SPATIAL BEHAVIOR IN THE VIBRATING THERMOVISCOELASTIC POROUS MATERIALS

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ABSTRACT. In this paper we study the spatial behavior of the amplitude of the steady-state vibrations in a thermoviscoelastic porous beam. Here we take into account the effects of the viscoelastic and thermal dissipation energies upon the corresponding harmonic vibrations in a right cylinder made of a thermoviscoelastic porous isotropic material. In fact, we prove that the positiveness of the viscoelastic and thermal dissipation energies are sufficient for characterizing the spatial decay and growth properties of the harmonic vibrations in a cylinder.

1. Introduction. Recently Ieşan [18] developed a linear theory of thermoviscoelastic porous materials, in which the set of independent constitutive variables includes the time derivative of the strain tensor, the time derivative of the volume fraction field and the time derivative of the gradient of the volume fraction field. The theory represents an extension of the Cowin and Nunziato’s theory of elastic materials with voids [9] and that of Ieşan’s theory of thermoelastic materials with voids [16] by incorporating the memory effects.

Recently, Sharma and Kumar [21] and Tomar et al. [23] study the time harmonic steady-state vibrations in an infinite thermoviscoelastic material with voids. It is shown that there are four basic waves traveling with distinct speeds, out of which one is a shear wave and the other three are dilatational waves.

Since the porous materials have many potential uses this has inspired much work into dynamical problems, see e.g. Ieşan [15], Ciarletta and Scalia [6], Chirîţă and Scalia [2], Scalia et al. [20], Ciarletta and Straughan [7], Ciarletta et al. [8] and into plane strain problems, see e.g. Ieşan and Nappa [19], D’Apice and Chirîţă [4], [11]. For a review of the literature on porous elastic materials the reader is referred to the books by Straughan [22] and Ieşan [17].

Within the context of the thermoviscoelastic model of materials with voids, in the present paper we take into consideration the dissipation mechanism due to the thermal and the memory effects in order to study the spatial behavior of the steady-state solutions. On this basis we establish the spatial decay and growth properties of the harmonic in time solutions in a finite as well as in a semi-infinite cylinder made of an isotropic thermoviscoelastic porous material. The presence of the thermal and

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memory dissipation energies allow us to establish the spatial behavior results for the entire class of thermoviscoelastic porous materials compatible with thermodynamics and for every value of the frequency of vibrations, without any positive definiteness upon the elastic coefficients. This is in essential contrast with the studies made in the classical elasticity by Flavin and Knops [12] and in the classical theory of thermoelasticity by Chirită [1], where the spatial estimates describing the spatial behavior are established in the class of the steady-state solutions whose frequencies are lower than a certain critical frequency, provided the assumptions are made relating the positive definiteness of the elastic coefficients. It becomes clear from the studies existing at this time in literature (see e.g. [1] and [10]) that the thermal dissipation only does not work for establishing the spatial behavior for any frequency of the classical thermoelastic harmonic vibrations.

On the other hand, it was shown in [3], [14] and [13] that the viscoelastic dissipation mechanism can furnish the advantage to establish the spatial behavior for the entire class of steady-state solutions, without any constraint upon their frequencies.

To establish the spatial behavior of the steady-state vibrations within the context of the linear theory of thermoviscoelastic porous materials, we introduce a natural functional and show that this is a measure of the amplitude of vibrations that exponentially decays or grows with respect to the distance to the loaded end of cylinder. We then modify the identity satisfied by the functional in concern to obtain in a natural way the spatial decay and growth properties of the steady-state vibrations for the whole class of thermoviscoelastic porous materials whose viscous dissipation energy satisfies a strong ellipticity condition.

Thus, we may conclude that the thermal dissipation effect in conjunction with the memory dissipation mechanism are able to assure a complete study of the spatial decay and growth properties of the steady-state vibrations.

2. Basic equations. Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes \( O_{x_k}, (k = 1, 2, 3) \). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers \( (1, 2, 3) \), whereas Greek subscripts are confined to the range \( (1, 2) \), summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation. Throughout this section we suppose that a regular region \( B \) is filled by a homogeneous and isotropic thermoviscoelastic material. Considering the linear theory of thermoviscoelastic materials with voids and assuming that the initial body is free from stresses and has zero intrinsic equilibrated body force and entropy, the system of field equations consists of (cf. Ieșan [18])

- the equations of motion
  \[
  t_{rs,r} + \varrho f_s = \varrho \ddot{u}_s, \\
  H_{r,r} + g + \varrho \ell = \varrho \kappa \ddot{\varphi}, 
  \text{ in } B \times (0, \infty),
  \tag{1}
  \]

- the equation of energy
  \[
  \varrho \mathcal{I} \ddot{\eta} = Q_{r,r} + \varrho s, 
  \text{ in } B \times (0, \infty),
  \tag{2}
  \]

- the constitutive equations
  \[
  t_{rs} = \lambda e_{mm} \delta_{rs} + 2\mu e_{rs} + b\varphi \delta_{rs} - \beta \theta \delta_{rs} + \lambda^* \dot{e}_{mm} \delta_{rs} + 2\mu^* \dot{e}_{rs} + b^* \dot{\varphi} \delta_{rs},
  \]
H_r = \alpha \varphi, r + \alpha^* \dot{\varphi}, r + \tau^* \theta, r, \\
g = -b e_{mm} - \xi \varphi + m \theta - \gamma^* e_{mm} - \xi^* \dot{\varphi}, \\
g \eta = \beta e_{mm} + a \theta + m \varphi, \\
Q_r = k \theta, r + \zeta \dot{\varphi}, r, \text{ in } B \times [0, \infty), \tag{3}

and

the geometrical relations

e_{rs} = \frac{1}{2} (u_{r,s} + u_{s,r}), \text{ in } B \times [0, \infty). \tag{4}

Here we have used the following notations: \( t_{rs} \) are the components of the stress tensor, \( H_r \) are the components of the equilibrated stress vector, \( g \) is the equilibrated body force, \( \eta \) is the entropy per unit mass, \( Q_r \) are the components of the heat flux vector, \( e_{rs} \) are the components of the strain tensor, \( \varrho \) is the mass density of the medium, \( \kappa \) is the equilibrated inertia, \( u_r \) are the components of the displacement vector, \( \varphi \) is the void volume fraction, \( \theta \) is the change in temperature from the constant ambient temperature \( T_0 > 0 \), \( \delta_{rs} \) are the components of the Kronecker delta, \( \lambda \) and \( \mu \) are well known Lame's constant parameters, \( b, \alpha, \xi \) and \( \xi^* \) are the constant parameters corresponding to voids present in the medium, \( \beta, \tau^* \) are the constant thermal parameters and \( \lambda^*, \mu^*, b^*, \alpha^* \) and \( \gamma^* \) are the constant viscoelastic parameters and \( f_r \) are the components of the body force per unit mass, \( \ell \) is the extrinsic equilibrated body force per unit mass, \( s \) is the heat supply per unit mass. Furthermore, in view of the second law of thermodynamics, the Clausius-Duhem inequality must be satisfied, which provides the positive semi-definiteness of the total dissipation energy \( \Lambda \), that is

\[ \Lambda = \lambda^* e_{mm} \dot{\varphi} + 2 \mu^* e_{rs} \dot{\varphi} + \alpha^* \dot{\varphi}, r + \xi^* \ddot{\varphi} + \frac{1}{T_0} k \theta, r \geq 0. \tag{5} \]

This implies the following inequalities among the various thermoviscoelastic moduli

\[ \mu^* \geq 0, \quad \xi^* \geq 0, \quad \frac{1}{4} (b^* + \gamma^*)^2 \leq \xi^* \left( \lambda^* + \frac{2}{3} \mu^* \right), \quad (6) \]

\[ k \geq 0, \quad T_0 \left( \tau^* + \frac{1}{T_0} \xi \right)^2 \leq 4 \alpha^* k. \]

By substituting the relation (4) into (3) and further, the result into (1) and (2) and assuming that the body and heat source density are zero, we obtain the following system of differential equations

\[ \mu u_{s,rr} + (\lambda + \mu) u_{m,ms} + b \varphi, s - \beta \theta, s = g \ddot{u}, \]

\[ \alpha \varphi, rr - \nu u_{r,rr} - \xi \varphi + \tau \theta, rr + m \theta = g \kappa \ddot{\varphi}, \tag{7} \]

\[ k \theta, rr - \beta T_0 \dot{u}_{r,rr} + \zeta \dot{\varphi}, rr - m T_0 \dot{\varphi} = \varphi, \text{ in } B \times (0, \infty), \]
where \( c = aT_0 \) and

\[
\lambda_\ast = \lambda + \lambda^* \frac{\partial}{\partial t}, \quad \mu_\ast = \mu + \mu^* \frac{\partial}{\partial t},
\]

\[
b_\ast = b + b^* \frac{\partial}{\partial t}, \quad \alpha_\ast = \alpha + \alpha^* \frac{\partial}{\partial t}, \quad \nu_\ast = b + \gamma^* \frac{\partial}{\partial t}, \quad \xi_\ast = \xi + \xi^* \frac{\partial}{\partial t}.
\]

3. **Formulation of the problem and constitutive hypotheses.** In this section we assume that \( B \) is the interior of a right cylinder of length \( L > 0 \) whose cross section is bounded by one or more piecewise smooth curves. We assume the Cartesian coordinates such that the origin lies in one end of the cylinder and such that the \( x_3 \)-axis is parallel to the generators. Let \( D(x_3) \) denote the cross section of the cylinder corresponding to the axial distance \( x_3 \), and let \( \partial D(x_3) \) denote the cross-sectional boundary. We denote by \( \pi \) the lateral surface of the cylinder, so that \( \pi = \partial D \times (0, L) \) and \( B = D \times (0, L) \).

For the steady-state time harmonic solutions in the cylinder \( B = \Sigma \times (0, L) \), we consider

\[
\begin{align*}
    u_r (x, t) &= v_r (x) e^{-i\omega t}, \\
    \varphi (x, t) &= \phi (x) e^{-i\omega t}, \\
    \theta (x, t) &= T (x) e^{-i\omega t},
\end{align*}
\]

where \( \omega \) is a prescribed positive frequency and \( i \) is the unit complex, that is \( i = \sqrt{-1} \).

Inserting \( u_r \), \( \varphi \) and \( \theta \) from (9) into equations (7), and assuming zero body forces and heat supply, we obtain for the amplitude \( \{v_r, \phi, T\} \) of the oscillation the following differential system

\[
\begin{align*}
    \mu_0 v_{r,rr} + (\lambda_0 + \mu_0) v_{r,rs} + \rho \omega^2 v_r + b_0 \phi_r - \beta T_r &= 0, \\
    \alpha_0 \phi_{rr} - \xi_0 \phi + \rho \omega^2 \phi - \nu_0 \phi_{mm} + \tau^* T_{rr} + mT &= 0, \\
    kT_{rr} - \iota \omega \xi \phi_{rr} + \iota \omega T_0 (\beta v_{mm} + aT + m\phi) &= 0, \quad \text{in} \quad B = \Sigma \times (0, L),
\end{align*}
\]

where

\[
\begin{align*}
    \lambda_0 &= \lambda - i\omega \alpha^*, \quad \mu_0 = \mu - i\omega \mu^*, \quad b_0 = b - i\omega b^*, \\
    \alpha_0 &= \alpha - i\omega \alpha^*, \quad \nu_0 = b - i\omega \nu^*, \quad \xi_0 = \xi - i\omega \xi^*.
\end{align*}
\]

We further assume that the lateral boundary surface of the cylinder is fixed and thermal insulated, that is we suppose the following lateral boundary conditions

\[
v_r (x) = 0, \quad \phi (x) = 0, \quad T (x) = 0 \quad \text{on} \quad \pi.
\]

Moreover, we assume the following end boundary conditions

\[
\begin{align*}
    v_r (x_1, x_2, 0) &= \bar{v}_r (x_1, x_2), \quad \phi (x_1, x_2, 0) = \bar{\phi} (x_1, x_2), \\
    T (x_1, x_2, 0) &= \bar{T} (x_1, x_2), \quad \text{for all} \quad (x_1, x_2) \in D(0),
\end{align*}
\]

and

\[
\begin{align*}
    v_r (x_1, x_2, L) &= 0, \quad \phi (x_1, x_2, L) = 0, \\
    T (x_1, x_2, L) &= 0, \quad \text{for all} \quad (x_1, x_2) \in D(L),
\end{align*}
\]

where \( \bar{v}_r (x_1, x_2), \bar{\phi} (x_1, x_2) \) and \( \bar{T} (x_1, x_2) \) are prescribed functions on \( D(0) \).
Throughout this paper we consider the boundary value problem \( \mathcal{P} \) defined by the differential system (10), the lateral boundary conditions (12) and the end boundary conditions (13) and (14), whose solution \( \{ v_r, \phi, T \} \) is the amplitude of oscillation (9). We are interested to describe the spatial decay and growth properties of the amplitude \( \{ v_r, \phi, T \} \) for both finite and semi-infinite cylinders made of a thermoviscoelastic porous material. To this aim we assume the constitutive profile characterized by

\[
\mu^* > 0, \quad \xi^* > 0, \quad \frac{1}{4} (b^* + \gamma^*)^2 < \xi^* \left( \lambda^* + \frac{2}{3} \mu^* \right), \quad (15)
\]

\[
k > 0, \quad T_0 \left( \tau^* + \frac{1}{T_0} \xi \right)^2 < 4 \alpha^* k. \quad (16)
\]

Obvious, such hypothesis agrees with the consequences of the Clausius-Duhem inequality as expressed by relation (6). In fact, the inequalities in (15) express the positive definiteness of the quadratic form

\[
\Lambda_1 (\xi_{ij}, \chi) = \lambda^* \xi_{mm} \xi_{nn} + 2 \mu^* \xi_{rs} \xi_{rs} + (b^* + \gamma^*) \xi_{mm} \chi + \xi^* \chi^2, \quad \xi_{rs} = \xi_{sr}. \quad (17)
\]

while the inequalities in (16) prove that

\[
\Lambda_2 (\zeta_p, \eta_q) = \alpha^* \omega^2 \zeta_p \zeta_r + \omega (\zeta + T_0 \tau^*) \zeta \eta_r + T_0 k \eta_r \eta_r, \quad (18)
\]

is a positive definite quadratic form. It follows then that we have

\[
\nu_1 (\xi_{rs} \xi_{rs} + \chi^2) \leq \Lambda_1 (\xi_{ij}, \chi) \leq \nu_2 (\xi_{rs} \xi_{rs} + \chi^2), \quad (19)
\]

and

\[
\pi_1 (\zeta_p \zeta_p + \eta_q \eta_q) \leq \Lambda_2 (\zeta_p, \eta_q) \leq \pi_2 (\zeta_p \zeta_p + \eta_q \eta_q), \quad (20)
\]

where \( \nu_1 \) and \( \nu_2 \) are the lowest and the greatest eigenvalues of the quadratic form \( \Lambda_1 \) and \( \pi_1 \) and \( \pi_2 \) are the lowest and the greatest eigenvalues of the quadratic form \( \Lambda_2 \).

4. Spatial behavior. In this section we shall study the spatial behavior of the amplitude \( \{ v_r, \phi, T \} \) of the considered vibrations under the assumptions (15) and (16) as suggested by the dissipation inequality (6). To this end we associate with the amplitude \( \{ v_r, \phi, T \} \) of the steady-state thermoviscoelastic vibration, the cross-sectional functional

\[
I(x_3) = - \int_{\partial(x_3)} \left\{ \text{i} \omega T_0 \left[ \mu_0 v_{r,3} \overline{v}_r - \overline{\mu}_0 v_r \overline{v}_{r,3} + (\lambda_0 + \mu_0) \overline{v}_{3} v_{r,r} ight. \right.
\]

\[
- \left( \overline{\lambda}_0 + \overline{\mu}_0 \right) v_3 \overline{v}_{r,r} + b_0 v_3 \overline{\phi} - \overline{b}_0 v_3 \overline{\phi} + \beta v_3 \overline{T} - \beta \overline{v}_3 T 
\]

\[
+ \text{i} \omega T_0 \left( \alpha_0 \phi,3 \overline{\phi} - \overline{\alpha}_0 \overline{\phi},3 \phi + \tau^* T_{3,3} \overline{\phi} - \tau^* \overline{T}_{3,3} \phi \right) + k T_{3,3} \overline{T} + k \overline{T}_{3,3} T + \text{i} \omega \zeta (T \overline{T}_{3,3} - \phi,3 \overline{T}) \right\} da,
\]

where the superposed bar denotes the complex conjugate.

**Theorem 4.1.** In the context of a finite cylinder made of a thermoviscoelastic porous material, the cross-sectional functional \( I(x_3) \) represents an acceptable measure of the amplitude \( \{ v_r, \phi, T \} \), in the sense that it is positive for all \( \{ v_r, \phi, T \} \) and it vanishes only when \( \{ v_r, \phi, T \} = 0 \). Moreover, there exists a computable positive
constant \( \sigma \), depending on the thermoviscoelastic porous profile, so that it holds true the following exponential decay estimate

\[
0 \leq I(x_3) \leq I(0)e^{-\sigma x_3} \quad \text{for all} \quad x_3 \in [0, L].
\]  

(22)

**Proof.** First of all we note that the equation (10) implies

\[
0 = v_s (\rho \omega^2 v_s) - \nabla_s \left( \rho \omega^2 v_s \right) = (\mu_0 v_{s,rr} - \nabla_0 v_s v_{s,rr})
\]

\[
+ \left[ \left( \lambda_0 + \mu_0 \right) \nabla_s v_{s,rr} - ( \lambda_0 + \nabla_0 ) v_s \nabla_{s,rr} \right] + \left( b_0 \nabla_s \phi_s - b_0 v_s \tilde{\phi}_s \right) + \left( \beta v_s T_s - \beta \nabla_s T_s \right),
\]  

(23)

and hence we have

\[
[(\mu_0 v_{s,rr} - \nabla_0 v_s v_{s,rr}) + \left( \lambda_0 + \mu_0 \right) \nabla_s v_{s,rr} - ( \lambda_0 + \nabla_0 ) v_s \nabla_{s,rr}]_{s}
\]

\[
+ (\nabla_0 - \mu_0) v_{s,r} \nabla_{s,r} + ( \lambda_0 + \nabla_0 - \lambda_0 - \mu_0 ) v_{r,r} \nabla_{s,s}
\]

\[
+ (b_0 \nabla_s \phi_s - b_0 v_s \tilde{\phi}_s) + (\beta v_s T_s - \beta \nabla_s T_s) = 0.
\]  

(24)

Thus, an integration of the identity (24) over \( D(x_3) \) followed by the use of the lateral boundary condition (12) leads to the following identity

\[
\frac{d}{dx_3} \int_{D(x_3)} \left[ (\mu_0 v_{s,3} \nabla_s - \nabla_0 v_s v_{s,3} + (\lambda_0 + \mu_0) \nabla_s v_{s,3} - ( \lambda_0 + \nabla_0 ) v_3 \nabla_{s,3} \right] da
\]

\[
= \int_{D(x_3)} \left[ (\mu_0 - \nabla_0) v_{s,r} \nabla_{s,r} + (\lambda_0 + \mu_0 - \lambda_0 - \nabla_0) v_{r,r} \nabla_{s,s}
\]

\[
+ \left( b_0 \nabla_s \phi_s - b_0 v_s \tilde{\phi}_s \right) + \left( \beta v_s T_s - \beta \nabla_s T_s \right) \right] da.
\]  

(25)

Similarly, from the equation (10) we get

\[
0 = (\rho \omega^2 \phi) - (\rho \kappa \omega^2 \phi) = \alpha_0 \phi_{,rr} \phi - \alpha_0 \phi_{,rr} \phi - \xi_0 \phi \phi - \nu_0 v_{m,m} \phi
\]

\[
+ \nabla_0 \nabla_{m,m} \phi + \tau^* T_{,rr} \phi - \tau^* T_{,rr} \phi + m \nabla \phi - m \nabla \phi,
\]  

(26)

so that we obtain

\[
\left( \alpha_0 \phi_{,rr} \phi - \alpha_0 \phi_{,rr} \phi + \tau^* T_{,rr} \phi - \tau^* T_{,rr} \phi \right)_{,r} = \alpha_0 \phi_{,rr} \phi - \alpha_0 \phi_{,rr} \phi
\]

\[
+ \nabla_0 \nabla_{m,m} \phi + \nabla_0 \nabla_{m,m} \phi - m \nabla \phi - m \nabla \phi.
\]  

(27)

By integrating (27) over \( D(x_3) \) and by using the lateral boundary condition (12) we obtain the following identity

\[
\frac{d}{dx_3} \int_{D(x_3)} \left( \alpha_0 \phi_{,3} \phi - \alpha_0 \phi_{,3} \phi + \tau^* T_{,3} \phi - \tau^* T_{,3} \phi \right) da
\]

\[
= \int_{D(x_3)} \left[ \alpha_0 \phi_{,rr} \phi + \alpha_0 \phi_{,rr} \phi + \left( \xi_0 - \xi_0 \right) \phi \phi + \nu_0 v_{m,m} \phi
\]

\[
- \nabla_0 \nabla_{m,m} \phi + \nabla_0 \nabla_{m,m} \phi + m \nabla \phi - m \nabla \phi \right] da.
\]  

(28)

Finally, from the equation (10) we have

\[
0 = (\omega T_0 a T) - (\omega T_0 a T) T = k T_{,rr} T + k T_{,rr} T + i \omega \zeta (T \nabla \phi - T \phi_{,rr} - T \phi_{,rr}) + \omega T_0 \left( \beta v_{m,m} T - \beta \nabla_{m,m} T + m \phi T - m \phi T \right),
\]  

(29)
and, therefore, we get

\[
[kT_r \rho + kT_r \rho + i\omega (T \bar{\phi}_r - \phi_r T)] = 2kT_r \rho + i\omega (T \bar{\phi}_r - \bar{T}_r \phi_r) - i\omega T_0 (\beta v_{m,m} \rho - \beta v_{m,m} \rho + m\phi \rho - m\bar{\phi} \rho).
\]

(30)

Therefore, from (30) and (12) we obtain the following identity

\[
\frac{d}{dx_3} \int_{D(x_3)} [kT_3 \rho + kT_3 \rho + i\omega (T \bar{\phi}_3 - \rho \phi_3)] da = \int_{D(x_3)} [2kT_r \rho + i\omega (T \bar{\phi}_3 - \rho \phi_3)] da.
\]

(31)

At this stage we multiply the identities (25) and (28) by \(i\omega T_0\) and then we add the result to the identity (31) to obtain

\[
\int_{D(x_3)} \left\{ i\omega T_0 \left[ \mu_0 v_{r,3} \bar{v}_r - \bar{v}_0 v_r \bar{v}_{r,3} + (\lambda_0 + \mu_0) \bar{v}_3 v_{r,r} - (\lambda_0 + \bar{\lambda}_0) \bar{v}_3 v_{r,r} + \beta v_{r,3} \rho - \beta v_{r,3} \rho + \tau^* T_{r,3} \bar{\phi} - \tau^* \bar{T}_{r,3} \phi \right] \right\} da = \int_{D(x_3)} \left\{ i\omega T_0 \left[ (\mu_0 - \bar{\mu}_0) v_{s,r} \bar{v}_{s,r} + (\lambda_0 + \mu_0) - \bar{\lambda}_0 - \bar{\lambda}_0 \right] v_{s,s} v_{r,r} + \beta v_{r,3} \rho - \beta v_{r,3} \rho + \tau^* T_{r,3} \bar{\phi} - \tau^* \bar{T}_{r,3} \phi \right\} da.
\]

(32)

In view of relations (11) and (21), we can write the identity (32) in the following form

\[
-\frac{dI}{dx_3}(x_3) = \int_{D(x_3)} \left\{ \omega^2 T_0 \left[ 2\mu^* v_{r,s} \bar{v}_{s,r} + 2(\lambda^* + \mu^*) v_{s,s} \bar{v}_{r,r} + 2\xi^* \bar{\phi} \bar{\phi} \right] + \left( b^* + \gamma^* \right) \left( v_{s,s} \bar{\phi} + v_{s,s} \bar{\phi} \right) \right\} da.
\]

(33)

On the other hand, in view of the lateral boundary conditions (12) and the integration by parts, we have

\[
\int_{D(x_3)} \epsilon_{r,s} \bar{v}_{r,s} da = \frac{1}{4} \int_{D(x_3)} (v_{r,s} + v_{s,r}) (\bar{v}_{r,s} + \bar{v}_{s,r}) da
\]
\[ \frac{dI}{dx_3}(x_3) = \int_{D(x_3)} \left\{ \omega^2 T_0 \left[ 2\lambda^* \varepsilon_{mm} \varepsilon_{nn} + 4\mu^* \varepsilon_{rs} \varepsilon_{rs} + 2\xi^* \phi \phi \right] \\
+ (b^* + \gamma^*) \left( \varepsilon_{ss} \phi + \varepsilon_{ss} \phi \right) + 2\omega^2 T_0 \alpha^* \phi_r \phi_r \\
+ 2kT_0 \bar{T}_r + i\omega (\zeta + T_0 \tau^*) \left( T_r \bar{T}_r - T_r \phi_r \right) \right\} da. \] (36)

Further, on the basis of the assumptions (15) and (16) we can use the relations (19) and (20) into (36), to obtain

\[ \frac{dI}{dx_3}(x_3) \geq \int_{D(x_3)} \left\{ 2\nu_1 \omega^2 T_0 \left( \varepsilon_{rs} \varepsilon_{rs} + \phi \bar{\phi} \right) \\
+ \int_{D(x_3)} 2\pi_1 T_0 \left( \omega^2 \phi_r \bar{\phi_r} + \frac{1}{T_0^2} T_r \bar{T}_r \right) da, \right\} \] (37)

so that, by means of the identity

\[ \varepsilon_{rs} \varepsilon_{rs} + \frac{1}{4} (v_{r,s} - v_{s,r}) (\bar{v}_{r,s} - \bar{v}_{s,r}) = v_{r,s} \bar{v}_{r,s}, \] (38)

we have

\[ \frac{dI}{dx_3}(x_3) \geq \int_{D(x_3)} \left\{ 2\nu_1 \omega^2 T_0 \left( v_{r,s} \bar{v}_{r,s} + \phi \bar{\phi} \right) \\
+ \int_{D(x_3)} 2\pi_1 T_0 \left( \phi_r \bar{\phi_r} + \frac{1}{T_0^2} T_r \bar{T}_r \right) da. \right\} \] (39)

This last relation proves that \( I(x_3) \) is a non-increasing function with respect to \( x_3 \) on \((0, L)\). On this basis and by taking into account that, in view of the end boundary condition (14), we have \( I(L) = 0 \), it follows that

\[ I(x_3) \geq 0 \quad \text{for all} \quad x_3 \in (0, L). \] (40)

Moreover, by integrating with respect to \( x_3 \) variable upon \((x_3, L)\) and by using the relation \( I(L) = 0 \), from (39) we obtain

\[ I(x_3) \geq \int_{B(x_3)} 2\nu_1 \omega^2 T_0 \left( v_{r,s} \bar{v}_{r,s} + \phi \bar{\phi} \right) dv \\
+ \int_{B(x_3)} 2\pi_1 T_0 \left( \omega^2 \phi_r \bar{\phi_r} + \frac{1}{T_0^2} T_r \bar{T}_r \right) dv, \] (41)

where

\[ B(x_3) = D \times (x_3, L). \] (42)
Thus, the relation (41), in conjunction with the lateral and the end boundary conditions, proves that \( I(x_3) \) is a measure for the amplitude \( \{v_r, \phi, T\} \) of the steady-state vibration.

On the other hand, in view of the lateral boundary conditions (12), we have

\[
\int_{D(x_3)} v_{r,\alpha} \overline{v}_{r,\alpha} \, da \geq \chi_0 \int_{D(x_3)} v_r \overline{v}_r \, da, \tag{43}
\]

and

\[
\int_{D(x_3)} \phi_{\alpha} \overline{\phi}_{\alpha} \, da \geq \chi_0 \int_{D(x_3)} \phi \overline{\phi} \, da, \tag{44}
\]

\[
\int_{D(x_3)} T_{\alpha} \overline{T}_{\alpha} \, da \geq \chi_0 \int_{D(x_3)} T \overline{T} \, da, \tag{45}
\]

where \( \chi_0 \) is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross section \( D \). On this basis and by means of the arithmetic-geometric mean inequality and the Cauchy-Schwarz inequality, it follows from (21) that there exist the computable positive constants \( m_1, m_2, m_3 \) and \( m_4 \) so that

\[
|I(x_3)| \leq m_1 \int_{D(x_3)} v_{r,s} v_{r,s} \, da + m_2 \int_{D(x_3)} \phi \phi \, da + m_3 \int_{D(x_3)} \phi_{r} \overline{\phi}_{r} \, da + m_4 \int_{D(x_3)} T_{r} \overline{T}_{r} \, da. \tag{46}
\]

Consequently, it follows from the relations (39) and (46) the following first-order differential inequality

\[
|I(x_3)| + \frac{1}{\sigma} \frac{dI}{dx_3}(x_3) \leq 0 \quad \text{for all} \quad x_3 \in (0, L), \tag{47}
\]

where

\[
\frac{1}{\sigma} = \max \left( \frac{m_1}{2 \nu_1 \omega^2 T_0}, \frac{m_2}{2 \nu_1 \omega^2 T_0}, \frac{m_3}{2 \pi_1 T_0}, \frac{m_4 T_0}{2 \pi_1} \right). \tag{48}
\]

By integration one obtains the estimate (22) and the proof is complete. \( \square \)

In order to treat the case of a semi-infinite cylinder we introduce the volume energetic measure

\[
E(x_3) = \int_{B(x_3)} \left\{ \omega^2 T_0 \left[ 2 \lambda^* \varepsilon_{mm} \varepsilon_{nn} + 4 \mu^* \varepsilon_{rs} \varepsilon_{rs} + 2 \xi^* \phi \overline{\phi} \right. \right. \\
\left. + (b^* + \gamma^*) \left( \varepsilon_{ss} \phi + \varepsilon_{ss} \overline{\phi} \right) \right. + 2 \lambda \omega^2 T_0 \alpha^* \phi_{r} \overline{\phi}_{r} \\
+ 2 k T_{r} \overline{T}_{r} + i \omega (\zeta + T_0 \tau^*) \left( T_{r} \overline{\phi}_{r} - T_{r} \phi_{r} \right) \left. \right\} \, dv. \tag{49}
\]

Then we have the following Phragmèn-Lindelöf alternative result.

**Theorem 4.2.** Within the context of a semi-infinite cylinder made of a thermoviscoelastic porous material we have the following alternative:

: (a) either the amplitude of the steady-state vibration has a finite volume energetic measure \( E(x_3) \) when the cross-sectional functional \( I(x_3) \) is equal to \( E(x_3) \) and it decays spatially faster than the exponential \( e^{-\alpha x_3} \), or

: (b) the amplitude of the steady-state vibration has an infinite volume energetic measure \( E(x_3) \) and then \(-I(x_3)\) grows spatially faster than the exponential \( e^{\alpha x_3} \).
Proof. We recall first that the relation (39) holds true for a semi-infinite cylinder and hence $I(x_3)$ remains a non-increasing function with respect to $x_3$ on $(0, \infty)$. Consequently, we have the only two possibilities: (i) $I(x_3) \geq 0$ for all $x_3 \in (0, \infty)$, or (ii) there exists the value $x_3^* \in (0, \infty)$ so that $I(x_3) < 0$.

Let us consider the case (i), when the differential inequality (47) becomes

$$\sigma I(x_3) + \frac{dI}{dx_3}(x_3) \leq 0 \quad \text{for all } x_3 \in (0, L),$$

and hence we have

$$0 \leq I(x_3) \leq I(0)e^{-\sigma x_3} \quad \text{for all } x_3 \in [0, \infty).$$

Thus, we have

$$I(\infty) = \lim_{x_3 \to \infty} I(x_3) = 0,$$

and so, by integrating (33) over $(x_3, \infty)$ we obtain that

$$I(x_3) = \mathcal{E}(x_3),$$

and, therefore, $\mathcal{E}(x_3)$ is finite and

$$0 \leq \mathcal{E}(x_3) \leq I(0)e^{-\sigma x_3} \quad \text{for all } x_3 \in [0, \infty).$$

In the case (ii) we can conclude that

$$I(x_3) < 0 \quad \text{for all } x_3 \in (x_3^*, \infty),$$

and then the differential inequality (47) becomes

$$\frac{dI}{dx_3}(x_3) - \sigma I(x_3) \leq 0 \quad \text{for all } x_3 \in (x_3^*, \infty).$$

By integration, from (56) we obtain the following spatial estimate

$$-I(x_3) \geq -I(x_3^*)e^{\sigma(x_3-x_3^*)} > 0, \quad \text{for all } x_3 \in (x_3^*, \infty),$$

and the relations (33) and (49) imply that the volume energetic measure $\mathcal{E}(x_3)$ is infinite. Thus, the proof is complete. \qed

Remark 1. We have established our results under the constitutive profile described by relations (15) and (16). It can be easily seen from our proofs that the above results hold true when we relax the constitutive hypothesis by replacing (15) with

$$\mu^* > 0, \quad \xi^* > 0, \quad \frac{1}{4}(b^* + \gamma^*)^2 < \xi^* \left(\lambda^* + \frac{4}{3}\mu^*\right).$$

It is obvious that the class of thermoviscoelastic porous materials characterized by (15) and (58) includes that described by (15) and (16).

In fact, the hypothesis expressed in (58) implies that

$$2\mu^* v_{s,r} v_{s,r} + 2(\lambda^* + \mu^*) v_{s,s} v_{r,r} + 2\xi^* \phi \bar{\phi} + (b^* + \gamma^*) (v_{s,s} \phi + v_{s,s} \bar{\phi}) \geq 0,$$

and hence from (33) we can deduce the inequality (39) and the proof follows the same line as in the previous theorems.

Remark 2. The class of thermoviscoelastic porous materials for which the established spatial estimates hold true can yet be extended as follows. First, we note
that, by means of the lateral boundary conditions (12) we can write the relation (33) in the form

\[-\frac{dI}{dx_3}(x_3) = \int_{D(x_3)} \left\{ \omega^2 T_0 \left[ 2\mu^* v_{s,r}\overline{v}_{s,r} + 2(\lambda^* + \mu^*) v_{s,r}\overline{v}_{r,s} + 2\xi^* \phi \overline{\phi} \right. \\
+ \left. (b^* + \gamma^*) (v_{s,s}\phi + v_{s,s}\phi) \right] + 2\omega^2 T_0 \alpha^* \phi,\overline{\phi} \right\}\, da.\]  

(60)

Now, we replace the constitutive hypothesis described by (58) by the following one

\[\mu^* > 0, \quad -2\mu^* < \lambda^* < 0, \quad (b^* + \gamma^*)^2 < 4\xi^* (\lambda^* + 2\mu^*),\]  

(61)

so that, from (60) we can deduce again the relation (39) and the proof continues as in the above two theorems.

We can conclude here that the results established in the present paper relating the spatial behavior of the amplitude of the harmonic in time vibrations refer to the whole class of thermoviscoelastic porous materials whose viscous dissipation energy satisfies a strong ellipticity condition (see e.g. Chiriță and Ghiba [5] for strongly elastic materials with voids).

REFERENCES


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