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ON THE SPATIAL BEHAVIOR OF THE STEADY-STATE VIBRATIONS IN THERMOVISCOELASTIC POROUS MATERIALS

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This paper reviews the spatial behavior of the amplitude of the steady-state vibrations in an anisotropic and homogeneous thermoviscoelastic porous beam. Here, we take into account the effects of the viscoelastic and thermal dissipation energies upon the corresponding harmonic vibrations in a right cylinder made of an anisotropic thermoviscoelastic porous material. In fact, we prove that the positiveness of the viscoelastic and thermal dissipation energies is sufficient for characterizing the spatial decay and growth properties of the harmonic vibrations in a cylinder, without any limiting restriction upon their frequencies.

Keywords: Growth and decay spatial estimates; Steady-state vibrations; Thermoviscoelastic porous bodies

INTRODUCTION

Ieșan [1] developed a linear theory of thermoviscoelastic porous materials, in which the set of independent constitutive variables includes the time derivative of the strain tensor, the time derivative of the volume fraction field and the time derivative of the gradient of the volume fraction field. The theory represents an extension of the Cowin and Nunziato’s theory of elastic materials with voids [2] and that of Ieșan’s theory of thermoelastic materials with voids [3] by incorporating the memory effects.

Since the porous materials have many potential uses this has inspired much work into dynamical problems, see, e.g., Ieșan [4], Ciarletta and Scalia [5], Chiriță and Scalia [6], Scalia et al. [7], Ciarletta and Straughan [8], Ciarletta et al. [9] and into plane strain problems, see, e.g., Ieșan and Nappa [10], D’Apice and Chiriță [11, 12]. For a review of the literature on porous elastic materials the reader is referred to the books by Straughan [13] and Ieșan [14].

Recently, Sharma and Kumar [15] and Tomar et al. [16] studied the time harmonic steady-state vibrations in an infinite thermoviscoelastic material with voids. It is shown that there are four basic waves traveling with distinct speeds, out of which one is a shear wave and the other three are dilatational waves. Svanadze
[17, 18] obtains the fundamental solution of the system of equations of steady-state vibrations and furnishes a representation of Galerkin-type of solution. The completeness of such representations of solutions is proved.

Within the context of the thermoviscoelastic model of materials with voids, in the present paper we take into consideration the dissipation mechanism due to the thermal and the memory effects to study the spatial behavior of the steady-state solutions. On this basis we establish the spatial decay and growth properties of the harmonic in time solutions in a finite as well as in a semi-infinite cylinder made of an anisotropic thermoviscoelastic porous material. The presence of the thermal and memory dissipation energies allows us to establish the spatial behavior results for the entire class of thermoviscoelastic porous materials compatible with thermodynamics and for every value of the frequency of vibrations, without any positive definiteness upon the elastic coefficients. This is in essential contrast with the studies made in the classical elasticity by Flavin and Knops [19] and in the classical theory of thermoelasticity by Chiriţă [20], where the estimates describing the spatial behavior are established in the class of the steady-state solutions whose frequencies are lower than a certain critical frequency, provided the assumptions are made relating the positive definiteness of the elastic coefficients. It becomes clear from the studies existing at this time in literature (see e.g., [20, 21]) that the thermal dissipation only does not work for establishing the spatial behavior for any frequency of the classical thermoelastic harmonic vibrations.

On the other hand, it was shown in [22–24] that the viscoelastic dissipation mechanism can furnish the advantage to establish the spatial behavior for the entire class of steady-state solutions, without any constraint upon their frequencies.

To establish the spatial behavior of the steady-state vibrations within the context of the linear theory of thermoviscoelastic porous materials, we introduce a natural functional and show that this is a measure of the amplitude of vibrations that exponentially decays or grows with respect to the distance to the loaded end of cylinder.

Thus, we may conclude that the thermal dissipation effect in conjunction with the memory dissipation mechanism are able to assure a complete study of the spatial decay and growth properties of the steady-state vibrations.

BASIC EQUATIONS

Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes $Ox_k, (k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, whereas Greek subscripts are confined to the range $(1, 2)$, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation. Throughout this section we suppose that a regular region $B$ is filled by a homogeneous and anisotropic thermoviscoelastic material. Considering the linear theory of thermoviscoelastic materials with voids and assuming that the initial body is free from stresses and has zero intrinsic equilibrated body force and entropy, the system of field equations consists of (c.f., Ieşan [1])
• the equations of motion

\[
\begin{align*}
t_{rs} + \varrho f_r &= \varrho \ddot{u}_r, \\
H_{r,s} + g + \varrho \dot{\mathcal{L}} &= \varrho \kappa \dot{\varphi}, \quad \text{in } B \times (0, \infty)
\end{align*}
\]  

(1)

• the equation of energy

\[
\varrho T_0 \dot{\eta} = Q_r + \varrho \lambda, \quad \text{in } B \times (0, \infty)
\]  

(2)

• the constitutive equations

\[
\begin{align*}
t_{rs} &= C_{rs pq} \varepsilon_{pq} + B_{rs} \varphi + D_{rs pq} \dot{\varphi}_p - \beta_{rs} \theta + S^*_{rs} \\
H_r &= A_{rs} \varphi_r + D_{pqrs} \varepsilon_{pq} + d_{rs} \varphi - a_{rs} \theta + H^* r \\
g &= -B_{rs} \varepsilon_{rs} - \zeta \dot{\varphi} - d_{rs} \varphi + m \theta + g^* \\
\varrho \eta &= \beta_{rs} \varepsilon_{rs} + a \theta + m \varphi + a_{rs} \varphi_r \\
Q_r &= k_{rs} \theta_{rs} + f_{pq} \dot{\varepsilon}_{pq} + b_{rs} \dot{\varphi} + a_{rs} \dot{\varphi}_r, \quad \text{in } \overline{B} \times [0, \infty)
\end{align*}
\]  

(3)

with

\[
\begin{align*}
S^*_{rs} &= C^*_{rs pq} \varepsilon_{pq} + B^*_{rs} \dot{\varphi} + D^*_{rs pq} \dot{\varphi}_p + M^*_{rs qr} \theta_p \\
H^* r &= A^*_{rs} \varphi_r + G^*_{pqrs} \varepsilon_{pq} + d^*_{rs} \dot{\varphi} + P^*_{rs} \theta_{rs} \\
g^* &= -F^*_{rs} \varepsilon_{rs} - \zeta^* \dot{\varphi} - \gamma^* \dot{\varphi}_r - K^* \theta_{rs}
\end{align*}
\]  

(4)

and

• the geometrical relations

\[
\begin{align*}
e_{rs} &= \frac{1}{2} (u_{r,s} + u_{s,r}), \quad \text{in } \overline{B} \times [0, \infty)
\end{align*}
\]  

(5)

Here, we have used the following notations: \( t_{rs} \) are the components of the stress tensor, \( H_{r,s} \) are the components of the equilibrated stress vector, \( g \) is the equilibrated body force, \( \eta \) is the entropy per unit mass, \( Q_r \) are the components of the heat flux vector, \( \varepsilon_{rs} \) are the components of the strain tensor, \( \varrho \) is the mass density of the medium, \( \kappa \) is the equilibrated inertia, \( \varphi \) is the void volume fraction, \( u_r \) are the components of the displacement vector, \( \varphi_r \) is the void volume fraction, \( \theta \) is the change in temperature from the constant ambient temperature \( T_0 > 0 \), \( f_r \) are the components of the body force per unit mass, \( \mathcal{L} \) is the extrinsic equilibrated body force per unit mass, \( s \) is the heat supply per unit mass. The constitutive coefficients have the following symmetries:

\[
\begin{align*}
C_{rs pq} &= C_{pqrs} = C_{sr pq}, \quad \beta_{rs} = \beta_{sr}, \quad D_{pqrs} = D_{qprs} \\
A_{rs} &= A_{sr}, \quad B_{rs} = B_{sr} \\
C^*_{rs pq} &= C^*_{pqrs} = C^*_{sr pq}, \quad B^*_{rs} = B^*_{sr}, \quad D^*_{pqrs} = D^*_{qprs} \\
M^*_{pqrs} &= M^*_{qprs}, \quad A^*_{rs} = A^*_{sr}, \quad G^*_{pqrs} = G^*_{sr pq}, \quad P^*_{rs} = P^*_{sr} \\
F^*_{rs} &= F^*_{sr}, \quad k_{rs} = k_{sr}, \quad f_{pq rs} = f_{pq sr}, \quad a_{rs} = a_{sr}
\end{align*}
\]  

(6)

(7)
Furthermore, in view of the second law of thermodynamics, the Clausius–Duhem inequality must be satisfied, which provides the positive semi-definiteness of the total dissipation energy $\Lambda$, that is,

$$\Lambda \left( \dot{e}_{rs}, \dot{\varphi}, \dot{\theta}, \varphi, \theta \right) \geq 0$$  \hspace{1cm} (8)

where

$$\Lambda \left( \dot{e}_{rs}, \dot{\varphi}, \dot{\theta}, \varphi, \theta \right) = C_{pqrs}^* \dot{e}_{pq} \dot{e}_{rs} + A_{qrs}^* \dot{\varphi} \dot{\varphi} + \dot{\theta}^2 + \frac{1}{T_0} k_{rs} \theta \theta + (B_{rs}^* + F_{rs}^*) \dot{\varphi} \dot{\varphi} + (D_{pqr}^* + G_{pqr}^*) \dot{e}_{pq} \dot{\varphi} \dot{\varphi}$$

$$+ \left( M_{pqrs}^* + \frac{1}{T_0 f_{pq}} \right) \dot{e}_{pq} \theta \theta + (d_{rs}^* + \gamma_{rs}^*) \dot{\varphi} \dot{\varphi}$$

$$+ \left( R_{rs}^* + \frac{1}{T_0 b_{rs}} \right) \dot{\varphi} \theta \theta + \left( P_{rs}^* + \frac{1}{T_0 a_{rs}} \right) \dot{\varphi} \theta \theta$$  \hspace{1cm} (9)

**FORMULATION OF THE PROBLEM**

In this section we assume that $B$ is the interior of a right cylinder of length $L > 0$ whose cross-section is bounded by one or more piecewise smooth curves. We assume the Cartesian coordinates such that the origin lies in one end of the cylinder and such that the $x_3$-axis is parallel to the generators. Let $D(x_3)$ denote the cross-section of the cylinder corresponding to the axial distance $x_3$, and let $\partial D(x_3)$ denote the cross-sectional boundary. We denote by $\pi$ the lateral surface of the cylinder, so that $\pi = \partial D \times (0, L)$ and $B = D \times (0, L)$.

For the steady-state time harmonic solutions in the cylinder $B = \Sigma \times (0, L)$, we consider

$$u_r \left( x, t \right) = u_r \left( x \right) e^{-i\omega t}$$

$$\varphi \left( x, t \right) = \phi \left( x \right) e^{-i\omega t}$$

$$\theta \left( x, t \right) = T \left( x \right) e^{-i\omega t}$$  \hspace{1cm} (10)

where $\omega$ is a prescribed positive frequency and $i$ is the unit complex, that is $i = \sqrt{-1}$. Inserting $u_r$, $\varphi$, and $\theta$ from (10) into Eqs. (1)–(5), and assuming zero body forces and heat supply, we obtain for the amplitude $\{u_r, \phi, T\}$ of the oscillation the following system of differential equations:

$$s_{rs} + \rho \omega^2 v_s = 0$$

$$h_{r,t} + G + \rho k \omega^2 \phi = 0$$

$$- i\omega T_0 \phi N = q_{r,t}, \text{ in } B = D \times (0, L)$$  \hspace{1cm} (11)

and

$$s_r = C_{pqrs}^* \epsilon_{pq} + B_{pq}^0 \phi + D_{pqrs}^* \phi, \phi_{,p} - \beta_{rs} T + M_{pqrs}^* T_{,p}$$

$$h_r = A_{pqrs}^0 \phi, \phi_{,s} + G_{pqrs}^0 \epsilon_{pq} + d_{pqrs}^0 \phi - a_{rs} T + P_{pqrs} T_{,s}$$
\begin{equation}
G = -F^0_{rs} e_{rs} - \zeta^0 \phi - \gamma^0_{,r} \phi_{,r} + mT - R^*_s T, \tag{12}
\end{equation}

\begin{align*}
\varrho N &= \beta_{rs} e_{rs} + aT + m\phi + a_r \phi_{,r}, \\
q_r &= k_r T_{,s} - i\omega \left( f_{pq} e_{pq} + b_r \phi + a_n \phi_{,r} \right)
\end{align*}

and

\begin{equation}
\varepsilon_{rs} = \frac{1}{2} (v_{r,s} + v_{s,r}) \tag{13}
\end{equation}

in \( D \times [0, L] \). In the preceding relations we have used the following notations:

\begin{align*}
C^0_{rspq} &= C_{rspq} - i\omega C^*_{rspq}, & B^0_r &= B_r - i\omega B^*_r, \\
D^0_{rsp} &= D_{rsp} - i\omega D^*_{rsp}, & A^0_r &= A_r - i\omega A^*_r, \\
G^0_{pq} &= D_{pq} - i\omega G^*_{pq}, & d^0_r &= d_r - i\omega d^*_r, \\
F^0_{rs} &= B^*_r - i\omega F^*_r, & \zeta^0 &= \zeta - i\omega \zeta^*, & \gamma^0_r &= d_r - i\omega \gamma^*_r
\end{align*}

\tag{14}

We further assume that the lateral boundary surface of the cylinder is fixed and thermal insulated, that is, we suppose the following lateral boundary conditions:

\begin{equation}
v_r (x) = 0, \quad \phi (x) = 0, \quad T (x) = 0 \quad \text{on} \quad \pi \tag{15}
\end{equation}

Moreover, we assume the following end boundary conditions:

\begin{align*}
v_r (x_1, x_2, 0) &= \tilde{v}_r (x_1, x_2), & \phi (x_1, x_2, 0) &= \tilde{\phi} (x_1, x_2) \\
T (x_1, x_2, 0) &= \tilde{T} (x_1, x_2), & \text{for all} \quad (x_1, x_2) \in D(0)
\end{align*}

\tag{16}

and

\begin{align*}
v_r (x_1, x_2, L) &= 0, & \phi (x_1, x_2, L) &= 0 \\
T (x_1, x_2, L) &= 0, & \text{for all} \quad (x_1, x_2) \in D(L)
\end{align*}

\tag{17}

where \( \tilde{v}_r (x_1, x_2), \tilde{\phi} (x_1, x_2) \) and \( \tilde{T} (x_1, x_2) \) are prescribed functions on \( D(0) \). We have to note that the end condition (17) has to be avoided when the case of a semi-infinite cylinder is to be considered.

Throughout this article we consider the boundary value problem \( \mathcal{P} \) defined by the differential system (11) to (13), the lateral boundary conditions (15) and the end boundary conditions (16) and (17), whose solution \( \{v_r, \phi, T\} \) is the amplitude of oscillation (10). We are interested in describing the spatial decay and growth properties of the amplitude \( \{v_r, \phi, T\} \) for both finite and semi-infinite cylinders made of an anisotropic thermoviscoelastic porous material.
CONSTITUTIVE HYPOTHESES: SOME AUXILIARY ESTIMATES

Throughout this paper we will assume the dissipation energy \(\Lambda(\zeta_{rs}, \sigma, v_p, \tau_q)\) to be a positive definite form in terms of \(\zeta_{rs}, \sigma, v_p\) and \(\tau_q\). This means that there exist the strictly positive constants \(\mu_m, a_m, \xi_m, k_m\) and \(\mu_M, a_M, \xi_M, k_M\) so that

\[
\mu_m \zeta_{rs} \zeta_{rs} + a_m \sigma^2 + \xi_m v_p v_p + k_m \tau_q \tau_q \leq \Lambda(\zeta_{rs}, \sigma, v_p, \tau_q)
\]

\[
\leq \mu_M \zeta_{rs} \zeta_{rs} + a_M \sigma^2 + \xi_M v_p v_p + k_M \tau_q \tau_q
\]

for all real \(\zeta_{rs} = \zeta_{sr}\), \(\sigma, v_p\) and \(\tau_q\). It follows then that \(C^*_{pqrs}\) is a positive definite tensor, and hence, we will have

\[
\mu^*_m \zeta_{rs} \zeta_{rs} \leq C^*_{pqrs} \zeta_{pq} \zeta_{rs} \leq \mu^*_M \zeta_{rs} \zeta_{rs}, \quad \text{for all real } \zeta_{rs} = \zeta_{sr}
\]

where \(\mu_m\) and \(\mu_M\) are the minimum and maximum elastic moduli for the elasticity tensor \(C^*_{pqrs}\) (see, e.g., Gurtin [25]).

We proceed now to establish some useful estimations. To this end, in what follows, we will use a superposed bar for denoting the complex conjugate of a quantity. Thus, we use the relation (12) and the Cauchy–Schwarz inequality to obtain

\[
s_{\nu s} s_{\nu s} = C^0_{3pq} e_{pq} s_{\nu s} + B^0_{3s} \phi s_{\nu s} + D^0_{3p} \phi_s s_{\nu s} - \beta \bar{s}_s T \bar{s}_s + M^*_{3p} T \bar{s}_s
\]

\[
\leq \left( C^0_{3pq} \right)^{1/2} \left( e_{kl} \bar{e}_{kl} \right)^{1/2} \left( s_{\nu s} \bar{s}_{\nu s} \right)^{1/2} + \left( B^0_{3s} \bar{B}^0_{3s} \right)^{1/2} \left( \phi \bar{\phi} \right)^{1/2} \left( s_{\nu s} \bar{s}_{\nu s} \right)^{1/2}
\]

\[
+ \left( D^0_{3p} \bar{D}^0_{3p} \right)^{1/2} \left( \phi_s \bar{\phi}_s \right)^{1/2} \left( s_{\nu s} \bar{s}_{\nu s} \right)^{1/2} + \left( \beta \bar{s}_s T \bar{s}_s \right)^{1/2} \left( s_{\nu s} \bar{s}_{\nu s} \right)^{1/2}
\]

\[
+ \left( M^*_{3p} \bar{M}^*_{3p} \right)^{1/2} \left( T \bar{T} \right)^{1/2} \left( s_{\nu s} \bar{s}_{\nu s} \right)^{1/2}
\]

and hence, we deduce the following estimate:

\[
(s_{\nu s} \bar{s}_{\nu s})^{1/2} \leq \beta \left( T \bar{T} \right)^{1/2} + \sigma_1 \Lambda^0_1 (e_{\nu s}, \phi, \phi_s, T_q)
\]

where

\[
\Lambda_1 (e_{\nu s}, \phi, \phi_s, T_q) = \mu_m e_{\nu s} \bar{e}_{\nu s} + a_M \phi \bar{\phi} + \xi_M \phi_s \bar{\phi}_s + k_M T_q \bar{T}_q
\]

and

\[
\beta = (\beta_3 \beta_s)^{1/2}
\]

\[
\sigma_1 = \left( \frac{1}{\mu_M} C^0_{3pq} \bar{C}^0_{3pq} \right)^{1/2} + \left( \frac{1}{a_M} B^0_{3s} \bar{B}^0_{3s} \right)^{1/2}
\]

\[
+ \left( \frac{1}{\xi_M} D^0_{3p} \bar{D}^0_{3p} \right)^{1/2} + \left( \frac{1}{k_M} M^*_{3pq} \bar{M}^*_{3pq} \right)^{1/2}
\]
In a similar way, we have

\[ h_3 \bar{h}_3 = A_{3i}^0 \phi_3 \bar{h}_3 + G_{pq3}^0 e_{pq} \bar{h}_3 + d_{3i}^0 \phi \bar{h}_3 - a_3 T \bar{h}_3 + P_{3i}^* T_{,i} \bar{h}_3 \]

\[ \leq \left[ \left( A_{3i}^0 \bar{A}_{3i}^0 \right)^{1/2} \left( \phi_3 \bar{\phi}_3 \right)^{1/2} + \left( G_{pq3}^0 \bar{G}_{pq3}^0 \right)^{1/2} \left( e_{rs} \bar{e}_{rs} \right)^{1/2} \right. \]

\[ + \left. \left( d_{3i}^0 \bar{d}_{3i}^0 \right)^{1/2} \left( \phi \bar{\phi} \right)^{1/2} + |a_3| \left( T T \right)^{1/2} + \left( P_{3i}^* P_{3i}^* \right)^{1/2} \left( T_{,i} T_{,i} \right)^{1/2} \right] (h_3 \bar{h}_3)^{1/2} \] (24)

and hence

\[ (h_3 \bar{h}_3)^{1/2} \leq |a_3| (T T)^{1/2} + \sigma_2 \Lambda_0^{1/2} (e_{rs}, \phi, \phi_{,r}, T_{,q}) \] (25)

where

\[ \sigma_2 = \left( \frac{1}{\mu_M} G_{pq3}^0 \bar{G}_{pq3}^0 \right)^{1/2} + \left( \frac{1}{a_3} d_{3i}^0 \bar{d}_{3i}^0 \right)^{1/2} + \left( \frac{1}{\sigma_M} A_{3i}^0 \bar{A}_{3i}^0 \right)^{1/2} + \left( \frac{1}{k_M} P_{3i}^* P_{3i}^* \right)^{1/2} \] (26)

Finally, we have

\[ q_3 \bar{q}_3 = k_3 T_{,r} \bar{q}_3 - i \omega f_{3pq} e_{pq} \bar{q}_3 - i \omega b_3 \phi \bar{q}_3 - i \omega a_3 \phi_{,r} \bar{q}_3 \]

\[ \leq \left[ (k_3 k_3)^{1/2} (T_{,r} T_{,r})^{1/2} + \omega \left[ \left( f_{3pq} f_{3pq} \right)^{1/2} \left( e_{rs} \bar{e}_{rs} \right)^{1/2} + |b_3| \left( \phi \bar{\phi} \right)^{1/2} \right. \right. \]

\[ + \left. \left. \left( a_3 a_3 \right)^{1/2} \left( \phi_{,r} \phi_{,r} \right)^{1/2} \right] (q_3 \bar{q}_3)^{1/2} \right] \] (27)

so that we obtain

\[ (q_3 \bar{q}_3)^{1/2} \leq \sigma_3 \Lambda_0^{1/2} (e_{rs}, \phi, \phi_{,r}, T_{,q}) \] (28)

where

\[ \sigma_3 = \left( \frac{1}{k_M} k_3 k_3 \right)^{1/2} + \omega \left[ \left( \frac{1}{\mu_M} f_{3pq} f_{3pq} \right)^{1/2} + \frac{|b_3|}{\sqrt{a_M}} + \left( \frac{1}{\sigma_M} a_3 a_3 \right)^{1/2} \right] \] (29)

**SPATIAL BEHAVIOR**

Here, we study the spatial behavior of the amplitude \{v_r, \phi, T\} of the considered vibrations under the assumption that the dissipation energy is a positive definite form. To this end we associate with the amplitude \{v_r, \phi, T\} of the steady-state thermoviscoelastic vibration, the cross-sectional functional

\[ I(x_3) = \int_{D(x_3)} \left[ i \omega T_{,0} (v_\bar{s}_{,3} - \bar{v}_{,3} s_3) + i \omega T_{,0} (\phi \bar{h}_3 - \bar{\phi} h_3) - T q_3 - T \bar{q}_3 \right] da \] (30)

where the superposed bar denotes the complex conjugate.

In what follows we establish some useful properties of the cross-sectional functional \( I(x_3) \). On this basis we then establish the spatial behavior of the amplitude \{v_r, \phi, T\} of the considered vibrations. We start our analysis with the following result:
Theorem 1. Consider a finite cylinder made of a thermoviscoelastic porous material and subjected to zero body loads and the boundary loads as specified previously. Then the cross-sectional functional $I(x_3)$ defined by (30) represents an acceptable measure of the amplitude $\{v_3, \phi, T\}$, in the sense that it is positive for all $\{v_3, \phi, T\}$ and it vanishes only when $\{v_3, \phi, T\} = 0$. Moreover, we have

$$-rac{dI}{dx_3}(x_3) \geq 2T_0 \int_{D(x_3)} \left\{ \omega^2 \left( \mu_n e_{xx} \tilde{e}_{xx} + a_m \tilde{\phi} + \epsilon_m \phi,_{,p} \tilde{\phi},_p \right) + k_m T_p \tilde{T}_p \right\} da \geq 0 \quad \text{for all } x_3 \in [0, L] \quad (31)$$

Proof. First of all we note that, by a direct differentiation with respect to $x_3$ of Eq. (30), we have

$$\frac{dI}{dx_3}(x_3) = \int_{D(x_3)} \left\{ i\omega T_0 \left( v_{3r} \tilde{\phi}_{3r} - \tilde{v}_{3r} \tilde{\phi}_{3r} \right) + i\omega T_0 \left( \phi_{3r} \tilde{h}_{3r} - \tilde{\phi}_{3r} \tilde{h}_{3r} \right) - T_3 q_3 - T_3 \tilde{q}_3 \right\} da + \int_{D(x_3)} \left\{ i\omega T_0 \left( v_{3r} \tilde{\phi}_{3r} - \tilde{v}_{3r} \tilde{\phi}_{3r} \right) + i\omega T_0 \left( \phi_{3r} \tilde{h}_{3r} - \tilde{\phi}_{3r} \tilde{h}_{3r} \right) - T_3 q_3 - T_3 \tilde{q}_3 \right\} da \quad (32)$$

and hence, by means of the relation (11) and by using the lateral boundary condition (15), we obtain

$$\frac{dI}{dx_3}(x_3) = \int_{D(x_3)} \left[ i\omega T_0 \left( v_{3r} \tilde{\phi}_{3r} - \tilde{v}_{3r} \tilde{\phi}_{3r} \right) + i\omega T_0 \left( \phi_{3r} \tilde{h}_{3r} - \tilde{\phi}_{3r} \tilde{h}_{3r} \right) - \tilde{T}_r q_r - T_r \tilde{q}_r + i\omega T_0 \left( \tilde{T}_0 N - T_0 \tilde{N} \right) \right] da \quad (33)$$

Further, by using the symmetry relations (6) and (7) and by substituting relations (12) to (14) into (33), we obtain

$$\frac{dI}{dx_3}(x_3) = - \int_{D(x_3)} \left\{ T_0 \omega^2 \left[ 2C_{rep} e_{rs} \tilde{e}_{rs} + 2A_{rep} \phi_r \tilde{\phi},_r + 2 \epsilon_{rs} \phi \tilde{\phi} \right. \right.$$  

$$\left. + \left(B^*_{rep} + F^*_{rep} \right) \left( e_{rs} \tilde{\phi} + \epsilon_{rs} \phi \right) + \left(D^*_{rep} + G^*_{rep} \right) \left( e_{rs} \tilde{\phi},_p + \epsilon_{rs} \phi,_p \right) \right. \right.$$  

$$\left. + \left( d^*_r + \gamma_r^* \right) \left( \phi_r \tilde{\phi} + \tilde{\phi}_r \phi \right) \right] + 2k_{rs} T_r \tilde{T},_s$$  

$$+ i\omega T_0 \left[ \left( M^*_{rep} + \frac{1}{T_0} f_{pr} \right) \left( T_{p \tilde{r}} e_{rs} - e_{rs} \tilde{T}_{p \tilde{r}} \right) \right.$$

$$\left. + \left( P^*_{pr} + \frac{1}{T_0} a_{sr} \right) \left( T_{,r} \tilde{\phi},_r - \tilde{T}_{,r} \tilde{\phi},_r \right) \right. \right.$$  

$$\left. + \left( R^*_{pr} + \frac{1}{T_0} h_r \right) \left( T_{,r} \tilde{\phi} - \tilde{T}_{,r} \tilde{\phi} \right) \right\] \right\} da \quad (34)$$

In view of the assumption (18), from (34) we deduce the inequality (31) and this last relation proves that $I(x_3)$ is a non-increasing function with respect to $x_3$ on $(0, L)$. On this basis and by taking into account that, in view of the end boundary condition (17), we have $I(L) = 0$, it follows that

$$I(x_3) \geq 0 \quad \text{for all } x_3 \in (0, L) \quad (35)$$
Moreover, by integrating with respect to $x_3$ variable upon $(x_3, L)$ and by using the relation $I(L) = 0$, from (31), we obtain

$$I(x_3) \geq 2T_0 \int_{D(x_3)} \left\{ \omega^2 (\epsilon_m \epsilon_{rs} \tilde{\varepsilon}_{rs} + a_m \phi \phi + \tilde{\varepsilon}_{rs} \phi_{r,p} \phi_{,r}) + k_m T_p T_{,p} \right\} dv \geq 0 \quad \text{for all} \quad x_3 \in [0, L]$$

(36)

where

$$B(x_3) = D \times (x_3, L)$$

(37)

Thus, the relation (36), in conjunction with the lateral and the end boundary conditions, proves that $I(x_3)$ is a measure for the amplitude $\{v_r, \phi, T\}$ of the steady-state vibration.

Next we proceed to obtain an upper bound for the cross-sectional measure $I(x_3)$.

**Theorem 2.** Within the context of the considered thermoviscoelastic porous cylinder with a positive definite dissipation energy, for any positive constant $\varepsilon$ there exists the positive constant $\sigma(\varepsilon)$ so that the positive cross-sectional measure $I(x_3)$ defined by (30) satisfies the following estimate:

$$I(x_3) \leq \frac{T_0}{2} \int_{D(x_3)} \left\{ \omega^2 v_r \tilde{\varepsilon}_r + \sigma(\varepsilon) \Lambda_0 \left( \epsilon_{rs}, \phi, \phi_{,r}, T_{,p} \right) \right\} da, \quad \text{for all} \quad x_3 \in [0, L]$$

(38)

**Proof.** First of all, we note that by means of the lateral boundary condition (15) we can write

$$\lambda \int_{D(x_3)} T T da \leq \int_{D(x_3)} T_{,p} T_{,p} da$$

(39)

where $\lambda$ is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross-section $D(x_3)$.

On the other hand, by using the Cauchy–Schwarz and arithmetic-geometric mean inequalities into relation (30) we deduce

$$I(x_3) \leq \frac{T_0}{2} \int_{D(x_3)} \left[ \frac{\epsilon_1}{\epsilon_2} (\omega^2 v_r \tilde{\varepsilon}_r) + \frac{\epsilon_1}{\epsilon_2} (\omega^2 \phi \phi) + \frac{\epsilon_3}{T_0} (T T) \right]$$

$$+ \frac{1}{\epsilon_1} (s_3 \tilde{s}_3) + \frac{1}{\epsilon_2} (h_3 \tilde{h}_3) + \frac{1}{T_0 \epsilon_3} (q_3 \tilde{q}_3) \right\} da$$

(40)

for all positive parameters $\epsilon_1, \epsilon_2, \epsilon_3$. In view of the estimates (21), (25), and (28), we have

$$I(x_3) \leq \frac{T_0}{2} \int_{D(x_3)} \left\{ \frac{\epsilon_1}{\epsilon_2} (\omega^2 v_r \tilde{\varepsilon}_r) + \frac{\epsilon_1}{\epsilon_2} (\omega^2 \phi \phi) + \left[ \frac{\epsilon_3}{T_0} + \frac{1 + \epsilon_4}{\epsilon_1} (\beta_{rs} \beta_{rs}) + \frac{1 + \epsilon_3}{\epsilon_2} \right] (T T) \right\}$$

$$+ \left[ \frac{\sigma_1^2}{\epsilon_1} (1 + \frac{1}{\epsilon_4}) + \frac{\sigma_2^2}{\epsilon_2} (1 + \frac{1}{\epsilon_5}) + \frac{\sigma_3^2}{T_0 \epsilon_3} \right] \Lambda_0 \left( \epsilon_{rs}, \phi, \phi_{,r}, T_{,p} \right) \right\} da$$

(41)
for all positive parameters $\varepsilon_4$ and $\varepsilon_5$. Further, we use the inequality (39) into (41) in order to obtain the relation (38) with $\varepsilon = \varepsilon_1$ and

$$
\sigma(\varepsilon) = \frac{\varepsilon_2 \omega^2}{\alpha_0} + \frac{1}{2k_M} \left[ \frac{\varepsilon_3}{T_0} + \frac{1}{\varepsilon_1} \left( \beta_\alpha \beta_\alpha + \frac{1}{\varepsilon_2} \right) \right] \cdot \frac{1}{1 + \frac{1}{\varepsilon_4}} \cdot \frac{1}{1 + \frac{1}{\varepsilon_5}} + \frac{\sigma_2 L}{\varepsilon_1} \left( 1 + \frac{1}{\varepsilon_4} \right) + \frac{\sigma_3 L}{\varepsilon_2} \left( 1 + \frac{1}{\varepsilon_5} \right) + \frac{\sigma_3 S}{\varepsilon_3}
$$

(42)

Thus, the proof is complete.

Finally, we have the following result describing the spatial behavior of the amplitude of vibrations in concern.

**Theorem 3.** The cross-sectional functional $I(x_3)$, associated with the amplitude $\{v_r, \phi, T\}$ by (30), satisfies the following spatial estimate:

$$0 \leq I(x_3) \leq I(0) \exp \left( -\frac{x_3 - \ell}{\tau_m(\ell, \omega)} \right), \quad \text{for all } x_3 \in [\ell, L]
$$

(43)

where $\ell$ is a value fixed in $(0, \frac{L}{2})$ and $\tau_m(\ell, \omega)$ is an appropriate computable constant.

**Proof.** We first introduce the cylinder slice $B(x_3, \ell) = D \times [x_3, x_3 + \ell]$, $0 \leq x_3 < x_3 + \ell \leq L$. Then, in view of the boundary condition (15) and by using the inequality (19), we have (see, e.g., Toupin [26] and Gurtin [25])

$$
\int_{B(x_3, \ell)} C_{pqrs}^* e_{pq} \tilde{e}_{rs} dv \geq q u_{0}^2(\ell) \int_{B(x_3, \ell)} v_{i} \tilde{v}_{i} dv
$$

(44)

where $u_{0}(\ell)/(2\pi)$ is the lowest frequency of vibration of the cylinder $B(x_3, \ell)$ filled by an elastic material whose components of the constant elasticity tensor are $C_{pqrs}^*$ and whose lateral surface is clamped and plane ends are free.

Next, we choose $\ell \in (0, \frac{L}{2})$ and we let

$$
Q(x_3, \ell) = \frac{1}{\ell} \int_{x_3}^{x_3 + \ell} I(s) ds, \quad x_3 \in [0, L - \ell]
$$

(45)

and observe that

$$
\frac{\partial Q}{\partial x_3}(x_3, \ell) = \frac{1}{\ell} [I(x_3 + \ell) - I(x_3)], \quad x_3 \in (0, L - \ell)
$$

(46)

and

$$
I(x_3 + \ell) \leq Q(x_3, \ell) \leq I(x_3), \quad x_3 \in [0, L - \ell]
$$

(47)

Further, we integrate the inequality (31) with respect to $x_3$ variable on the interval $[x_3, x_3 + \ell]$ and we use relation (46) to obtain

$$
\frac{\partial Q}{\partial x_3}(x_3, \ell) \leq -\frac{2T_0}{\ell} \int_{B(x_3, \ell)} \left\{ \omega^2 \left( \mu_m e_{rs} \tilde{e}_{rs} + a_m \phi \tilde{\phi} + \xi_m \phi \tilde{\phi} \right) + k_m T_p \tilde{T}_p \right\} dv \leq 0 \quad \text{for all } x_3 \in (0, L - \ell)
$$

(48)
and hence
\[
\frac{\partial Q}{\partial x_3}(x_3, \ell) \leq -\frac{2T_0}{\ell} \sum \int_{B(\ell, x_3)} \Lambda_0 (e_{ir}, \phi, \phi_p, T_q) \, dv, \quad \text{for all } x_3 \in (0, L - \ell) \tag{49}
\]
where
\[
\Sigma_1 = \min \left( \frac{\omega^2 \mu_m}{\mu M}, \frac{\omega^2 a_m}{a M}, \frac{\omega^2 \varepsilon_m}{\varepsilon M}, \frac{k_m}{k M} \right) \tag{50}
\]

On the other hand, from relations (19), (22), (38), (44), and (45), we get
\[
Q(x_3, \ell) = \frac{1}{\ell} \int_{x_3}^{x_3+\ell} \lambda(s) \, ds \leq \frac{T_0}{2\ell} \int_{B(\ell, x_3)} \left[ e\omega^2 v \Phi_v + \sigma(\varepsilon) \Lambda_0 (e_{ir}, \phi, \phi_p, T_q) \right] \, dv \\
\leq \frac{T_0}{2\ell} \int_{B(\ell, x_3)} \left[ \frac{e\omega^2}{qu_0^2(\ell)} C_{pqr} e_{pq} \Phi_{rs} + \sigma(\varepsilon) \Lambda_0 (e_{ir}, \phi, \phi_p, T_q) \right] \, dv \\
\leq \frac{T_0}{2\ell} \int_{B(\ell, x_3)} \Sigma_2 (\varepsilon, x_3, \ell) \Lambda_0 (e_{ir}, \phi, \phi_p, T_q) \, dv, \quad \text{for all } x_3 \in [0, L - \ell] \tag{51}
\]
where
\[
\Sigma_2 (\varepsilon, \ell) = \frac{e\omega^2 \mu^2_m}{qu_0^2(\ell) \mu M} + \sigma(\varepsilon) \tag{52}
\]

Combining the prior relations (48) and (51), we obtain the following differential inequality:
\[
\frac{\Sigma_2 (\varepsilon, \ell)}{4\Sigma_1} \frac{\partial Q}{\partial x_3}(x_3, \ell) + Q(x_3, \ell) \leq 0, \quad \text{for all } x_3 \in (0, L - \ell) \tag{53}
\]
when integrated, furnishes
\[
Q(x_3, \ell) \leq Q(0, \ell) \exp \left( -\frac{x_3}{\tau(\omega, \ell, \varepsilon)} \right), \quad \text{for all } x_3 \in [0, L - \ell] \tag{54}
\]
where
\[
\tau(\omega, \ell, \varepsilon) = \frac{\Sigma_2 (\varepsilon, \ell)}{4\Sigma_1} \tag{55}
\]

By taking into account the inequality (47), from (54) we obtain the estimate (43), provided that \( \tau_m (\ell, \omega) \) is the minimum value of \( \tau(\omega, \ell, \varepsilon) \) on varying the parameters \( \varepsilon_1 \) to \( \varepsilon_5 \). Thus, the proof is complete. \( \square \)

The preceding results may be easily extended to a semi-infinite cylinder, namely the case when \( L \to \infty \). Then, we have the following Phragmèn–Lindelöf alternative result.
Theorem 4. Within the context of a semi-infinite cylinder made of a thermoviscoelastic porous material we have the following alternative:

(a) either the amplitude of the steady-state vibration has a finite volume energetic measure when the cross-sectional functional $I(x_3)$ decays spatially faster than the exponential $\exp\left(-\frac{x_3 - \ell}{\tau_m(\ell, \omega)}\right)$ or

(b) the amplitude of the steady-state vibration has an infinite volume energetic measure when there exists $x_3^* \in [0, \infty)$ such that $I(x_3^*) < 0$ and then $-I(x_3)$ grows spatially faster than the exponential $\exp\left(-\frac{x_3 - \ell - x_3^*}{\tau_m(\ell, \omega)}\right)$.

Proof. We recall first that relation (31) holds true for a semi-infinite cylinder and hence $I(x_3)$ remains a non-increasing function with respect to $x_3$ on $(0, \infty)$. Consequently, we have the only two possibilities: (i) either $I(x_3) \geq 0$ for all $x_3 \in (0, \infty)$, or (ii) there exists the value $x_3^* \in (0, \infty)$ so that $I(x_3^*) < 0$.

Let us consider the case (i), when since $I(x_3) \geq 0$ for all $x_3 \in (0, \infty)$, we deduce the same differential inequality (53), and therefore the estimate (54) holds true and hence the decay estimate (43) holds.

Considering now the case (ii) we deduce that

$$I(x_3) < 0 \text{ for all } x_3 \in [x_3^*, \infty)$$

and hence, $-I(x_3)$ represents an acceptable measure of the amplitude on $[x_3^*, \infty)$. Then we can repeat the reasoning presented in the preceding analysis with $I(x_3)$ replaced by $-I(x_3)$ and so we deduce the following estimate:

$$-I(x_3) \geq -I(x_3^*) \exp\left(\frac{x_3 - \ell - x_3^*}{\tau_m(\ell, \omega)}\right), \text{ for all } x_3 \in [x_3^* + \ell, \infty)$$

and this concludes the proof. \qed

CONCLUDING REMARKS

We can conclude here that our analysis developed in the above sections proves the following:

1. The positive definiteness of the total dissipation energy is sufficient for a complete description of the spatial behavior of the amplitude of the steady-state vibrations.
2. The spatial decay and growth properties of the amplitude are established without any limiting upon the frequencies of the steady-state vibrations.
3. In view of the results established by Chiriţă [20] in the classical linear thermoelasticity, we can conclude that the thermal dissipation alone is not sufficient for establishing the spatial behavior for any value of the frequency of vibrations.

REFERENCES

