i}(t) \right] da$$

$$- \int_{\Sigma_0} \left[ t_{3i} (t) \dot{u}_i(t) - \frac{1}{T_0} q_3 (t) T(t) + \sigma w_i(t) q_{3i}(t) \right] da \quad (77)$$
Let us further introduce the function

\[ I (x_3) = \int_0^t \int_{\Sigma_{x_3}} e^{\alpha t} \left[ r_3 (t) \dot{u}_i (t) - \frac{1}{T_0} q_3 (t) T (t) + \sigma w_i (t) q_3 (t) \right] dadt \tag{78} \]

where \( \sigma \in (\sigma_1, \sigma_2) \) and \( \omega \) is a positive parameter at our disposal, whose values will be explicitly given later. Then, from (77) and by taking into account relation (71), we can deduce that

\[ I (x_3) = I (0) + \frac{1}{2} (e^{\alpha t} - \delta^2) \int_{B_{x_3}} q \dot{u}_i (\tau) \dot{u}_i (\tau) d\nu + \frac{1}{2} (e^{\alpha t} - \xi^2) \int_{B_{x_3}} aT^2 (\tau) d\nu \\
+ \frac{1}{2} (e^{\alpha t} - v^2) \int_{B_{x_3}} \sigma b w_i (\tau) w_i (\tau) d\nu + (e^{\alpha t} - \gamma^2) \int_{B_{x_3}} W (e_{pq} (t)) d\nu \\
- \int_0^t \int_{\Sigma_{x_3}} e^{\alpha t} \left\{ \frac{\alpha}{2} [q \dot{u}_i (t) \dot{u}_i (t) + aT^2 (t) + \sigma b w_i (t) w_i (t) + 2W (e_{pq} (t))] \right\} d\nu dt \tag{79} \]

Therefore, from (79) we deduce that

\[ \frac{dI}{dx_3} (x_3) = \frac{1}{2} (e^{\alpha t} - \delta^2) \int_{\Sigma_{x_3}} q \dot{u}_i (\tau) \dot{u}_i (\tau) d\alpha + \frac{1}{2} (e^{\alpha t} - \xi^2) \int_{\Sigma_{x_3}} aT^2 (\tau) d\alpha \\
+ \frac{1}{2} (e^{\alpha t} - v^2) \int_{\Sigma_{x_3}} \sigma b w_i (\tau) w_i (\tau) d\alpha + (e^{\alpha t} - \gamma^2) \int_{\Sigma_{x_3}} W (e_{pq} (t)) d\alpha \\
- \int_0^t \int_{\Sigma_{x_3}} e^{\alpha t} \left\{ \frac{\alpha}{2} [q \dot{u}_i (t) \dot{u}_i (t) + aT^2 (t) + \sigma b w_i (t) w_i (t) + 2W (e_{pq} (t))] \right\} d\alpha dt \tag{80} \]

At this instant we recall the assumption (72) so that it becomes clear that it is possible to choose the parameter \( \omega \) in such a way to have

\[ e^{\alpha t} - \delta^2 \leq 0, \quad e^{\alpha t} - \xi^2 \leq 0, \quad e^{\alpha t} - v^2 \leq 0, \quad e^{\alpha t} - \gamma^2 \leq 0 \quad \tag{81} \]

that is, we assume that \( \omega \) ranges into the set

\[ 0 < \omega \leq 2 \min (\ln |\gamma|, \ln |\delta|, \ln |\xi|, \ln |v|) \tag{82} \]

Furthermore, in view of the assumption (75), we can see that \( W(e_{pq}) \) is a positive definite quadratic form. Moreover, we can write

\[ \mu_m e_{ij} e_{ij} \leq W (e_{pq}) \leq \mu_M e_{ij} e_{ij} \quad \tag{83} \]
where $2\mu_m = \min (2\mu, 3\lambda + 2\mu) > 0$, $2\mu_M = \max (2\mu, 3\lambda + 2\mu) > 0$. Finally, assumption (75) implies that $D_2 (w_p)$ is a positive definite quadratic form; moreover, we can write

$$\kappa_0^2 w_{i,j} w_{i,j} \leq D_2 (w_p) \leq \bar{\kappa}_0^2 w_{i,j} w_{i,j} \tag{84}$$

where $\kappa_0^2$ and $\bar{\kappa}_0^2$ are the minimum and the maximum eigenvalues of $D_2 (w_p)$. With these choices, we can conclude that

$$- \frac{dI}{dx_3} (x_3) \geq \int_0^x \int_{\Sigma_{x_3}} e^{\sigma \tau} \left\{ \frac{\omega_0}{2} \left[ \dot{q} \dot{u}_i (t) u_i (t) + aT^2 (t) + \sigma \dot{e}_i (t) w_i (t) + 2W (e_{pq} (t)) \right] \right. \tag{85}$$

$$+ D_1 (T (t), w_p (t), \sigma) + \sigma D_2 (w_p (t)) \right\} d\sigma dt$$

and hence $I(x_3)$ is a non-increasing function with respect to $x_3$ on $(0, L)$.

On the other hand, from relations (1) and (78) we deduce that

$$I(x_3) = \int_0^x \int_{\Sigma_{x_3}} e^{\sigma \tau} \left\{ [\dot{q} e_{mn} (t) \delta_{3i} + 2\mu e_{3i} (t)] \dot{u}_i (t) - \beta T (t) u_3 (t) - \frac{k}{T_0} T (t) T_3 (t) \right. \tag{86}$$

$$- \frac{k_1}{T_0} T (t) w_3 (t) - \sigma [\kappa_0 w_{i,3} + \kappa_3 w_{3,i} + \kappa_4 w_{m,m} \delta_{3i}] w_i \left. \right\} d\sigma dt$$

By means of Schwarz’s inequality and the arithmetic-geometric mean inequality, we have

$$[\dot{q} e_{mn} (t) \delta_{3i} + 2\mu e_{3i} (t)] \dot{u}_i (t) \leq \sqrt{\frac{\mu M}{q}} \left[ W (e_{pq} (t)) + \frac{1}{2} q \dot{u}_i (t) \dot{u}_i (t) \right] \tag{87}$$

$$|\beta T (t) u_3 (t)| \leq \left| \frac{\beta}{\sqrt{a q}} \right| \frac{1}{2} q \dot{u}_i (t) \dot{u}_i (t) + \frac{1}{2} aT^2 (t) \right\} \tag{88}$$

$$\left| \frac{k}{T_0} T (t) T_3 (t) \right| \leq \left| \frac{k}{\sqrt{a q}} \right| \frac{1}{2} aT^2 (t) + \left| \frac{k}{2T_0} \right| \sqrt{\frac{q k}{a}} T_3^2 (t) \tag{89}$$

$$\left| \frac{k_1}{T_0} T (t) w_3 (t) \right| \leq \left| \frac{k_1}{\sqrt{a q}} \right| \frac{1}{2} aT^2 (t) + \left| \frac{k_1}{2T_0} \right| \sqrt{\frac{q k_1}{a}} w_3^2 (t) \tag{90}$$

$$\left| \sigma \left[ \kappa_0 w_{i,3} + \kappa_3 w_{3,i} + \kappa_4 w_{m,m} \delta_{3i} \right] w_i \right| \leq \frac{\sigma}{2} \sqrt{\frac{\kappa_0^2}{B}} \left[ D_2 (w_p (t)) + bw_i (t) w_i (t) \right] \tag{91}$$

Finally, we substitute the estimates (87)–(91) into relation (86) and then use the estimate (85) in order to obtain the following first-order differential inequality

$$|I (x_3)| \leq -\kappa \frac{dI}{dx_3} (x_3) \quad \text{for all } x_3 \in (0, L) \tag{92}$$
where

\[ \kappa = \max \left\{ \frac{1}{\omega} \left( \frac{\sqrt{\mu M}}{a} + \frac{1}{\sqrt{\alpha Q}} \right), \frac{3}{\omega \sqrt{\alpha Q}}, \frac{1}{\omega} \sqrt{\frac{\kappa_0}{B}}, \frac{1}{\omega} \sqrt{\frac{\kappa_0}{b}}, \frac{1}{\omega} \sqrt{\frac{2}{T_0^2 |\beta|}} \sqrt{\frac{a}{\kappa_0}} \left( \frac{k^2}{\kappa_0} + \kappa_0^2 \right) \right\} \]

(93)

Next we discuss the consequences of this differential inequality. To this end we recall that \( I(x_3) \) is a non-increasing function on (0, \( L \)); hence, we have the only following two possibilities: (i) \( I(x_3) > 0 \) for all \( x_3 \in (0, L) \), or (ii) there is \( x_3^0 \in (0, L) \) so that \( I(x_3^0) < 0 \).

Let us first consider the case when \( I(x_3) > 0 \) for all \( x_3 \in (0, L) \). Then the relation (92) implies the first-order differential inequality

\[ \frac{dI}{dx_3} (x_3) + \frac{1}{\kappa} I(x_3) \leq 0 \quad \text{for all } x_3 \in (0, L) \]

(94)

which, when integrated, gives the following decay estimate

\[ 0 \leq I(x_3) \leq I(0) e^{-\frac{x_3}{\kappa}} \quad \text{for all } x_3 \in [0, L) \]

(95)

For the case of a semi-infinite cylinder, that is when \( L \) tends to infinity, relation (95) proves that

\[ \lim_{x_3 \to \infty} I(x_3) = 0 \]

(96)

and then the relation (80), when integrated on \((x_3, \infty)\), implies that the total energy \( \mathcal{E}(x_3) \), associated with the semi-infinite cylinder, defined by

\[
\mathcal{E}(x_3) = \frac{1}{2} (\delta^2 - e^{\sigma t}) \int_{R(x_3, \infty)} \tilde{q} \tilde{u}_t (\tau) \tilde{u}_t (\tau) d\tau + \frac{1}{2} (\zeta^2 - e^{\sigma t}) \int_{R(x_3, \infty)} aT^2 (\tau) d\tau
+ \frac{1}{2} (\nu^2 - e^{\sigma t}) \int_{R(x_3, \infty)} \sigma b w_1 (\tau) w_1 (\tau) d\tau + (\gamma^2 - e^{\sigma t}) \int_{R(x_3, \infty)} W (e_{pq} (\tau)) d\tau
+ \int_{0}^{\tau} \int_{R(x_3, \infty)} e^{\sigma t} \left[ \frac{a}{2} [\tilde{q} \tilde{u}_t (\tau) \tilde{u}_t (\tau) + aT^2 (\tau) + \sigma b w_1 (\tau) w_1 (\tau) + 2W (e_{pq} (\tau))] \right]
+ D_1 (T (t), w_p (t), \sigma) + \sigma D_2 (w_p (t)) \right] d\tau dt, \quad \text{with } B(x_3, \infty) \equiv \Sigma \times (x_3, \infty)
\]

(97)

exists and it satisfies

\[ \mathcal{E}(x_3) \leq I(0) e^{-\frac{x_3}{\kappa}} \quad \text{for all } x_3 \in [0, \infty) \]

(98)

We next consider the case (ii) when we have

\[ I(x_3) \leq I(x_3^0) < 0 \quad \text{for all } x_3 \in [x_3^0, L) \]

(99)
In such a case, the relation (92) implies the following differential inequality
\[
\frac{dI}{dx_3}(x_3) - \frac{1}{\kappa} I(x_3) \leq 0 \quad \text{for all } x_3 \in [x_3^0, L]
\] (100)
and hence we deduce that
\[
-I(x_3) \geq -I(x_3^0) e^{\frac{\alpha x_3^0}{\omega}} \quad \text{for all } x_3 \in [x_3^0, L]
\] (101)
When a semi-infinite cylinder is considered, relation (101) proves that \(I(x_3)\) becomes unbounded when \(x_3\) takes asymptotically large values and, therefore, \(\mathcal{E}(x_3)\) becomes unbounded when \(L \to \infty\).

We may summarize this analysis in the following alternative of a Phragmén–Lindelöf-type result.

**Theorem 3.** In the context of a semi-infinite cylinder made of a thermoelastic material with microtemperatures, the solution \(S = (u, T, w)(x, t)\) of the forward-in-time problem \((\mathcal{P}_0^*)\) either has a finite energetic measure \(\mathcal{E}(x_3)\), which decays to zero faster than the exponential \(e^{-\frac{x_3}{\kappa}}\), or it has an infinite energetic measure, and then \(-I(x_3)\) goes to infinity faster than the exponential \(e^{-\frac{x_3}{\omega}}\).

**REFERENCES**


