On the forward and backward in time problems in the Kelvin–Voigt thermoviscoelastic materials

Stan Chiriţă a, b, Vittorio Zampoli c, *  

a Faculty of Mathematics, Al. I. Cuza University of Iaşi, 700506 Iaşi, Romania  
b Octav Mayer Mathematics Institute, Romanian Academy, 700505 Iaşi, Romania  
c Università della Calabria, Via Pietro Bucci, 87036 Arcavacata di Rende (CS), Italy

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A B S T R A C T

This paper studies the uniqueness of solutions to the forward and backward in time boundary value problems associated with the Kelvin–Voigt viscoelastic model of the thermoelastic materials. For thermo-viscoelastic materials with a center of symmetry, it is shown the uniqueness of solutions to the forward in time boundary value problems without any assumptions upon the thermoviscoelastic constitutive coefficients other than the symmetry properties and those induced by the dissipation inequality. While for the final boundary value problems two uniqueness theorems are presented: the first one is essentially based on the assumption that the specific heat is of negative definite sign, while the second is established in the class of displacement–temperature variation fields whose dissipation energy has a temporal behavior lower than an appropriate growing exponential.

1. Introduction

The viscoelastic thermal stresses and deformations in a body have been first discussed in the book by Boley and Weiner [1], where a manner in which temperature effects can be combined with the viscoelastic model has also been presented. Some of the available methods of solution of thermoviscoelastic problems are introduced and the viscoelastic–elastic analogy is discussed in details. Since then many achievements have been obtained in the literature on subject (see, for example, the book by Amendola et al. [2] and the references cited therein). Recently, leşan [3] developed a linear theory of thermoviscoelastic porous materials in which the microstructure of a body is taken also in consideration.

On the other hand, the study of improperly posed (or ill-posed) problems has received considerable recent attention in response to the realization that a number of physical situations lead to these kind of mathematical models. A backward in time problem is a final boundary value problem associated with a linear model of materials presenting dissipation. The final data are given at $t = 0$ and we are interested in extrapolating to previous times. It is like we see a movie from the end to the beginning, but it is a problem, we can imagine different beginnings for the same end. It is well known that this type of problem is ill-posed (it fails to have a global solution, or it fails to have a unique solution or the solution does not depend continuously on the data). In order to stabilize such kind of problems many different techniques have been developed in literature such of those of “solving” ill-posed problems for equations of evolution. Some of these involve the altering of governing equations in such a way as to make such problems well-posed. Others involve changing the initial and/or boundary conditions again in such way as to make the problems well-posed. Still others involve constraining solutions to lie in a certain constraint set. This class of problems was studied by Ames and Payne [4] who derived stabilizing criteria for solutions of the Cauchy problem for the standard equations of dynamical linear thermoelasticity backward in time. They also obtained inequalities establishing continuous dependence on the coupling parameter that links the elastic displacement field and the temperature field. Ciarletta [5] and Chiriţă [6] provide a closely related discussion.

This type of problems has been initially considered by Serrin [7] who established uniqueness results for the Navier–Stokes equations backward in time. Explicit uniqueness and stability criteria for the classical Navier–Stokes equations backward in time were obtained by Knops and Payne [8] and Galdi and Straughan [9]. For an overview of improperly posed problems, the reader may look in the important study made by Ames and Straughan [10].

* Corresponding author.  
E-mail addresses: schirita@uaic.ro (S. Chiriţă), zampoli@gmail.com (V. Zampoli).

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2. Basic equations

Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes $Ox_k, (k = 1, 2, 3).$ We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3),$ whereas Greek subscripts are confined to the range $(1, 2),$ summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation. Throughout this section we assume that a regular region $B$ is filled by an inhomogeneous and anisotropic thermoviscoelastic material. Considering the linear theory of thermoviscoelastic materials and assuming that the initial body is free from stresses and has zero entropy, the field equations consists of (see, for example, Leșan [3]).

- the equations of motion

$$t_{rs, r} + \varrho \ddot{s}_r = \varrho \dot{s}_r, \quad \text{in } B \times (0, \infty),$$

(1)

- the equation of energy

$$\varrho \ddot{\theta} + \dot{Q}_r + \dot{Q}_s, \quad \text{in } B \times (0, \infty),$$

(2)

- the constitutive equations

$$t_{rs} = C_{pqrs} \dot{e}_{pq} - \beta_0 \dot{s} + \frac{1}{2} M_{rs} \dot{\theta},$$

$$\ddot{Q}_r = k_s \dot{\theta}_r + \dot{f}_{pq} \dot{e}_{pq}, \quad \text{in } B \times (0, \infty),$$

(3)

and

- the geometrical relations

$$\dot{e}_{rs} = \frac{1}{2} (\dot{u}_{rs} + \dot{u}_{sr}), \quad \text{in } \overline{B} \times (0, \infty).$$

(4)

Here we have used the following notations: $t_{rs}$ are the components of the stress tensor, $\eta$ is the entropy per unit mass, $Q_r$ are the components of the heat flux vector, $e_{rs}$ are the components of the strain tensor, $\varrho$ is the mass density of the medium, $u_r$ are the components of the displacement vector, $\dot{\theta}$ is the change in temperature from the constant ambient temperature $T_0 > 0,$ $f_r$ are the components of the body force per unit mass, $s$ is the heat supply per unit mass. The constitutive coefficients are continuous differentiable functions of the spatial variables and satisfy the following symmetries

$$C_{pqrs} = C_{pqrs}, \quad C_{pqrs} = C_{pqrs},$$

$$M_{rs} = M_{rs}, \quad \beta_0 = \beta_0, \quad k_s = k_s, \quad f_{pq} = f_{pq}. \quad (5)$$

Furthermore, in view of the second law of thermodynamics, the Clausius–Duhem inequality must be satisfied, which provides the positive semi-definiteness of the total dissipation energy $\Lambda,$ that is

$$\Lambda \left( \dot{e}_{rs}, \dot{\theta}_r \right) = C_{pqrs} \dot{e}_{pq} \dot{e}_{rs} + k_s \dot{\theta}_r, \quad \text{in } B \times (0, \infty),$$

$$+ \left( M_{rs} + f_{pq} \right) \dot{e}_{pq} \dot{\theta}_r \geq 0. \quad (6)$$

Throughout this paper we will consider inhomogeneous and anisotropic Kelvin–Voigt thermoviscoelastic materials with a center of symmetry, that is we will assume that

$$M_{pqrs} = \frac{1}{\varrho} f_{pq} = 0. \quad (7)$$

In terms of the displacement field and the temperature variation, the basic equations (1)–(4) lead to the following differential system

$$\left( C_{pqrs} \ddot{u}_{p,q} \right)_{, r} - \left( \beta_0 \ddot{\theta} \right)_{, r} + \left( \frac{1}{\varrho} \ddot{\varrho} \right)_{, r} = 0,$$

$$\frac{1}{\varrho} \left( k_s \dot{\theta}_r + \dot{f}_{pq} \dot{e}_{pq} \right)_{, r} - \beta_0 \dot{u}_r + \varrho \dot{\varrho} = 0, \quad \text{in } B \times (0, \infty).$$

(8)

3. Forward in time boundary value problems

In this section we consider the forward in time boundary value problem $(T_0)$ defined by the differential system (8) and the boundary conditions

$$\ddot{u}_i (x,t) = \ddot{u}_i (x,t) \quad \text{on } \Sigma_1 \times [0, \infty),$$

$$\ddot{u}_i (x,t) = \ddot{u}_i (x,t) \quad \text{on } \Sigma_2 \times [0, \infty),$$

$$\ddot{u}_i (x,t) = \ddot{u}_i (x,t) \quad \text{on } \Sigma_3 \times [0, \infty),$$

$$\ddot{u}_i (x,t) = \ddot{u}_i (x,t) \quad \text{on } \Sigma_4 \times [0, \infty),$$

(9)

and the initial conditions

$$\ddot{u}_i (x,t) = 0 \quad \ddot{u}_i (x,0) = 0, \quad \ddot{u}_i (x,0) = 0,$$

$$\ddot{\theta} (x,0) = \ddot{\theta} (x,0), \quad x \in \overline{B},$$

(10)

where $\ddot{u}_i (x,t), \ddot{u}_i (x,t), \ddot{\theta} (x,t), \ddot{\theta} (x,t), \ddot{u}_i (x,t), \ddot{u}_i (x,t)$ and $\ddot{\theta} (x,t)$ are prescribed functions and

$$t_i (x,t) = t_i (x,t), \quad q (x,t) = q (x,t), \quad \ddot{q} (x,t) = q (x,t) \mid \Sigma_i,$$

(11)

$n_i$ are the components of the outward unit normal vector to the boundary surface and $\Sigma_1, \Sigma_2, \Sigma_3$ and $\Sigma_4$ are subsurfaces of $\partial B,$ such that $\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \partial B$ and $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset.$

In this section we are interested in establishing the uniqueness of solutions of the forward in time boundary value problem $(T_0)$ under mild conditions upon the thermoviscoelastic coefficients. Thus, we have

Theorem 1. Suppose that $q > 0$ and $a > 0$ on $B.$ Then the forward in time boundary value problem $(T_0)$ has at most one solution.
**Proof.** In order to prove the uniqueness result it is sufficient to prove that the zero external given data, that is $D = \{ \mathbf{f}_{i}, \mathbf{s}^{0}, \mathbf{u}^{0}_{i}, \mathbf{q}^{0}_{i}; \mathbf{t}_{i}, \mathbf{ar{t}}, \hat{\theta}, \overline{\theta} \} = 0$, implies that the corresponding solution $\mathbf{U} = \{ \mathbf{u}_{i}, \theta \} (\mathbf{x}, t)$ is vanishing on $\overline{B} \times [0, \infty)$. In this aim we use the following Lagrange identity

$$\frac{\partial}{\partial s} \{ \rho u_{i}(t + s)u_{i}(t - s) + \rho \dot{u}_{i}(t + s)u_{i}(t - s) \} = \rho u_{i}(t - s)\dot{u}_{i}(t - s) - \rho u_{i}(t + s)\dot{u}_{i}(t - s),$$

for all $s \in (0, t)$, $t > 0$, (12)

which, when integrated for $(s, \mathbf{x}) \in (0, t) \times B$ and by means of the basic equation (1), furnishes

$$2 \int_{B} \rho u_{i}(t)\dot{u}_{i}(t) dv = \int_{0}^{t} \int_{B} \{ u_{i}(t + s)T_{i,j}(t - s) - u_{i}(t - s)T_{i,j}(t + s) \} ds dv.$$  

(13)

Further, we use the divergence theorem and the boundary conditions and the constitutive equations for $T_{ij}$ and $Q\eta$, in order to obtain

$$2 \int_{B} \rho u_{i}(t)\dot{u}_{i}(t) dv = \int_{0}^{t} \int_{B} \left[ \frac{\partial}{\partial s} \{ C_{ijkl}e_{j}(t + s)e_{l}(t - s) \} \right] ds dv,$$

$$+ \int_{0}^{t} \int_{B} \left[ \theta(t - s)Q\eta(t + s) - \theta(t + s)Q\eta(t - s) \right] \times Q\eta(t - s) \right] ds dv.$$  

(14)

Now, by an integration with respect to time variable of the equation of energy (2) and by using the zero initial conditions, we have

$$Q\eta(t) = \frac{1}{T_{0}} \int_{0}^{t} Q_{\eta,r}(z) dz,$$  

(15)

which, when substituted into (14) and the divergence theorem is applied, gives

$$2 \int_{B} \rho u_{i}(t)\dot{u}_{i}(t) dv = \int_{0}^{t} \int_{B} \left[ \frac{\partial}{\partial s} \{ C_{ijkl}e_{j}(t + s)e_{l}(t - s) \} \right] ds dv,$$

$$+ \int_{0}^{t} \int_{B} \left[ \theta(t - s)Q\eta(t + s) - \theta(t + s)Q\eta(t - s) \right] \times Q\eta(t - s) \right] ds dv.$$  

(16)

Thus, we have the following identity

$$2 \int_{B} \rho u_{i}(t)\dot{u}_{i}(t) dv + \int_{B} C_{ijkl}e_{j}(t)e_{l}(t) dv,$$

$$+ \int_{0}^{t} \int_{B} k_{ij}\theta_{i}(t + s)dv \int_{0}^{t} \theta_{j}(z)dz = 0,$$  

(17)

which, when integrated with respect to time variable and when the zero initial conditions are used, implies

$$\int_{B} \rho u_{i}(t)\dot{u}_{i}(t) dv + \int_{0}^{t} \int_{B} C_{ijkl}e_{j}(t)e_{l}(t) dv,$$

$$+ \int_{0}^{t} \int_{0}^{t} k_{ij}\theta_{i}(z)dz \int_{0}^{t} \theta_{j}(z)dz dv = 0.$$  

(18)

In view of the positiveness of the tensors $C_{ijkl}$ and $k_{ij}$, from (18) we deduce that

$$u_{i}(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \overline{B} \times (0, \infty),$$  

(19)

and then the identity

$$\frac{1}{T_{0}} \int_{\overline{B}} \alpha \theta^{2}(t) dv + \int_{0}^{t} \int_{\overline{B}} k_{ij}\theta_{i}(s)\theta_{j}(s) dv,$$

$$= - \int_{0}^{t} \int_{\overline{B}} \beta_{ij}\theta(s)\dot{u}_{ij}(s) dv,$$  

(20)

implies

$$\theta(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \overline{B} \times (0, \infty)$$  

(21)

and this together with relation (19) proves the result.

**Remark 1.** The above result of uniqueness continues to be true when the hypothesis $a > 0$ is replaced by the assumption that $k_{ij}$ is positive definite and the measure of $\Sigma_{3}$ is different from zero.

### 4. Final boundary value problems

Throughout this section we will consider the final boundary value problem ($P_{B}$) defined by the differential system (8), considered for all $(\mathbf{x}, t) \in \overline{B} \times (0, \infty)$, together with the final conditions

$$u_{i}(\mathbf{x}, 0) = u_{i}^{0}(\mathbf{x}), \quad \dot{u}_{i}(\mathbf{x}, 0) = \dot{u}_{i}^{0}(\mathbf{x}),$$

(22)

and the boundary conditions

$$u_{i}(\mathbf{x}, t) = \bar{u}_{i}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{1} \times (-\infty, 0],$$

$$t_{i}(\mathbf{x}, t) = \bar{t}_{i}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{2} \times (-\infty, 0],$$

(23)

$$\theta(\mathbf{x}, t) = \bar{\theta}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{3} \times (-\infty, 0].$$

In the present section we are interested in establishing the uniqueness of solutions of the final boundary value problem ($P_{B}$) under mild conditions upon the thermoviscoelastic characteristics. To this end, we transform ($P_{B}$), by making $t \rightarrow -t$ and by using appropriate notations, in the initial boundary value problem ($P_{BP}$) defined by the equations

$$\left( C_{pqrs}u_{p,q} \right)_{r} - \left( \beta_{rs} \theta \right)_{r} - \left( C_{pqsp}u_{p,q} \right)_{s} + \theta_{s} = \rho \dot{u}_{s},$$

(24)

$$\frac{1}{T_{0}} \left( k_{rs} \theta_{r} \right)_{r} + \beta_{rs} \dot{u}_{r} + s = -a \theta,$$

for all $(\mathbf{x}, t) \in \overline{B} \times (0, \infty)$, together with the initial conditions

$$u_{i}(\mathbf{x}, 0) = u_{i}^{0}(\mathbf{x}), \quad \dot{u}_{i}(\mathbf{x}, 0) = \dot{u}_{i}^{0}(\mathbf{x}),$$

(25)

and the boundary conditions

$$u_{i}(\mathbf{x}, t) = \bar{u}_{i}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{1} \times [0, \infty),$$

$$t_{i}(\mathbf{x}, t) = \bar{t}_{i}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{2} \times [0, \infty),$$

(26)

$$\theta(\mathbf{x}, t) = \bar{\theta}(\mathbf{x}, t) \quad \text{on} \quad \Sigma_{3} \times [0, \infty).$$

In order to establish uniqueness of solutions to the initial boundary value problem ($P_{BP}$) it is sufficient to prove that the initial boundary value problem ($P_{BP}$) with the zero external given data, $D = \{ f_{i}, s; u^{0}_{i}, \mathbf{q}^{0}_{i}; \mathbf{t}_{i}, \mathbf{ar{t}}, \hat{\theta}, \overline{\theta} \} = 0$, implies that the corresponding solution $\mathbf{U} = \{ u_{i}, \theta \} (\mathbf{x}, t)$ is vanishing on $\overline{B} \times [0, \infty)$. Concerning the solutions of the initial boundary value problem
\( (\tau_{bw}^0) \) we can establish a series of auxiliary identities. Thus, we have

**Theorem 2.** Let \( U = \{ u_{i}, \theta \} (x, t) \) be a solution of the initial boundary value problem \((\tau_{bw}^0)\). Then the following identities hold true

\[
\int_{B}^{} \int_{0}^{t} \left[ Q \dot{u}_{i}(t) \dot{u}_{i}(t) + C_{ijkl}(t) \dot{e}_{kl}(t) \right] dv - 2 \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds,
\]

\[
= \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds,
\]

\[
= -2 \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds,
\]

\[
= -2 \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds.
\]

\[
\int_{B}^{} \int_{0}^{t} \left[ Q \dot{u}_{i}(t) \dot{u}_{i}(t) + C_{ijkl} \dot{e}_{kl}(t) + a \dot{\theta}^2(t) \right] dv
\]

\[
= 2 \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds.
\]

\[
= 2 \int_{0}^{t} \int_{B} \frac{1}{\theta_{i,j}} k_{i,j} \dot{e}_{i,j}(s) \dot{\theta}(s) dvds.
\]

\[
for \ all \ t \in [0, \infty).
\]

**Proof.** It is a simple matter to obtain the first three identities. In order to establish the identity (30) we start with the identity

\[
\frac{\partial}{\partial s} (Q \dot{u}_{i}(t + s) \dot{u}_{i}(t - s)) = Q \ddot{u}_{i}(t + s) \dot{u}_{i}(t - s) - Q \dot{u}_{i}(t + s) \ddot{u}_{i}(t - s),
\]

which, integrated with respect to \( s, x \) over \([0, t] \times B\) and with the aid of zero initial conditions, furnishes

\[
\int_{B} \int_{0}^{t} [Q \ddot{u}_{i}(t + s) \dot{u}_{i}(t - s) - Q \dot{u}_{i}(t + s) \ddot{u}_{i}(t - s)] dvds.
\]

Further, we use the equation (24), the divergence theorem and zero initial and boundary conditions in order to deduce

\[
\int_{B} \int_{0}^{t} [Q \ddot{u}_{i}(t + s) \dot{u}_{i}(t - s) - Q \dot{u}_{i}(t + s) \ddot{u}_{i}(t - s)] dvds.
\]

Finally, we use the equation (24), the divergence theorem and zero initial and boundary conditions to obtain

\[
\int_{B} \int_{0}^{t} [Q \ddot{u}_{i}(t + s) \dot{u}_{i}(t - s) - Q \dot{u}_{i}(t + s) \ddot{u}_{i}(t - s)] dvds.
\]

\[
\Phi(t) = \left[ 2 \int_{0}^{t} \psi(z) \dot{\psi}(z) dz \right]^{1/2}.
\]

From this last equality we get

\[
2 \Phi(t) \Phi(t) = 2 \int_{0}^{t} \psi(z) \dot{\psi}(z) dz \leq 2 \Phi(t) \Phi(t),
\]

and hence we deduce that

\[
\Phi(t) \leq \Phi(t), \quad t \in [0, \infty),
\]

which implies that

\[
\Phi(t) \leq \int_{0}^{t} \psi(z) dz.
\]

Consequently, from (43) and the Cauchy–Schwarz inequality we obtain

\[
\Phi(t) \leq \left[ \int_{0}^{t} \psi(z) dz \right]^{2} \leq \left( \int_{0}^{t} dz \right) \left( \int_{0}^{t} \psi^{2}(z) dz \right)
\]

\[
= t \int_{0}^{t} \psi^{2}(z) dz,
\]

and so we obtain the estimate (36) and the proof is complete.

We have now all the auxiliary material to establish the expected uniqueness result.

**Theorem 3.** Suppose that the density mass \( Q \) is strictly positive on \( B \) and that the dissipation energy \( \Lambda \) is a positive definite quadratic form and mass \( \Sigma_{3} \neq 0 \). Under the hypothesis that \( a \) is negative on \( B \), the final boundary value problem \((\tau_{fh})\) has at most one solution.
Proof. From the relations (30) and (31) we deduce the following identity
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu = \int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(t) \hat{e}_k^j(t) + a^2(t) \right] d\nu.
\]
(45)

In view of our hypotheses it follows that \( C_{ijkl}^* \) is a positive definite tensor and hence we have
\[
\mu_m \hat{e}_i^j \leq C_{ijkl}^* \hat{e}_i^j \hat{e}_j^i \leq \mu_M \hat{e}_i^j \hat{e}_j^i,
\]
(46)
where \( \mu_m \) and \( \mu_M \) are appropriate constants related to the lower and upper eigenvalues of \( C_{ijkl}^* \). Moreover, by means of the Cauchy–Schwarz inequality, we have
\[
\left| C_{ijkl}^* \hat{e}_i^j(t) \hat{e}_k^j(t) \right| \leq M_{en}(t) \hat{e}_n(t),
\]
(47)
where
\[
M = \max \left\{ \frac{C_{ijkl}^*}{\hat{B}} \right\} \frac{1}{2},
\]
(48)
Further, we use the relations (36), (46) and (47) in order to obtain the following estimate
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(t) \hat{e}_k^j(t) \right] d\nu d\nu \leq \frac{TM}{\mu_m} \int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu,
\]
(49)
and consequently, from (45), we obtain
\[
\left( 1 - \frac{TM}{\mu_m} \right) \int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \leq 0,
\]
(50)
for
\[
t \in [0, \infty).
\]
(51)
Finally, we set
\[
\tau = \frac{\mu_m}{M},
\]
(52)
and note that (50) implies that
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \leq 0, \quad t \in [0, \tau),
\]
(53)
and hence, with the aid of identity (30), we deduce that
\[
\{ u_i, \theta \} (x, t) = 0 \quad (x, t) \in B \times [0, \tau).
\]
(54)

Furthermore, we repeat the above procedure on the interval \([\tau, \infty)\) and so we prove that (53) is true on \( B \times [0, 2\tau) \) and so we can continue in order to obtain the required uniqueness result.

Remark 2. We can avoid the assumption that \( C_{ijkl}^* \) is a positive definite tensor when the elasticity tensor \( C_{ijkl} \) is a negative semi-definite tensor.

Remark 3. When \( a(x) \) is of indefinite sign on \( B \) we can prove the following uniqueness result.

Theorem 4. Suppose that the density mass \( \rho \) is strictly positive on \( B \) and that the dissipation energy \( \Lambda \) is a positive definite quadratic form and means \( \Sigma_3 \neq 0 \). Moreover, we assume that \( C_{ijkl}^* \) is a negative semi-definite tensor on \( B \). Then there exists a strictly positive constant \( \alpha \) so that, in the class of displacement–temperature fields \( \{ \{ u_i, \theta \} \} \), defined on \( B \times [0, \infty) \) and satisfying
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \leq M^2 e^{\alpha t},
\]
(55)
with \( M^2 = \text{constant} \), the final boundary value problem \( \{ T_{bw} \} \) has at most one solution.

Proof. In view of the hypothesis, from the identity (45) we obtain
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \leq \int_0^t e^{a^2(t)} d\nu.
\]
(56)
Since \( \Lambda \) is a positive definite quadratic form, it follows that \( (1/T_0)k_{ij}^2 \) is a positive definite tensor and hence we have
\[
k_{ij}^2 \hat{e}_i^j \hat{e}_j^i \leq \frac{1}{T_0} k_{ij} \hat{e}_i^j \hat{e}_j^i \leq k_m \hat{e}_i^j \hat{e}_j^i,
\]
(57)
where \( k_m \) and \( k_M \) are positive constants related to the lower and upper eigenvalues of \( (1/T_0)k_{ij} \).

On the other side, since means \( \Sigma_3 \neq 0 \), it follows that
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(t) \hat{e}_k^j(t) \right] d\nu d\nu \geq \lambda \int_0^t e^{a^2(t)} d\nu,
\]
(58)
where \( \lambda \) is the smallest eigenvalue for the fixed membrane problem for \( B \).

If we use the relations (56) and (57) into (55), then we obtain
\[
\int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \leq \sigma^2 \int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(t) \hat{e}_k^j(t) \right] d\nu d\nu,
\]
(59)
where
\[
\sigma^2 = \frac{1}{\lambda k_m} \max_{x, \beta} \left| a(x) \right|.
\]
(60)
If we set
\[
\chi(t) = \left\{ \int_0^t \int _B \left[ C_{ijkl}^* \hat{e}_i^j(z) \hat{e}_k^j(z) \right] d\nu d\nu \right\}^{\frac{1}{2}},
\]
(61)
then the relation (58) implies
\[
\chi(t) \leq 2\sigma^2 \chi(t), \quad t \in [0, \infty).
\]
(62)
Suppose that \( \chi(t) = 0 \) for all \( t \in [0, \infty) \). Then, by using the positive definiteness of the dissipation energy, from (60) and (31) and the zero initial and boundary conditions we deduce that
\[
\{ u_i, \theta \} (x, t) = 0 \quad (x, t) \in B \times [0, \infty),
\]
(63)
that is the required uniqueness result.

Let us now suppose that there exists \( \tau \in (0, \infty) \) so that \( \chi(\tau) > 0 \). Then we have
\[
\chi(t) > 0, \quad t \in [\tau, \infty),
\]
(64)
and then the inequality (61) furnishes
\[
\frac{d}{dt} \left[ \chi(t)e^{-t/(2\sigma^2)} \right] \geq 0 \quad \text{for all} \quad t \in [\tau, \infty),
\]
(65)
which, integrated with respect to \( t \), gives
\[
\chi(t)e^{-t/(2\sigma^2)} \leq \chi(t)e^{-t/(2\sigma^2)} \leq \lim_{t \to \infty} \left[ \chi(t)e^{-t/(2\sigma^2)} \right],
\]
(66)
Consequently, if we choose \( \alpha \) so that
\[
0 \leq \alpha < \frac{1}{\sigma^2},
\]
(67)
then, by the hypothesis expressed by (54), we deduce that
\[
\lim_{t \to \infty} \left[ \chi(t)e^{-t/(2\sigma^2)} \right] = 0.
\]
and hence the inequality (65) implies that
\[ \chi(t) = 0 \quad \text{for all} \quad t \in [0, \infty), \]  
(68)
a relation entering in contradiction with our initial assumption (63).
Thus, the assumption (63) cannot be possible and hence we have
the uniqueness result and the proof is complete.

5. Concluding remarks

We can conclude here that our analysis developed in the above
sections proves that:
1. The positive definiteness of the total dissipation energy is
   sufficient for establishing the uniqueness of solutions to the cor-
   responding forward in time problems.
2. Under the hypothesis that the specific heat is of negative definite
   sign on \( B \), the final boundary problem has at most one solution.
3. When the specific heat is of indefinite sign on \( B \) the unique-
   ness result can be established in an appropriate class of
   displacement–temperature variation fields characterized by a
   dissipation energy having a growth in time lower than an appro-
   priate growing exponential.

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