Thermoelastic surface waves on an exponentially graded half-space

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In this paper, we consider the propagation of Rayleigh surface waves in a functionally graded isotropic thermoelastic half-space, in which all thermoelastic characteristic parameters exponentially change along the depth direction. The propagation condition is established in the form of a bicubic equation whose coefficients are complex numbers while the analytical solutions (eigensolutions) of the thermoelastic system are explicitly obtained in terms of the characteristic solutions. The concerned solution of the Rayleigh surface wave problem is subsequently expressed as a linear combination of the three eigensolutions while the secular equation is established in an implicit form. The explicit secular equation is written when an isotropic and homogeneous thermoelastic half-space is considered and some numerical simulations are given for a specific material.

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1. Introduction

The theory of Rayleigh surface waves finds its principal application in the interpretation of ground motions resulting from earthquakes and explosions. The surface waves have been also successfully applied in many technological fields, such as the non-destructive evaluation of materials, filters and sensors, soil dynamics, nuclear reactors, electronic components, resonators, high energy particle accelerators, driving microfluidic actuation, among many others. Such applications involve the effect of the interaction between the elastic deformation and the heat conduction on the wave propagation. The study of the propagation of thermoelastic Rayleigh waves within the framework of linear theory of thermoelasticity for isotropic and homogeneous thermoelastic half-space was first initiated by Lockett (1958). It has been shown that the waves, in the general case, exhibit dispersion and are accompanied by attenuation. Moreover, Chadwick (1960) considers the modification aspect and he studies thoroughly the quasi-elastic and quasi-thermal modes both for waves of assigned frequency and for waves of assigned length. The secular equation was further studied by Chadwick and Windle (1964) (see also Atkin and Chadwick, 1981) for the special cases in which the stress-free surface is either held at constantly uniform temperature or is thermally insulated. It was shown by Currie (1979) that elliptical paths, followed by material particles on free surface, are described in reverse direction. Ivanov (1988) discussed some specific behavior criteria at infinity of the thermoelastic Rayleigh waves considered by Chadwick (1960).

On the other hand, it was shown recently (see, for example, Destre, 2007; Ting, 2011a,b) that in case of seismic Rayleigh wave propagation in an anisotropic, inhomogeneous elastic Earth, analytical and exact results can be obtained when stiffness and mass density similarly vary with exponential change along the depth direction. The propagation of anti-plane shear waves in an elastic half-space when shear modulus and mass density have an arbitrary dependence on the distance from the free surface has been studied by Achenbach and Balogun (2010).

This paper is concerned with the propagation of Rayleigh surface waves in an isotropic and inhomogeneous thermoelastic half-space whose thermoelastic characteristic parameters exponentially change along the depth direction. A sagittally polarized (Rayleigh type) wave travels along a symmetry axis (while being attenuated along another) of an isotropic material with thermoelastic coefficients and mass density similarly and exponentially varying with depth, proportional to a common factor $\exp(-2\pi x_3/\sigma)$ say, where $\sigma$ is the inverse
of an inhomogeneity characteristic length, and $x_2$ is the coordinate along the normal to the free surface. By using an idea developed by Destrade (2007), we seek wave solutions to the basic differential system of the linear thermoelastodynamics and the propagation condition is consequently established as a bicubic equation with complex coefficients. The analytical solutions (eigensolutions) of the thermoelastodynamic system are further explicitly obtained in terms of characteristic solutions to propagation equation. Finally, the concerned solution of the Rayleigh surface wave problem is expressed as a linear combination of the three eigensolutions while the secular equation is implicitly established. The explicit expression for the secular equation of a homogeneous thermoelastic half-space is also obtained and some numerical simulations are given for a specific material.

2. Basic equations

Throughout this section we assume that a regular region $B$ is filled by an isotropic thermoelastic material. According to the linear theory of thermoelastodynamics, the fundamental system of field equations, in the absence of the supply terms, consists (see, e.g., Carlson, 1972) of the strain–displacement relation

$$ e_{rs} = \frac{1}{2}(u_{r,s} + u_{s,r}) \quad \text{in} \quad B \times [0, \infty), $$

(1)

the thermal gradient–temperature relation

$$ g_r = \theta_r \quad \text{in} \quad B \times [0, \infty), $$

(2)

the stress–strain–temperature relation

$$ S_{rs} = \lambda e_{mm}\delta_{rs} + 2\mu e_{rs} + m\theta\delta_{rs} \quad \text{in} \quad B \times [0, \infty), $$

(3)

the heat conduction equation

$$ q_r = -k\theta_r \quad \text{in} \quad B \times [0, \infty), $$

(4)

the equations of motion

$$ S_{rs},r = \varrho u_r \quad \text{in} \quad B \times (0, \infty), $$

(5)

and the energy equation

$$ -q_{r,r} + \theta_0 m\theta_r = c\theta \quad \text{in} \quad B \times (0, \infty). $$

(6)

Here $u_r$ are the components of the displacement vector, $\theta$ is the temperature variation from the uniform reference temperature $\theta_0 > 0$, $e_{rs}$ are the components of the strain tensor, $g_r$ are the components of the thermal gradient vector, $S_{mm}$ are the components of the stress tensor and $q_r$ are the components of the heat flux vector. Further, we assume that the mass density $\varrho$, the Lamé moduli $\lambda$ and $\mu$, the stress-temperature modulus $m$, the specific heat $c = \theta_0 a$ and the conductivity $k$ are prescribed fields and, moreover, that $\lambda$, $\mu$, $m$ and $k$ are smooth on $B$, while $\varrho$ and $c$ are continuous on $B$. We also assume that the mass density $\varrho$, the specific heat $c$ and the conductivity $k$ are strictly positive quantities and the Lamé moduli $\lambda$ and $\mu$ satisfy

$$ \mu > 0, \quad \lambda + \frac{2}{3}\mu > 0. $$

When the isotropic thermoelastic material is homogeneous, the basic equations can be expressed in terms of the displacements $u_r$ and the temperature variation $\theta$ as

$$ \mu u_{r,ss} + (\lambda + \mu) u_{s,ss} + m\theta_r = \varrho u_r, $$

$$ k\theta_{rr} + \theta_0 m\theta_r = c\theta. $$

(8)

3. Plane harmonic waves in an infinite thermoelastic medium

In this section we give a brief account of the harmonic plane progressive wave solutions for homogeneous and isotropic bodies with zero body force and heat supply. When the thermal dissipative properties of the elastic material are taken into consideration, then the thermoelastic processes are irreversible and the plane waves solutions are expected to be damped in time (see, for example, Lebeau and Zuazua, 1999). That means we have to seek solutions of the displacement–temperature equations of motion (8) in the form (Cf. Carlson, 1972)

$$ u_r(x, t) = \text{Re}[A_r e^{i(\alpha x - vt)}], $$

$$ \theta(x, t) = \text{Re}[B e^{i(\alpha x - vt)}]. $$

(9)

Here $\text{Re} \{ \cdot \}$ is the real part, $i = \sqrt{-1}$ is the imaginary unit, $A = (A_1, A_2, A_3)$ is a constant complex vector and $B$ is a complex constant with $|A| \neq 0$ or $|B| \neq 0, \alpha$ is the wave number, $n$ is a real unit vector giving the direction of propagation. Further,

$$ v = \text{Re}(v) + i\text{Im}(v) $$

(10)

is a complex constant so that

$$ \text{Re}(v) \geq 0 $$

(11)

is giving the wave speed and $\text{exp}[^{i\text{Im}(v)t}]$ is giving the damping in time of the wave and hence we assume that

$$ \text{Im}(v) \leq 0. $$

(12)
If the imaginary part of \( \nu \) happens to vanish, that is \( \text{Im}(\nu) = 0 \), then we have an undamped harmonic in time wave. Otherwise, that is when \( \text{Im}(\nu) < 0 \) we have a damped wave.

In what follows we are searching displacements and temperature variations of the form (9) satisfying the basic equations (8). To this end we substitute (9) into (8) to obtain
\[
(\mu - \theta \nu^2) c^2 \frac{\partial^2 v}{\partial t^2} + (\lambda + \mu) c^2 \frac{\partial^2 n}{\partial t^2} A_3 - i \varepsilon m \frac{\partial v}{\partial t} B = 0,
\]
and hence, by taking into account that \( |A| \neq 0 \) or \( |B| \neq 0 \), we obtain the following characteristic equation
\[
(c_1^2 - \nu^2)^2 \left[ (c_1^2 - \nu^2) \left( 1 - \frac{iv}{k\lambda} \right) - \frac{i \theta m^2}{k\lambda} \nu \right] = 0,
\]
where
\[
c_1 = \sqrt{\frac{\lambda + 2\mu}{Q}}, \quad c_2 = \sqrt{\frac{\mu}{Q}}.
\]
Thus, we have the following solutions satisfying restrictions (11) and (12),
\[
v_3 = v_4 = c_2,
\]
while the other admissible solutions satisfy the equation
\[
(v^2 - c_1^2) \left( v + \frac{ib\lambda}{k} \right) - \varepsilon c_1^2 v = 0,
\]
where
\[
\varepsilon = \frac{\theta_0 m^2}{Q c_1^2}.
\]
For \( v = v_3 = v_4 = c_2 \) there are transverse waves of the form
\[
\varrho^{(T)} = \text{Re}[A e^{i(n x - c_2 t)}],
\]
\[
\theta^{(T)} = 0, \quad \text{with } A \cdot n = 0.
\]
These transverse waves are independent of the thermal effects, they are undamped in time and they propagate with speed \( c_2 \).

Let us now consider \( v \) a solution of Eq. (17) that satisfies the conditions (11) and (12). We set
\[
v = ik
\]
in the relation (17) so that \( k \) has to satisfy the equation
\[
k^3 + \frac{xk}{c} + c_1^2(1 + \varepsilon)k + \frac{ixk\varepsilon}{c} = 0,
\]
that is a cubic having a negative real solution \( k_2 < 0 \) while the other two are of the following form
\[
k_1 = -a - ib, \quad k_3 = -a + ib,
\]
where \( a \) and \( b \) are positive numbers that can be explicitly determined. In what follows we are going to take into account conditions (11) and (12) and we have chosen \( v_1 \) and \( v_2 \) as the complex solutions of Eq. (17) so that
\[
v_1 = b - ia, \quad v_2 = ik_2.
\]
We note that the wave solution corresponding to \( v = v_1 \) is the longitudinal quasi-elastic wave of the form
\[
\varrho^{(Le)} = \text{Re} \left\{ \frac{1}{\varepsilon(c_1^2 - v_1^2)} \left( c_1^2 - c_1^2 + \frac{iv_1 cc_1^2}{xk - iv_1 c} \right) e^{i(n x - c_1 t)} \right\} n_1 e^{-\varepsilon x t},
\]
\[
\theta^{(Le)} = \text{Re} \left\{ \frac{\theta_0 m v_1}{xk - iv_1 c} e^{i(n x - c_1 t)} \right\} e^{-\varepsilon x t},
\]
while the quasi-thermal mode solution corresponding to \( v = v_2 \) is of the form
\[
\varrho^{(Qm)} = \text{Re} \left\{ \frac{im \theta_0}{Q c_1^2 - v_2^2} e^{i(n x)} \right\} n_1 e^{\varepsilon x t},
\]
\[
\theta^{(Qm)} = \text{Re} \left\{ \theta_0 e^{i(n x)} \right\} e^{\varepsilon x t}.
\]
Thus, we have the damped longitudinal quasi-elastic waves whose amplitudes of oscillation decrease exponentially to zero when the time is going to infinity like \( e^{-\varepsilon x t} \). The quasi-thermal mode, like the purely thermal disturbance, is a standing wave decaying to zero like \( e^{\varepsilon x t} \) when the time goes to infinity. This agrees with results in specialized literature concerning the time decay of solutions in one-dimensional linear thermoelasticity (see, for example, Lebeau and Zuazua, 1999).

Table 1 contains basic data for the four common metals at 20 °C as considered by Chadwick (1960) and the corresponding values for \( v_1 \) and \( v_2 \) when \( \varepsilon = 1 \, \text{cm}^{-1} \). It should be therefore noticed here that the coupled thermoelastic model explains two aspects: one proves that the
Table 1

<table>
<thead>
<tr>
<th>Material</th>
<th>ε</th>
<th>$v_1^0$ cm/s</th>
<th>$v_2^0$ cm/s</th>
<th>$v_1$ cm/s</th>
<th>$v_2$ cm/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium</td>
<td>3.56 × 10⁻²</td>
<td>6.32 × 10³</td>
<td>-0.85713 i</td>
<td>643151 – 0.0150418 i</td>
<td>-0.845046 i</td>
</tr>
<tr>
<td>Copper</td>
<td>1.68 × 10⁻²</td>
<td>4.36 × 10³</td>
<td>-1.0838 i</td>
<td>439647 – 0.0089535 i</td>
<td>-1.06589 i</td>
</tr>
<tr>
<td>Iron</td>
<td>2.97 × 10⁻⁴</td>
<td>5.80 × 10³</td>
<td>-0.19223 i</td>
<td>580086 – 0.0000285376 i</td>
<td>-0.192173 i</td>
</tr>
<tr>
<td>Lead</td>
<td>7.33 × 10⁻²</td>
<td>2.14 × 10³</td>
<td>-0.23977 i</td>
<td>221704 – 0.00818743 i</td>
<td>-0.223395 i</td>
</tr>
</tbody>
</table>

decay rate of the quasi-thermal mode component tends to become lower than that of the purely thermal mode and the other proves that, for the quasi-elastic mode, the wave speed of the longitudinal quasi-elastic wave is greater than that of the purely elastic and it displays the damping characteristic.

4. Thermoelastic Rayleigh surface waves on an exponentially graded half-space

In this section we consider the propagation of an inhomogeneous plane wave with wavenumber $\nu$ in the $x_1$–direction and with attenuation in the $x_2$–direction in the half-space $x_2 \geq 0$ made of an isotropic thermoelastic material with an exponential depth profile of the form

$$
\lambda(x) = \lambda_0 e^{-2\sigma x_2}, \quad \mu(x) = \mu_0 e^{-2\sigma x_2}, \quad m(x) = m_0 e^{-2\sigma x_2},
$$

$$
k(x) = k_0 e^{-2\sigma x_2}, \quad c(x) = c_0 e^{-2\sigma x_2}, \quad \varrho(x) = \varrho_0 e^{-2\sigma x_2}.
$$

(26)

Here $\sigma$ is a prescribed real number and $\lambda_0, \mu_0, m_0, c_0$ and $\varrho_0$ are constants satisfying the conditions

$$
\mu_0 > 0, \quad \lambda_0 + 2/3\mu_0 > 0, \quad k_0 > 0, \quad c_0 > 0, \quad \varrho_0 > 0.
$$

(27)

The half-space is free of supply loads and free of traction on its surface $x_2 = 0$. It is free to exchange heat with the contents of the region $x_2 < 0$ and both media are at constant temperature $\theta_0$ in all places while preceding emergence of a disturbance. For a surface wave propagating in the direction of the $x_1$–axis in the half-space $x_2 \geq 0$, the surface traction at $x_2 = 0$ must vanish and zero radiative boundary condition should be assumed. Thus, we have the following boundary conditions

$$
S_{2\nu}(x, t) = 0, \quad q_2(x, t) + h\theta(x, t) = 0 \text{ at } x_2 = 0 \quad \text{for all } t \geq 0,
$$

(28)

where $h > 0$ is a prescribed constant. To these boundary conditions we have to adjoin the asymptotic conditions

$$
\lim_{x_2 \to -\infty} u_i(x, t) = 0, \quad \lim_{x_2 \to -\infty} \theta(x, t) = 0,
$$

$$
\lim_{x_2 \to +\infty} S_{2\nu}(x, t) = 0, \quad \lim_{x_2 \to +\infty} q_i(x, t) = 0,
$$

(29)

for all $x_1, x_3 \in \mathbb{R}$ and $t \geq 0$.

The Rayleigh surface wave propagation problem for the isotropic and inhomogeneous thermoelastic half-space $x_2 \geq 0$ consists of finding solutions with finite energy for the basic equations (1)–(6) and the boundary conditions (28) and the asymptotic conditions (29). To solve this problem we first seek solutions of the basic equations (1)–(6) in the form

$$
u_0(x, t) = U_0 e^{i\nu(x_1 - \nu t - (i/\nu)x_2)},
$$

$$
u_3(x, t) = 0,
$$

$$
\theta(x, t) = T e^{i\nu(x_1 - \nu t - (i/\nu)x_2)},
$$

(30)

where $U_0$ and $T$ are constant complex parameters with $|U| \neq 0$ or $|T| \neq 0$, $\nu$ is the wave number, $\nu$ is a complex parameter subject to the conditions (11) and (12) and $r$ is a complex parameter so that

$$
\text{Im}(r) > \frac{|\nu|}{\nu}.
$$

(31)

Last condition is required to satisfy (29).

A substitution of relation (30) into (1)–(4) gives

$$
S_{2\nu} = bx T_{2\nu} e^{i\nu(x_1 - \nu t + (i/\nu)x_2)},
$$

$$
q_{2\nu} = bx Q_{2\nu} e^{i\nu(x_1 - \nu t + (i/\nu)x_2)},
$$

(32)

and

$$
S_{33} = bx \left[ c_{12} \left( U_1 + \left( \frac{r}{\nu} \frac{i}{r} U_2 \right) - \frac{i\nu_0}{\nu} \right) \right] e^{i\nu(x_1 - \nu t + (i/\nu)x_2)},
$$

$$
S_{13} = S_{23} = 0, \quad q_3 = 0,
$$

(33)
where
\[ T_{11} = c_{11} U_1 + c_{12} \left( r - \frac{\sigma}{\kappa} \right) U_2 - \frac{\iota \phi_0}{\kappa} T, \]
\[ T_{22} = c_{12} U_1 + c_{11} \left( r - \frac{\sigma}{\kappa} \right) U_2 - \frac{\iota \phi_0}{\kappa} T, \]
\[ T_{12} = c_{66} \left( r - \frac{\sigma}{\kappa} \right) U_1 + c_{66} U_2, \]
\[ Q_1 = -k_0 T, \]
\[ Q_2 = -k_0 \left( r - \frac{\sigma}{\kappa} \right) T, \]
and
\[ c_{11} = \lambda_0 + 2\mu_0, \quad c_{12} = \lambda, \quad c_{66} = \mu. \]

(34)

It should be noticed that attenuation factors for displacement and temperature variation amplitudes in (30) are distinct from those related to stress and heat flux amplitudes in (32) and (33) and this is clearly due to the inhomogeneity of thermoelastic material.

Furthermore, if we use relations (32)–(35) into the equations of motion (5) and (6), we shall then obtain
\[ \left[ c_{66} \left( r^2 + \frac{\sigma^2}{\kappa^2} \right) + c_{11} C_1 \right] U_1 + \left[ c_{66} \left( r + \frac{\sigma}{\kappa} \right) + c_{12} \left( r - \frac{\sigma}{\kappa} \right) \right] U_2 - \frac{\iota \phi_0}{\kappa} T = 0, \]
\[ \left[ c_{66} \left( r - \frac{\sigma}{\kappa} \right) + c_{12} \left( r + \frac{\sigma}{\kappa} \right) \right] U_1 + \left[ c_{11} \left( r^2 + \frac{\sigma^2}{\kappa^2} \right) + c_{66} C_2 \right] U_2 - \frac{\iota \phi_0}{\kappa} \left( r + \frac{\sigma}{\kappa} \right) T = 0, \]
\[ \theta_0 \psi_{m0} U_1 + \theta_0 \psi_{m0} \left( r - \frac{\sigma}{\kappa} \right) U_2 - k_0 \left( r^2 + \frac{\sigma^2}{\kappa^2} + 1 - \frac{i c_0 \phi}{\kappa \psi_{m0}} \right) T = 0, \]
where
\[ C_1 = 1 - \frac{\iota \phi_0^2}{c_{11}}, \quad C_2 = 1 - \frac{\iota \phi_0^2}{c_{66}}. \]

(35)

A necessary and sufficient condition if this system is to have a nontrivial solution for \( U_1, U_2 \) and \( T \) is the following
\[ \left| c_{66} \left( r^2 + \frac{\sigma^2}{\kappa^2} \right) + c_{11} C_1 \right| \left| c_{66} \left( r + \frac{\sigma}{\kappa} \right) + c_{12} \left( r - \frac{\sigma}{\kappa} \right) \right| - \frac{\iota \phi_0}{\kappa} = 0, \]
\[ \left| c_{66} \left( r - \frac{\sigma}{\kappa} \right) + c_{12} \left( r + \frac{\sigma}{\kappa} \right) \right| \left| c_{11} \left( r^2 + \frac{\sigma^2}{\kappa^2} \right) + c_{66} C_2 \right| - \frac{\iota \phi_0}{\kappa} \left( r + \frac{\sigma}{\kappa} \right) = 0, \]
\[ \left| \theta_0 \psi_{m0} \right| \left| \theta_0 \psi_{m0} \left( r - \frac{\sigma}{\kappa} \right) \right| - k_0 \left( r^2 + \frac{\sigma^2}{\kappa^2} + 1 - \frac{i c_0 \phi}{\kappa \psi_{m0}} \right) = 0, \]
and that is the propagation condition we get here
\[ r^6 + P_1 r^4 + P_2 r^2 + P_3 = 0, \]

(39)

where
\[ P_1 = C_1 + C_2 + 1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} \frac{i \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} + 3 \frac{\sigma^2}{\kappa^2}, \]
\[ P_2 = C_1 C_2 + (C_1 + C_2) \left( 1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} \right) \left( 1 + C_2 \right) - \frac{i \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} + 2 \frac{\sigma^2}{\kappa^2} \left( C_1 + C_2 + 1 + \frac{2 \psi_{12}}{c_{11}} - \frac{i c_0 \psi}{\kappa \psi_{m0}} \right) - \frac{\iota \phi_0}{\kappa} \left( C_1 + C_2 \right) \frac{\iota \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} + 3 \frac{\sigma^4}{\kappa^4}, \]
\[ P_3 = C_2 \left[ C_1 \left( 1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} \right) \frac{i \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} + \frac{\sigma^2}{\kappa^2} \left( C_1 C_2 + (C_1 + C_2 + 4 \frac{C_{12}}{c_{11}}) - \frac{i c_0 \psi}{\kappa \psi_{m0}} \right) \right] \frac{\iota \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} \]
\[ + \frac{\sigma^4}{\kappa^4} \left( C_1 + C_2 + 4 \frac{C_{12}}{c_{11}} + 1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} - \frac{i \theta_0 \psi_{m0}^2}{\kappa \psi_{m0} \kappa_{c11}} \right) + \frac{\sigma^6}{\kappa^6}. \]

(40)

The solutions of the propagation equation are clearly dependent on the parameters \( \sigma \) and \( \varepsilon \) (or \( m_0 \)).

At \( \sigma = 0 \), the thermoelastic material is homogeneous, and the associated determinantal equation (39) or the propagation condition (40) reduces to
\[ \left( r^2 + C_2 \right) \left\{ r^4 + \left[ 1 + C_1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} \left( 1 + \varepsilon \right) \right] r^2 + C_1 - \frac{i c_0 \psi}{\kappa \psi_{m0}} (C_1 + \varepsilon) \right\} = 0, \]
whose solutions satisfy
\[ r^2 = -C_2, \]

(41)
and

\[ r^4 + \left[ C_1 + 1 - \frac{iC_0 \nu}{\nu_0} (1 + \epsilon) \right] \left( r^2 + C_1 \right) - \frac{iC_0 \nu}{\nu_0} (C_1 + \epsilon) = 0. \] (46)

Moreover, when there is no coupling between the mechanical and thermal effects, that is when \( \epsilon = 0 \), Eq. (46) reduces to

\[ (r^2 + C_1) \left( r^2 + 1 - \frac{iC_0 \nu}{\nu_0} \right) = 0 \] (47)

with solutions

\[ r^2 = -C_1, \] (48)

and

\[ r^2 = -1 + \frac{iC_0 \nu}{\nu_0}. \] (49)

We shall further mark the solutions of Eqs. (48), (45) and (49), respectively, whose imaginary parts are positive, by \( p_1, p_2 \) and \( p_3 \).

Let us mark the three solutions of the propagation condition (40) by \( r_{1}^{2}, r_{2}^{2} \) and \( r_{3}^{2} \). They depend on \( \sigma \) and \( \epsilon \), and for \( \sigma = 0, \epsilon = 0 \), we have

\[ r_{1}^{2}(\sigma = 0, \epsilon = 0) = -C_1, \]
\[ r_{2}^{2}(\sigma = 0, \epsilon = 0) = -C_2, \]
\[ r_{3}^{2}(\sigma = 0, \epsilon = 0) = -1 + \frac{iC_0 \nu}{\nu_0}. \] (50)

We choose the roots \( r_1, r_2 \) and \( r_3 \) so that

\[ r_1(\sigma = 0, \epsilon = 0) = p_1, \quad r_2(\sigma = 0, \epsilon = 0) = p_2, \quad r_3(\sigma = 0, \epsilon = 0) = p_3. \] (51)

Consequently, for \( r = r_n, n = 1, 2 \), the solution \( U^{(\alpha)} = (U_1^{(\alpha)}, U_2^{(\alpha)}, T^{(\alpha)}) \) of the system (37) is

\[
U_1^{(\alpha)} = \frac{1}{x} \left[ \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} + \frac{C_{66}}{C_{11}} \right) \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) - \frac{i\theta \nu \nu_0^{2}}{\nu_0 C_{11}} \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} \right) \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} \right) \left( r_{1}^{2} + \frac{\sigma_{1}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) \right].
\]

\[
U_2^{(\alpha)} = \frac{1}{x} \left[ \left( \frac{C_{66}}{C_{11}} \left( r_{2}^{2} + \frac{\sigma_{2}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) \left( r_{2}^{2} + \frac{\sigma_{2}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) \left( r_{2}^{2} + \frac{\sigma_{2}^{2}}{x^2} + 1 - \frac{iC_0 \nu}{\nu_0} \right) \right].
\]

\[ T^{(\alpha)} = \frac{i\theta \nu \nu_0 C_{66}}{C_{11}} \left( 2 \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) - \left( r_{3} + \frac{i\sigma}{x} \right) \right) \left( r_{3} - \frac{i\sigma}{x} \right) \] (no sum over \( \alpha \)).

while, for \( r = r_3 \), we have the solution \( U^{(3)} = (U_1^{(3)}, U_2^{(3)}, T^{(3)}) \) of the system (37) given by

\[
U_1^{(3)} = \frac{im_0 \theta_0}{C_{11}} \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} + 1 + C_{3} \right) \left( r_{3} + \frac{i\sigma}{x} \right) \left( r_{3} - \frac{i\sigma}{x} \right).
\]

\[
U_2^{(3)} = \frac{im_0 \theta_0}{C_{11}} \left( 2 \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) + \left( C_{1} + C_{2} \right) \right) \left( r_{3} + \frac{i\sigma}{x} \right) \left( r_{3} - \frac{i\sigma}{x} \right).
\]

\[ T^{(3)} = \frac{i\theta_0 \nu \nu_0 C_{66}}{C_{11}} \left( 2 \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) + \left( C_{1} + C_{2} \right) \right) \left( r_{3} + \frac{i\sigma}{x} \right) \left( r_{3} - \frac{i\sigma}{x} \right) \] (no sum over \( \alpha \)).

The corresponding state of stress and heat flux is obtained by substituting relations (52) and (53) into relations (34), (35) and (32). Thus, we obtain

\[ \tau_{ij}^{(3)} = \frac{C_0}{x} \left[ \left( -r_{ij}^{2} \right) + \frac{\Delta \sigma_{ij}^{2}}{C_{11}} \left( r_{ij} - \frac{i\sigma}{x} \right) \left( r_{ij} - \frac{i\sigma}{x} \right) \left( r_{ij} - \frac{i\sigma}{x} \right) \right]. \]

\[ \tau_{ij}^{(3)} = \frac{\Delta \sigma_{ij}^{2}}{C_{11}} \left( r_{ij}^{2} - \frac{i\sigma}{x} \right) \left( r_{ij} - \frac{i\sigma}{x} \right) \left( r_{ij} - \frac{i\sigma}{x} \right) \right]. \]

\[ Q_0^{(3)} = \frac{i\theta_0 \nu \nu_0 C_{66}}{C_{11}} \left( \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) + \left( C_{1} + C_{2} \right) \right) \left( r_{3} + \frac{i\sigma}{x} \right) \left( r_{3} - \frac{i\sigma}{x} \right) \] (no sum over \( \alpha \)).

and

\[ \tau_{ij}^{(3)} = \frac{i\theta_0 \nu \nu_0 C_{66}}{C_{11}} \left( \left( 2 \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) - \left( r_{3} + \frac{i\sigma}{x} \right) \right) \left( r_{3} - \frac{i\sigma}{x} \right) \right]. \]

\[ Q_0^{(3)} = \frac{i\theta_0 \nu \nu_0 C_{66}}{C_{11}} \left( \left( r_{3}^{2} + \frac{\sigma_{3}^{2}}{x^2} \right) + \left( C_{1} + C_{2} \right) \right) \left( r_{3} + \frac{i\sigma}{x} \right) \left( r_{3} - \frac{i\sigma}{x} \right) \] (no sum over \( \alpha \)).
Let us now consider the Rayleigh surface wave propagation problem. In view of the above analysis we will seek a solution of this problem in the following form

$$ u_0(x, t) = \sum_{n=1}^{3} y_n T^{(n)}_{12} \phi_{12}(x_1 - \nu t + (n_0/\nu)x_2), $$

$$ u_3(x, t) = 0, $$

$$ T(x, t) = \sum_{n=1}^{3} y_n T^{(n)}_{12} \phi_{12}(x_1 - \nu t + (n_0/\nu)x_2), $$

(56)

where $y_n$, $n = 1, 2, 3$, are complex parameters (at least one different from zero) to be determined later. Eq. (56) obviously satisfies the basic equations (1)–(6) and the asymptotic conditions (29). Such shall we determine the parameters $\gamma_1$, $\gamma_2$ and $\gamma_3$ that the boundary conditions (28) be satisfied. By substituting (56) into (1)–(4), we obtain

$$ S_{12}(x, t) = \sum_{n=1}^{3} y_n T^{(n)}_{12} \phi_{12}(x_1 - \nu t + (n_0/\nu)x_2), $$

$$ S_{22}(x, t) = \sum_{n=1}^{3} y_n T^{(n)}_{22} \phi_{22}(x_1 - \nu t + (n_0/\nu)x_2), $$

$$ q_{12}(x, t) = \sum_{n=1}^{3} y_n q^{(n)}_{12} \phi_{12}(x_1 - \nu t + (n_0/\nu)x_2), $$

(57)

so that the boundary conditions (28) imply the following homogeneous algebraic linear system

$$ \gamma_1 T^{(1)}_{12} + \gamma_2 T^{(2)}_{12} + \gamma_3 T^{(3)}_{12} = 0, $$

$$ \gamma_1 T^{(1)}_{22} + \gamma_2 T^{(2)}_{22} + \gamma_3 T^{(3)}_{22} = 0, $$

$$ \gamma_1 (\nu_0 q^{(1)}_{12} + h T^{(1)}_{12}) + \gamma_2 (\nu_0 q^{(2)}_{12} + h T^{(2)}_{12}) + \gamma_3 (\nu_0 q^{(3)}_{12} + h T^{(3)}_{12}) = 0, $$

(58)

where $T^{(n)}_{12}$, $T^{(n)}_{22}$ and $q^{(n)}_{12}$, $n = 1, 2, 3$, are given by (54) and (55). In order to have a non-trivial solution for $\gamma_1$, $\gamma_2$ and $\gamma_3$, we get to the following secular equation to determine the complex parameter $\nu$

$$ \Delta(\nu) = \left| \begin{array}{ccc}
T^{(1)}_{12} & T^{(2)}_{12} & T^{(3)}_{12} \\
T^{(1)}_{22} & T^{(2)}_{22} & T^{(3)}_{22} \\
h - i\nu\kappa_0 (r_1 - i\sigma r) & h - i\nu\kappa_0 (r_2 - i\sigma r) & h - i\nu\kappa_0 (r_3 - i\sigma r) \end{array} \right| = 0. $$

(59)

It can be noticed, from (56), that every point of the thermoelastic half-space moves on an elliptical orbit in plane $x_1 Ox_2$. As in Currie (1979), it can be shown that the thermoelastic Rayleigh surface wave is retrograde at the free surface $x_2 = 0$.

It should be underlined that, when a homogeneous isotropic thermoelastic half-space is considered (that is $\sigma = 0$), we have

$$ r_2^2 = -C_2 $$

(60)

and, based on Eq. (46), the relations (52)–(55) would therefore imply

$$ \gamma^{(1)}_{12} = \frac{2c_{66}}{x_0} \left( c_{12} + c_{66} \right) \left( r_1^2 + 1 - \frac{ic_3 \nu}{x_0} \right) - \frac{\theta_3 v_m n^2}{x_0} r_1, $$

$$ \gamma^{(1)}_{22} = -\frac{c_{66}}{x_0} \left( 1 + C_2 \right) \left( c_{12} + c_{66} \right) \left( r_1^2 + 1 - \frac{ic_3 \nu}{x_0} \right) - \frac{\theta_3 v_m n^2}{x_0} \right), $$

$$ \gamma^{(1)} = \frac{\theta_3 v_m n^2 c_{66}}{x_0} \left( r_1^2 + C_2 \right), $$

(61)

$$ \gamma^{(2)}_{12} = \frac{c_{66}}{x_0} \left( 1 + C_2 \right) \left( c_{12} + c_{66} \right) \left( r_2^2 + 1 - \frac{ic_3 \nu}{x_0} \right) - \frac{\theta_3 v_m n^2}{x_0} r_2, $$

$$ \gamma^{(2)}_{22} = \frac{2c_{66}}{x_0} \left( c_{12} + c_{66} \right) \left( r_2^2 + 1 - \frac{ic_3 \nu}{x_0} \right) - \frac{\theta_3 v_m n^2}{x_0} \right), $$

$$ \gamma^{(2)} = 0, $$

(62)
Therefore, the secular equation (59) reduces to the simplified form

\[ (h - i\kappa_0 r_1)(r_1^2 + C_1)[4C_2r_1 - (1 + C_2)\tilde{r}_2] \left[ C_{12} + \frac{c_{66}}{c_{66}} \left( \frac{r_1^2}{2} + 1 - \frac{i\kappa_0 v}{2} \right) - \frac{\theta_0 v m_0^2}{\kappa_0 c_{66}} \right] \]

\[ + \frac{\theta_0 v m_0^2}{\kappa_0 c_{66}} (h - i\kappa_0 r_1)(r_1^2 + C_2) \left[ 4C_2 r_1 - (1 + C_2)\tilde{r}_2 \right] = 0. \]  

Note that, when there is no coupling between the mechanical and thermal effects, the secular equation (64) reduces to that for isothermal isotropic elastic materials, that is

\[ 4C_2 r_1 - (1 + C_2)\tilde{r}_2 = 0. \]  

5. Numerical simulations

In order to get an idea upon the behavior of the complex parameter \( \nu \) of the Rayleigh surface wave with respect to the coupling parameter \( \kappa \), we will try to compute it numerically for a specific model. For this purpose, we consider the half-space \( x_2 \geq 0 \), made of an isotropic and homogeneous thermoelastic material, \( (\sigma = 0) \), whose boundary surface is insulated \( (t = 0) \). Furthermore, we will take the following values of the relevant parameters for copper material: \( c_1 = 4631.0 \text{ m/s}, c_2 = 2280.1 \text{ m/s}, c/k = 8066.8 \text{ s/m}^2, \kappa = 1.68 \times 10^{-2} \) and we will have the wave number set to \( \kappa = 1 \text{ m}^{-1} \). Recalling that \( C_2 = \frac{c_2}{c_1} \) and by setting

\[ \nu = i\kappa_0 w, \]  

the secular equation (64) could be then written in the following form, convenient for computing reasons,

\[ \hat{\varepsilon}(w) = R_1 \left( R_3 + \frac{\kappa}{c_2 c_1} C_2 \right) \left[ 4R_1 r_2 + \sqrt{\frac{\kappa/k}{c_2 c_1}} (1 + C_2)^2 \right] \left[ C_1 + \frac{c_1^2}{c_2^2} \left( R_1^2 + w \right) + \frac{c_1^2}{c_2^2} \frac{\kappa/k}{c_2 c_1} + \frac{c_1^2}{c_2^2} w \right] \]

\[ - \varepsilon wR_1 \left( R_3 + \frac{\kappa/k}{c_2 c_1} C_2 \right) \left[ 4R_1 r_2 + \sqrt{\frac{\kappa/k}{c_2 c_1}} (1 + C_2)^2 \right] = 0, \]  

where

\[ \tilde{C}_1 = 1 + \frac{c_1^2}{c_2^2} w^2, \quad \tilde{C}_2 = 1 + w^2, \]  

and

\[ r_{1,3}^2 = \frac{c_2^2 R_1^2}{\kappa/k}, \]

\[ R_{1,3}^2 = \frac{1}{2} \left\{ -\frac{\kappa/k}{c_2 c_1} (C_1 + 1) - (1 + \varepsilon) w + \sqrt{\frac{\kappa/k}{c_2 c_1} (C_1 - 1) - (1 - \varepsilon) w} \right\}^2 + 4\varepsilon w^2. \]  

The solutions of the secular equation (67) have to be selected so that

\[ \text{Re}(w) \leq 0, \quad \text{Im}(w) < 0, \]
and

$$\text{Im}(R_1) > 0, \quad \text{Im}(R_2) > 0, \quad \text{Im}(R_3) > 0.$$  \hfill (71)

When we substitute the assigned values of the relevant parameters in the above equations, the secular equation (67), for \( w \), is solved by means of the graphical method with the software package Wolfram Mathematica version 7.0.1.0. For computing convenience and in order to outline the expected solution (near to that isothermal, when \( w = 0 - 0.933528i \)), we introduce the following function

$$j(\text{Re}(w), \text{Im}(w)) = |10^{20} \times \text{E}(w)|^2 - |10^{20} \times \text{Re}[\text{E}(w)]|^2 + |10^{20} \times \text{Im}[\text{E}(w)]|^2,$$

and, with the software package Wolfram Mathematica version 7.0.1.0, we generate it graphically for \( \text{Re}(w) \in (-0.4, 0.0) \), \( \text{Im}(w) \in (-1.0, 0.0) \). As it can be seen in Figs. 1 and 2, there is a point \( w^* = w_1 + w_2i \), where \( \text{E}(w^*) = 0 \) and \( w_2 \) is near to \(-0.933528 \) and \( w_1 \) is near to zero. So, the attenuation in time factor is very small and the surface wave therefore travels at a speed and with an amplitude slightly lower than that in the isothermal case. This can be explained by the fact that the thermoelastic dissipation is not sufficiently strong. Although the numerical examples suggest results near those of isothermal case, the thermoelastic model exhibits the attenuation in time effect upon the Rayleigh surface waves.

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