On the wave propagation in the time differential dual-phase-lag thermoelastic model

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We study the propagation of plane time harmonic waves in the infinite space filled by a time differential dual-phase-lag thermoelastic material. There are six possible basic waves travelling with distinct speeds, out of which, two are shear waves, and the remaining four are dilatational waves. The shear waves are found to be uncoupled, undamped in time and travels independently with the speed that is unaffected by the thermal effects. All the other possible four dilatational waves are found to be coupled, damped in time and dispersive due to the presence of thermal properties of the material. In fact, there is a damped in time longitudinal quasi-elastic wave whose amplitude decreases exponentially to zero when the time is going to infinity. There is also a quasi-thermal mode, like the classical purely thermal disturbance, which is a standing wave decaying exponentially to zero when the time goes to infinity. Furthermore, there are two possible longitudinal quasi-thermal waves that are damped in time with different decreasing rates or there is one plane harmonic in time longitudinal thermal wave, depending on the values of the time delays. The surface wave problem is studied for a half space filled by a dual-phase-lag thermoelastic material. The surface of the half space is free of traction and it is free to exchange heat with the ambient medium. The dispersion relation is written...
in an explicit way and the secular equation is established. Numerical computations are performed for a specific model, and the results obtained are depicted graphically.

1. Introduction

The dual-phase-lag model developed by Tzou [1,2] describes the thermal relaxation and thermalization behaviours that are interweaving in the ultrafast process of heat transport in the electron gas. Two intrinsic delay times, called phase-lags denoted by $\tau_T$ and $\tau_q$, were introduced to account for the finite times required for the thermal equilibrium ($\tau_T$) and effective collisions ($\tau_q$) between electrons and phonons to take place. In fact, Tzou [1,2] proposed the following universal constitutive equation between the heat flux vector and the temperature gradient

$$q_i (t + \tau_q) = -k T_j (t + \tau_T), \quad \tau_q > 0, \quad \tau_T > 0,$$

(1.1)
to cover the fundamental behaviours of diffusion (macroscopic in both space and time), wave (macroscopic in space but microscopic in time), phonon–electron interactions (microscopic in both space and time) and pure phonon scattering. A refined structure of the lagging response was depicted by Tzou [3] by expanding the above equation in terms of the Taylor’s series with respect to time and an entire discussion was given for the orders of approximations leading to many representative models.

Such a theory of heat conduction replaces the Fourier law with one in which the gradient of the temperature at a point $x$ in the material at time $t + \tau_T$ corresponds to the heat flux vector at the same point at time $t + \tau_q$. The delay time $\tau_T$ is caused by microstructural interactions, such as phonon scattering or phonon–electron interactions. The delay time $\tau_q$ is interpreted as the phase lag of the heat flux vector due to fast-transient effects of thermal inertia. The phase lags $\tau_T$ and $\tau_q$ are treated as two additional intrinsic thermal properties characterizing the energy-bearing capacity of the material.

Recently, the capability of the dual-phase-lag model was tested to simulate the heat transport in some special cases such as micro/nanoscales [4,5], ultrafast laser-pulse processes [6,7], living tissues [8] and carbon nanotube [9].

In this paper, we consider a thermomechanical model based on an idea proposed by Tzou [1,10], in which the heat flux vector is given by the following constitutive law

$$q_i (x, t) + \tau_q \ddot{q}_i (x, t) + \frac{1}{2} \tau_q^2 \dot{q}_i (x, t) = -k (x) T_j (x, t) - \tau_T k (x) \dot{T}_j (x, t).$$

(1.2)

We note that with an integration of the second-order differential equation (1.2) in terms of $q_i$, (considering as a known function its right-hand side), we can see that $q_i$ results to be determined by a constitutive equation of memory type like that introduced by Gurtin & Pipkin [11] (see also Joseph & Preziosi [12] and Straughan [13]). Thus, it results that the equation (1.2) is a particular case of the theory developed by Gurtin & Pipkin [11], but with specific memory kernels generated by the special form of the differential equation (1.2).

In what follows, we study the propagation of plane time harmonic waves in the infinite space filled by a time differential dual-phase-lag thermoelastic material. We find six possible basic waves travelling with distinct speeds, out of which, two are shear waves, and the remaining four are dilatational waves. The shear waves are found to be uncoupled, undamped in time and travel independently with the speed that is unaffected by the thermal effects. All the possible four dilatational waves are found to be coupled, damped in time and dispersive due to the presence of thermal properties of the material. Thus, there is a damped in time longitudinal quasi-elastic wave whose amplitude decreases exponentially to zero when the time is going to infinity. There is also a quasi-thermal mode, like the classical purely thermal disturbance, which is a standing wave decaying exponentially to zero when the time goes to infinity. Furthermore, there are two possible longitudinal quasi-thermal waves that are damped in time with different decreasing rates or a possible quasi-thermal wave. The surface wave problem is studied for a half space filled by
a dual-phase-lag thermoelastic material. The surface of the half space is free of traction and it is free to exchange heat with the ambient medium. The dispersion relation is written in an explicit form and the secular equation is presented. Numerical computations are performed for a specific model, and the results obtained are depicted graphically.

It is worth recalling that the dual-phase-lag model has been studied and analyzed by Quintanilla [14,15], Horgan & Quintanilla [16], Quintanilla & Racke [17,18], and Quintanilla & Jordan [19]. The wave propagation problem has been studied by Prasad et al. [20], Kumar & Chawla [21] and Abouelregal [22] by using methods different from that used in this paper.

2. Basic equations

Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes \( Ox_k, \ (k = 1, 2, 3) \). We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers \( (1, 2, 3) \), summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate and a superposed dot denotes time differentiation. Throughout this paper, we suppose that a regular region \( B \) is filled by a homogeneous and isotropic thermoelastic material with phase lag. The basic equations for the linear theory of dual phase-lag thermoelasticity for an isotropic and homogeneous material are

- the equations of motion
  \[ t_{mn,m} + \rho F_n = \rho \ddot{u}_n, \]  
  (2.1)

- the equation of energy
  \[ \rho T \dot{\eta} = -\dot{q}_r + \rho S, \]  
  (2.2)

in \( B \times (0, \infty) \),

- the constitutive equations
  \[ t_{mn} = \lambda e_{rr} \delta_{mn} + 2\mu e_{mn} - \beta \eta \delta_{mn} T, \]  
  (2.3)

\[ \rho \eta = \beta e_{rr} + a T, \]  
  (2.4)

\[ \dot{q}_r + \tau \dot{q}_r + \frac{1}{2} \tau^2 \ddot{q}_r = -k T + \tau T, \]  
  (2.5)

in \( \bar{B} \times [0, \infty) \) and

- the displacement-strain equations
  \[ e_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}), \]  
  (2.6)

in \( \bar{B} \times [0, \infty) \). Here we have used the following notations: \( t_{mn} \) are the components of the stress tensor, \( \eta \) is the entropy per unit mass, \( \dot{q}_r \) are the components of the heat flux vector, \( e_{mn} \) are the components of the strain tensor, \( \rho \) is the mass density of the medium, \( u_r \) are the components of the displacement vector, \( T \) is the change in temperature from the constant ambient temperature \( T_0 > 0 \), \( F_r \) are the components of the body force per unit mass, \( S \) is the heat supply per unit mass and \( \lambda, \mu, \beta, a \) and \( k \) are the constant thermoelastic constitutive coefficients of the material. Furthermore, \( \tau_T > 0 \) and \( \tau_q > 0 \) are the constant time delays, intrinsic thermal characteristics of the dual-phase-lag thermoelastic material.

Throughout this paper, we will assume that

\[ \mu > 0, \quad \lambda + 2\mu > 0, \quad a > 0, \quad k > 0, \quad \tau_q > 0, \quad \tau_T > 0. \]  
(2.7)

For an isotropic and homogeneous thermoelastic material with dual-phase-lag, the above system of basic field equations is equivalent with the following system of linear partial differential
equations in terms of displacement and temperature variation \( (u, T)(x, t) \)

\[
\mu u_{n,mn} + (\lambda + \mu)u_{m,mn} - \beta \tau_{mn} T_{n,\mu} + \varrho F_{n} = \varrho \ddot{u}_{n}
\]

and

\[
\left( 1 + \tau_i \frac{\partial}{\partial t} + \frac{1}{2} \tau_q \frac{\partial^2}{\partial t^2} \right) (\beta T_{0r,r} + aT_{0} - \varrho S) = kT_{rr} + \tau_{r} \ddot{T}_{rr}.
\]

(2.8)

3. Plane harmonic waves in the entire space filled by a homogeneous dual-phase-lag thermoelastic material

In this section, we give an analysis of the harmonic plane wave solutions for homogeneous and isotropic thermoelastic bodies with zero body force and heat supply. When the thermal dissipative properties of the elastic material are taken into consideration, then the thermoelastic processes are irreversible and the plane waves solutions are expected to be damped in time (see, for example, Lebeau & Zuazua [23]). That means we have to seek solutions of the displacement–temperature equations of motion (2.8) in the form

\[
u_r(x, t) = \text{Re} \left\{ A_r e^{i\kappa(x-x-\nu t)} \right\}
\]

and

\[
T(x, t) = \text{Re} \left\{ C e^{i\kappa(x-x-\nu t)} \right\}.
\]

(3.1)

Here Re is the real part, \( i = \sqrt{-1} \) is the imaginary unit, \( A = (A_1, A_2, A_3) \) is a constant complex vector and \( C \) is a constant complex scalar with \( |A| + |C| \neq 0 \), \( \kappa \) is the real wavenumber and \( n \) is a real unit vector giving the direction of propagation. Moreover, in view of the dissipative character of the dual-phase-lag thermoelastic model, we assume \( \nu \) to be a complex number, that is

\[
v = \text{Re}(\nu) + i \text{Im}(\nu),
\]

(3.2)

with

\[
\text{Re}(\nu) \geq 0,
\]

(3.3)

giving the wave speed, while

\[
\text{Im}(\nu) \leq 0,
\]

(3.4)

describes the damping in time of the wave. Obviously, when \( \text{Im}(\nu) = 0 \), we have an undamped in time wave, whereas when \( \text{Im}(\nu) < 0 \), the corresponding wave is damped in time and it decays to zero like the exponential \( \exp(\kappa \text{Im}(\nu)t) \) as time tends to infinity. Moreover, when \( \text{Re}(\nu) = 0 \) and \( \text{Im}(\nu) < 0 \), then we have a standing wave whose amplitude decays exponentially with time.

Substitution of (3.1) into relations (2.8) leads to the following linear and homogeneous algebraic system for \( A_1, A_2, A_3 \) and \( C \),

\[
\left[ \left( \mu - \varrho \nu^2 \right) \delta_{rs} + (\lambda + \mu) n_r n_s \right] A_r + \frac{i\beta}{\kappa} n_s C = 0
\]

and

\[
\beta T_{0} \left( 1 - i\nu \tau_{q} - \frac{1}{2} \kappa^2 \nu^2 \tau_{q}^2 \right) A_s n_s
\]

\[
+ \left[ k - i\nu r \kappa \tau_{T} - \frac{i}{\kappa} \alpha T_{0} \left( 1 - i\nu \tau_{q} - \frac{1}{2} \kappa^2 \nu^2 \tau_{q}^2 \right) \right] C = 0.
\]

(3.5)

By taking into account that \( |A| + |C| \neq 0 \), it follows that the determinant of the algebraic system is vanishing and hence it implies

\[
(\mu - \varrho \nu^2)^2 \Delta(\nu) = 0,
\]

(3.6)

where

\[
\Delta(\nu) \equiv (\lambda + 2\mu - \varrho \nu^2) \left[ k - i\nu r \kappa \tau_{T} - \frac{i}{\kappa} \alpha T_{0} \left( 1 - i\nu \tau_{q} - \frac{1}{2} \kappa^2 \nu^2 \tau_{q}^2 \right) \right]
\]

\[
- \frac{1}{\kappa} \beta^2 T_{0} \nu \left( 1 - i\nu \tau_{q} - \frac{1}{2} \kappa^2 \nu^2 \tau_{q}^2 \right).
\]

(3.7)
Thus, we have the following solutions satisfying restrictions (3.3) and (3.4),

\[ v_1 = v_2 = c_2, \quad c_2 \equiv \sqrt{\frac{\mu}{\rho}}, \tag{3.8} \]

while the other admissible solutions satisfy the following equation

\[ \Delta(v) = 0. \tag{3.9} \]

For convenience, we suppose here that

\[ \Delta(c_2) \neq 0. \tag{3.10} \]

Such assumption is common in the study of wave propagation in the classical theories of elasticity and thermoelasticity.

On the other hand, from the algebraic system (3.5) we deduce that

\[
\begin{aligned}
&\left( \lambda + 2\mu - \rho v^2 \right) A_3 n_s + \frac{\beta}{\lambda} C = 0 \\
&\beta T_0 v \left( 1 - i x v \tau_q - \frac{1}{2} \chi^2 v^2 \tau_q^2 \right) A_3 n_s \\
&+ \left[ k - i x v k T - \frac{i}{\lambda} a T_0 v \left( 1 - i x v \tau_q - \frac{1}{2} \chi^2 v^2 \tau_q^2 \right) \right] C = 0,
\end{aligned}
\tag{3.11} \]

so that, in view of the hypothesis (3.10), we get

\[ A_3 n_s = 0, \quad C = 0, \tag{3.12} \]

for all solutions of the algebraic system (3.5), corresponding to the values \( v = v_\alpha, \alpha = 1, 2 \). Since we have a free choice of coordinate axes no loss of generality is involved in taking the \( x_1 \)-axis to coincide with the direction of propagation. Then we can associate the displacement transverse components \( u_2 \) and \( u_3 \) with shear waves. It becomes clear from the above relations that the shear waves are not subjected to any thermal modifications. Thus, for \( v = v_1 = v_2 = c_2 \), we have the undamped in time waves \( U^{(1)} = \{u_r^{(1)}, \tau^{(1)}\} \) and \( U^{(2)} = \{u_r^{(2)}, \tau^{(2)}\} \) as given by

\[
\begin{array}{l}
\begin{aligned}
&u_1^{(1)} = 0, \quad u_2^{(1)} = \frac{1}{\chi} e^{i x (x_1 - c_2 t)}, \quad u_3^{(1)} = 0, \quad \tau^{(1)} = 0 \\
&u_1^{(2)} = 0, \quad u_2^{(2)} = 0, \quad u_3^{(2)} = \frac{1}{\chi} e^{i x (x_1 - c_2 t)}, \quad \tau^{(2)} = 0.
\end{aligned}
\end{array}
\tag{3.13} \]

Let us now study the solutions of the equation (3.9). From the algebraic system (3.5), we deduce

\[
(\mu - \rho v^2) A_5 = - \left[ \left( \lambda + \mu \right) A_r n_r + \frac{i \beta}{\lambda} C \right] n_s
\]

that proves that all the corresponding waves are longitudinal. For further convenience, we introduce the following notations

\[
v = i c_2 w, \quad c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \varepsilon = \frac{\beta^2 T_0}{\rho c_1^2 c}, \quad c = a T_0, \tag{3.15} \]

so that the equation (3.9) becomes

\[
\begin{aligned}
&\left( 1 + \frac{c_2^2}{c_1^2} w^2 \right) \left[ 1 + \chi \tau_T c_2 w + \frac{c_2 c}{\chi k} w \left( 1 + \chi \tau_q c_2 w + \frac{1}{2} \chi^2 \tau_q^2 c_2^2 w^2 \right) \right] \\
&+ \frac{\varepsilon c_2 c}{\chi k} w \left( 1 + \chi \tau_q c_2 w + \frac{1}{2} \chi^2 \tau_q^2 c_2^2 w^2 \right) = 0.
\end{aligned}
\tag{3.16} \]

We have here a polynomial algebraic equation of fifth degree with all positive coefficients. Thus, it can be easily seen that this last equation has always a negative solution that depends, among other parameters, upon the coupling thermoelastic coefficient \( \varepsilon \) and on the delay times \( \tau_q \) and \( \tau_T \).
We denote this solution by $w_3 = -\Gamma(\varepsilon, \tau_q, \tau_T) < 0$ and conclude that we have for $v$ the following value

$$v_3 = ic_2 w_3. \tag{3.18}$$

The corresponding longitudinal solution takes the form

$$u_1^{(3)} = -\frac{ic_1}{c_2} e^{ic_1 x} e^{-x c_2 \Gamma t},$$

$$u_2^{(3)} = u_3^{(3)} = 0$$

and

$$T^{(3)} = T_0 \left(1 + \frac{c_2^2 \Gamma^2}{c_1^2}\right) e^{ic_1 x} e^{-x c_2 \Gamma t}. \tag{3.19}$$

It can be seen that it is a quasi-thermal mode, like the classical purely thermal disturbance, which is a standing wave decaying exponentially to zero like $e^{-x c_2 \Gamma t}$ when the time goes to infinity. In fact, this quasi-thermal mode can be set in connection with that of the classical thermoelasticity.

To this aim, we observe that the solutions of the equation (3.17) depend upon the coupling thermoelastic coefficient $\varepsilon$ and, moreover, when $\varepsilon$ vanishes, these solutions satisfy

$$1 + \frac{c_2^2}{c_1^2} w^2 = 0 \tag{3.20}$$

and

$$D(w) = 0, \tag{3.21}$$

where

$$D(w) \equiv \frac{x c}{2k} \frac{\tau_q}{\tau_q} c_2^3 w^3 + \frac{c q}{k} c_2^2 w^2 + \left(\frac{c}{x k} + x \tau_T\right) c_2 w + 1. \tag{3.22}$$

The equation (3.20) leads to the solution

$$w_4(\varepsilon = 0) = -\frac{c_1}{c_2} \tag{3.23}$$

which gives the value

$$v_4(\varepsilon = 0) = c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \tag{3.24}$$

and which leads to the classical elastic longitudinal wave propagating with velocity $c_1$.

Let us now consider the equation (3.21). It is easy to see that it has always a negative solution, denoted by $-\gamma$ and note that

$$\gamma(\tau_q, \tau_T) = \Gamma(\varepsilon = 0, \tau_q, \tau_T). \tag{3.25}$$

Furthermore, we note that, for $\tau_q = 0$ and $\tau_T = 0$, the equation (3.21) becomes

$$\frac{c_2 c}{x k} w + 1 = 0, \tag{3.26}$$

a relation that recalls the corresponding solution of the classical heat conduction theory

$$v_3(\varepsilon = 0, \tau_q = 0, \tau_T = 0) = -\frac{x k}{c} i, \tag{3.27}$$

with the purely thermal mode

$$T_0 e^{-(x k/c) t}. \tag{3.28}$$

Thus, the standing longitudinal wave can be considered like an extension of the classical thermal mode for the model in concern. To have a complete discussion on the equation (3.21), we have to note that it has two other complex conjugate solutions. In order to see this, we observe that

$$\frac{D}{dw}(w) = \frac{3x c}{2k} \frac{\tau_q^2}{\tau_q} c_2^2 w^2 + \frac{2c q}{k} c_2^2 w + \left(\frac{c}{x k} + x \tau_T\right) c_2 \geq 0, \quad \text{for all } w \in \mathbb{R}, \tag{3.29}$$

and hence $D(w)$ is an increasing function with respect to $w \in \mathbb{R}$ and, therefore, the equation (3.21) has only one real solution, namely, $-\gamma$, and the other two are complex conjugates.
Turning to the algebraic equation (3.17), it is important to note that one can have the following two cases of interest: (a) two negative solutions, \( w_{5,6} < 0 \), when the corresponding longitudinal solutions are standing waves decaying exponentially to zero when the time goes to infinity and (b) two complex conjugate solutions having a negative real part when we have a plane harmonic longitudinal quasi-thermal wave which is damped in time. Thus, in the case (a), we will have two standing waves of the type (3.28), while in the case (b), we will have the solutions

\[
w_{5,6} = -\alpha \pm i\delta, \quad \alpha \geq 0, \quad \delta \geq 0
\]

(3.30)

that characterize a plane harmonic in time longitudinal quasi-thermal wave decaying exponentially to zero like \( e^{-\alpha t} \) when time goes to infinity. It should be expected that the amplitude of these longitudinal waves decrease more rapidly to zero when time increases to infinity, with respect to the case of the quasi-thermal longitudinal wave.

Concluding, in what follows we will denote by \( v_4 = ic_2 w_4 \), \( v_5 = ic_2 w_5 \) and \( v_6 = ic_2 w_6 \), the solutions of the equation (3.9) which satisfy the conditions

\[
v_4(\epsilon = 0) = \sqrt{\frac{\lambda + 2\mu}{\epsilon}}
\]

(3.31)

and

\[
v_5(\epsilon = 0) = ic_2 w_5(\epsilon = 0), \quad v_6(\epsilon = 0) = ic_2 w_6(\epsilon = 0),
\]

(3.32)

when we have two negative solutions for \( w \), or

\[
v_5(\epsilon = 0) = -c_2(\delta + i\alpha), \quad v_6(\epsilon = 0) = c_2(\delta - i\alpha),
\]

(3.33)

when we have two complex conjugate solutions for \( w \). Thus, the corresponding longitudinal waves are given by

\[
\begin{align*}
  u_1^{(4)} &= -\frac{1}{\kappa} \left[ 1 - i\kappa \tau q v_4 - \frac{i\beta T_0 v_4}{\kappa} \left( 1 - i\kappa \tau q v_4 - \frac{1}{2} \kappa^2 \tau q^2 v_4^2 \right) \right] e^{i\kappa(\tau - t Re(v_4))} e^{\kappa t} Im(v_4), \\
  u_2^{(4)} &= u_3^{(4)} = 0, \\
  T^{(4)} &= \frac{\beta T_0 v_4}{\kappa} \left( 1 - i\kappa \tau q v_4 - \frac{1}{2} \kappa^2 \tau q^2 v_4^2 \right) e^{i\kappa(\tau - t Re(v_4))} e^{\kappa t} Im(v_4),
\end{align*}
\]

(3.34)

and

\[
\begin{align*}
  u_1^{(5,6)} &= -\frac{i\beta T_0}{\kappa(\lambda + 2\mu)} e^{i\kappa \tau q} e^{-\kappa c_2 w_{5,6} t}, \\
  u_2^{(5,6)} &= u_3^{(5,6)} = 0, \\
  T^{(5,6)} &= T_0 \left( 1 - \frac{v_{5,6}^2}{c_1^2} \right) e^{i\kappa \tau q} e^{-\kappa c_2 w_{5,6} t},
\end{align*}
\]

(3.35)

when the equation (3.17) has two negative solutions, or

\[
\begin{align*}
  u_1^{(5)} &= -\frac{i\beta T_0}{\kappa(\lambda + 2\mu)} e^{i\kappa(\tau - t Re(v_5))} e^{\kappa t} Im(v_5), \\
  u_2^{(5)} &= u_3^{(5)} = 0, \\
  T^{(5)} &= T_0 \left( 1 - \frac{v_5^2}{c_1^2} \right) e^{i\kappa(\tau - t Re(v_5))} e^{\kappa t} Im(v_5),
\end{align*}
\]

(3.36)

and

In conclusion, in a thermoelastic space with dual-phase-lag there are two transverse plane harmonic wave solutions and, possibly, three or four longitudinal wave solutions. More precisely, we have a damped in time longitudinal quasi-elastic wave as given by the relation (3.34) whose amplitude decreases exponentially to zero when the time is going to infinity. Always, we have a quasi-thermal mode, as given by the relation (3.19), which is a standing wave decaying
exponentially to zero when the time goes to infinity. Moreover, there are possibly two other quasi-thermal modes as given by the relation (3.35) or one plane harmonic in time wave solution as given by the relation (3.36). These last types of wave solutions are due to the presence of the delay time $\tau_q$, while the presence of the delay time $\tau_T$ influences only the decay rates with respect to time.

**Remark 3.1.** For $\tau_q = 0$, the equation (3.9) becomes
\[
(\lambda + 2\mu - \rho v^2) \left[ k - \frac{iv}{\kappa} (\alpha^2 k \tau_T + aT_0) \right] - \frac{i}{\kappa} \beta^2 T_0 v = 0,
\]
when we have only a damped in time longitudinal quasi-elastic wave and a standing wave decaying exponentially to zero when the time goes to infinity.

**Remark 3.2.** When the term $\tau_q^2$ is negligible, the equation (3.22) becomes
\[
c_2^2 c \tau_q w^2 + \left( \frac{c_2 \kappa}{\alpha} + \kappa \tau_T c_2 k \right) w + k = 0.
\]
Since the discriminant of this equation is
\[
\left( \frac{c_2 \kappa}{\alpha} - \kappa \tau_T c_2 k \right)^2 + 4c_2^2 k (\tau_T - \tau_q),
\]
it follows that for materials with $\tau_T \geq \tau_q$ (when the heat flux vector precedes the temperature gradient in the time-history, implying that the heat flux vector is the cause, while the temperature gradient is the effect of the heat flow), we have two negative solutions for $w$ leading to two standing waves decaying exponentially to zero when the time goes to infinity.

However, when
\[
\tau_q > \frac{\alpha^2 k}{4c} \left( \tau_T + \frac{c}{\alpha^2 k} \right)^2,
\]
there should be only a thermal plane harmonic wave. Thus, in this case, the thermoelastic model can admit only a damped in time longitudinal quasi-elastic wave and a damped in time longitudinal quasi-thermal wave.

### 4. Surface waves on a half-space

In this section, we consider that the dual-phase-lag thermoelastic material occupies the half space $x_2 \geq 0$ and is free of supply loads. Moreover, we assume that the surface $x_2 = 0$ is free of traction and free to exchange heat with the contents of the region $x_2 < 0$ and prior to the appearance of a disturbance both media are everywhere at constant temperature $T_0$. Therefore, for a surface wave propagating in the direction of the $x_1$-axis in the half space $x_2 > 0$, in what follows we will consider the following boundary conditions
\[
t_{2r} (x_1,0,x_3,t) = 0, \quad q_2 (x_1,0,x_3,t) = 0, \quad \text{for all } x_1, x_3 \in \mathbb{R}, t \geq 0.
\]

As a first step, we search for solutions of the basic equations (2.8) in the class of displacements and temperature variations $\{u_r, T\}$ characterized by the following requirements: (i) $\{u_r, T\}$, together with the corresponding state of stress and heat flux, decay asymptotically with the depth into the considered half space, that is they decay to zero when $x_2$ tends to infinity; (ii) the internal and dissipation energies associated with $\{u_r, T\}$ have to be finite; (iii) $\{u_r, T\}$ is damped in time that is it decreases with respect to the time variable.
In order to satisfy (i) we require that
\[ \sigma \] and, when the coupling thermoelastic coefficient
\[ \varepsilon \] and \( v \) the same meanings as in the above section.
In order to satisfy (i) we require that \( \sigma \) be a complex parameter so that
\[ \text{Im}(\sigma) < 0. \] (4.3)

It is obvious that the points (ii) and (iii) will be fulfilled if we require that \( v \) satisfies the relations
(3.3) and (3.4). We proceed to determine the parameter \( v \) in order that the displacement and temperature variation given by (4.2) satisfy the basic equations (2.8). Thus, by replacing (4.2) into relation (2.8), we find
\[ (\lambda + 2\mu - \varrho v^2 + \mu \sigma^2) A_1 - (\lambda + \mu) \sigma A_2 + \frac{i\beta}{\kappa} C = 0, \]
\[ - (\lambda + \mu) \sigma A_1 + [(\lambda + 2\mu)\sigma^2 + \mu - \varrho v^2] A_2 - \frac{i\beta}{\kappa} \sigma C = 0, \]
\[ [\mu(\sigma^2 + 1) - \varrho v^2] A_3 = 0 \] (4.4)

and
\[ \nu \beta T_0 \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) (A_1 - \sigma A_2) \]
\[ + \left[ k(1 + \sigma^2)(1 - i\kappa \tau_T v) - \frac{icv}{\kappa} \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) \right] C = 0. \]

Assuming that the wave propagation happens into the plane \( x_1Ox_2 \), then we can assume that \( \nu_3 = 0 \) and hence \( A_3 = 0 \). Moreover, by combining the first two equations of (4.4), we are led to the following algebraic system
\[ \left[ \mu(\sigma^2 + 1) - \varrho v^2 \right] (\sigma A_1 + A_2) = 0, \]
\[ \left[ (\lambda + 2\mu)(\sigma^2 + 1) - \varrho v^2 \right] (A_1 - \sigma A_2) + \frac{i\beta}{\kappa} (\sigma^2 + 1) C = 0 \]
and
\[ \nu \beta T_0 \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) (A_1 - \sigma A_2) \]
\[ + \left[ k(1 + \sigma^2)(1 - i\kappa \tau_T v) - \frac{icv}{\kappa} \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) \right] C = 0. \] (4.5)

For non-trivial solutions, the determinant of this system’s coefficient matrix must be zero. Thus, we are led to consider the dispersion relation
\[ \left( \sigma^2 + 1 - \frac{v^2}{c_1^2} \right) \left( \sigma^2 + 1 - \frac{v^2}{c_2^2} \right) \left[ (\sigma^2 + 1)(1 - i\kappa \tau_T v) - \frac{icv}{\kappa^2} \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) \right] \]
\[ - \frac{icv}{\kappa^2} (\sigma^2 + 1) \left( 1 - i\kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) = 0. \] (4.6)

Let us denote by \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) the roots of the dispersion relation (4.6) which satisfy the condition expressed by (4.3). Obviously, we have
\[ \sigma_3 = -i \sqrt{1 - \frac{v^2}{c_1^2}}, \] (4.7)
and, when the coupling thermoelastic coefficient \( \varepsilon \) is vanishing, we must have
\[ \sigma_1(\varepsilon = 0) = -i \sqrt{1 - \frac{v^2}{c_1^2}}, \quad \sigma_2(\varepsilon = 0, \tau_q = 0, \tau_T = 0) = -i \sqrt{1 - \frac{icv}{\kappa k}}. \] (4.8)
For \( \sigma = \sigma_1 \), we have the eigensolution \( U^{(1)} = \{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \theta^{(1)}\} \) as given by

\[
\begin{align*}
    v_1^{(1)} &= \frac{1}{\chi} \left[ (1 - i\chi \tau v) - \frac{i\chi v}{\chi k(\sigma_1^2 + 1)} \left(1 - i\chi \tau q v - \frac{1}{2} \chi^2 \tau q^2 v^2\right) \right] e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    v_2^{(1)} &= -\frac{\sigma_1}{\chi} \left[ (1 - i\chi \tau v) - \frac{i\chi v}{\chi k(\sigma_1^2 + 1)} \left(1 - i\chi \tau q v - \frac{1}{2} \chi^2 \tau q^2 v^2\right) \right] e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    v_3^{(1)} &= 0
\end{align*}
\]

and

\[
\theta^{(1)} = -\frac{v\beta T_0}{\chi k} \left(1 - i\chi \tau q v - \frac{1}{2} \chi^2 \tau q^2 v^2\right) e^{i\chi(x_1 - \sigma_1 x_2 - vt)},
\]

for \( \sigma = \sigma_2 \) we have the eigensolution \( U^{(2)} = \{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \theta^{(2)}\} \) as given by

\[
\begin{align*}
    v_1^{(2)} &= -\frac{i\chi T_0 \beta}{\chi (\lambda + 2\mu)} e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    v_2^{(2)} &= \frac{i\chi T_0 \beta}{\chi (\lambda + 2\mu)} \sigma_2 e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    v_3^{(2)} &= 0
\end{align*}
\]

and

\[
\theta^{(2)} = T_0 \left(\sigma_2^2 + 1 - \frac{v^2}{c_T^2}\right) e^{i\chi(x_1 - \sigma_2 x_2 - vt)},
\]

while for \( \sigma = \sigma_3 \) we have the eigensolution \( U^{(3)} = \{v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, \theta^{(3)}\} \) as given by

\[
\begin{align*}
    v_1^{(3)} &= \frac{\sigma_3}{\chi} e^{i\chi(x_1 - \sigma_3 x_2 - vt)}, \\
    v_2^{(3)} &= \frac{1}{\chi} e^{i\chi(x_1 - \sigma_3 x_2 - vt)}, \\
    v_3^{(3)} &= 0 \\
    \theta^{(3)} &= 0.
\end{align*}
\]

The relevant stress components corresponding to the eigensolutions (4.9)–(4.11) are, respectively, given by

\[
\begin{align*}
    f_{12}^{(1)} &= T_{12} e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    f_{12}^{(2)} &= T_{12} e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    f_{12}^{(3)} &= T_{12} e^{i\chi(x_1 - \sigma_3 x_2 - vt)}, \\
    q_2^{(1)} &= Q_2 e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    q_2^{(2)} &= Q_2 e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    q_2^{(3)} &= Q_2 e^{i\chi(x_1 - \sigma_3 x_2 - vt)}
\end{align*}
\]

and

\[
\begin{align*}
    f_{22}^{(1)} &= T_{22} e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    f_{22}^{(2)} &= T_{22} e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    f_{22}^{(3)} &= T_{22} e^{i\chi(x_1 - \sigma_3 x_2 - vt)}, \\
    q_2^{(1)} &= Q_2 e^{i\chi(x_1 - \sigma_1 x_2 - vt)}, \\
    q_2^{(2)} &= Q_2 e^{i\chi(x_1 - \sigma_2 x_2 - vt)}, \\
    q_2^{(3)} &= Q_2 e^{i\chi(x_1 - \sigma_3 x_2 - vt)}
\end{align*}
\]
where

\[
T_{12}^{(1)} = -2i\mu\sigma_1 \left[ 1 - i\tau T v - \frac{icv}{\kappa'(s_1^2 + 1)} \left( 1 - i\tau T v - \frac{1}{2}x^2\tau_0^2 v^2 \right) \right],
\]

\[
T_{22}^{(1)} = i[\lambda + (\lambda + 2\mu)s_1^2] \left[ 1 - i\tau T v - \frac{icv}{\kappa'(s_1^2 + 1)} \left( 1 - i\tau T v - \frac{1}{2}x^2\tau_0^2 v^2 \right) \right] + \frac{v\beta^2 T_0}{\kappa'} \left( 1 - i\tau T v - \frac{1}{2}x^2\tau_0^2 v^2 \right),
\]

\[
Q_{2}^{(1)} = -iv\beta T_0 (1 - i\tau T v)\sigma_1,
\]

\[
T_{12}^{(2)} = -\frac{2\mu\beta T_0}{\lambda + 2\mu}\sigma_2,
\]

\[
T_{22}^{(2)} = T_0\beta \left( -\frac{2\mu}{\lambda + 2\mu} + \frac{v^2}{c_1^2} \right),
\]

\[
Q_{2}^{(2)} = \frac{ixk T_0 \sigma_2 (1 - i\tau T v)}{1 - i\tau T v - (1/2)x^2\tau_0^2 v^2} \left( \sigma_2^2 + 1 - \frac{v^2}{c_1^2} \right),
\]

\[
T_{12}^{(3)} = i\mu(1 - \sigma_3^2),
\]

\[
T_{22}^{(3)} = -2i\mu\sigma_3,
\]

\[
Q_{2}^{(3)} = 0.
\]

The second step is to search for a solution of our surface wave problem as a linear combination of the eigensolutions \(U^{(1)}, U^{(2)}\) and \(U^{(3)}\), that is

\[
U = \text{Re} \{d_1 U^{(1)} + d_2 U^{(2)} + d_3 U^{(3)}\}, \quad U = \{u_1, u_2, u_3, T\}, \quad (4.18)
\]

where \(d_1, d_2\) and \(d_3\) are complex parameters to be determined in order that the boundary conditions expressed by (4.1) be fulfilled. It is clear that (4.18) satisfies the basic differential system (2.8). Moreover, we have

\[
t_{12} (d_1 U^{(1)} + d_2 U^{(2)} + d_3 U^{(3)}) = d_1 t_{12}^{(1)} + d_2 t_{12}^{(2)} + d_3 t_{12}^{(3)},
\]

\[
t_{22} (d_1 U^{(1)} + d_2 U^{(2)} + d_3 U^{(3)}) = d_1 t_{22}^{(1)} + d_2 t_{22}^{(2)} + d_3 t_{22}^{(3)},
\]

\[
q_2 (d_1 U^{(1)} + d_2 U^{(2)} + d_3 U^{(3)}) = d_1 q_2^{(1)} + d_2 q_2^{(2)} + d_3 q_2^{(3)},
\]

so that the boundary conditions (4.1) are satisfied if

\[
d_1 T_{12}^{(1)} + d_2 T_{12}^{(2)} + d_3 T_{12}^{(3)} = 0,
\]

\[
d_1 T_{22}^{(1)} + d_2 T_{22}^{(2)} + d_3 T_{22}^{(3)} = 0
\]

and

\[
d_1 Q_{2}^{(1)} + d_2 Q_{2}^{(2)} + d_3 Q_{2}^{(3)} = 0.
\]

For non-trivial solution of the Rayleigh wave problem, the determinant of this system’s coefficient matrix must be zero, that is

\[
\mathcal{D}(v) = \begin{vmatrix}
T_{12}^{(1)} & T_{12}^{(2)} & T_{12}^{(3)} \\
T_{22}^{(1)} & T_{22}^{(2)} & T_{22}^{(3)} \\
Q_{2}^{(1)} & Q_{2}^{(2)} & Q_{2}^{(3)}
\end{vmatrix} = 0, \quad (4.21)
\]

which represents the secular equation for the complex parameter \(v\) whose real part gives the wave speed and whose imaginary part gives the rate of damping in time. Moreover, we can write (4.21)
in the following form
\[
\frac{\epsilon}{\kappa} \left\{ 4 \sigma_1 \sigma_3 \left( 2 - \frac{v^2}{c_1^2} \right) \left[ \frac{c_1^2}{c_2^2} (\sigma_1^2 + 1) - 2 \right] \right\} 
\times \left[ 1 - i \kappa \tau_T v - \frac{ic v}{\kappa k (\sigma_1^2 + 1)} \left( 1 - i \kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) \right] 
+ \frac{\epsilon c^2}{\kappa k c_2^2} v \left( 1 - i \kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) 
\times \left\{ \frac{4 c^2}{c_1^2} \sigma_2 \sigma_3 + \left( 2 - \frac{v^2}{c_1^2} \right) \left[ \frac{2 c_2^2}{c_1^2} - \frac{v^2}{c_1^2} - \sigma_2 \left( \sigma_2^2 + 1 - \frac{v^2}{c_1^2} \right) \right] \right\} = 0.
\]

(4.22)

So we have to select the solutions of the secular equation (4.22) that satisfy the conditions (3.3) and (3.4).

Remark 4.1. It is easy to get from the relation (4.22), the secular equation of the corresponding Rayleigh wave problem of the classical elasticity. In fact, if we set \( \epsilon = 0 \) and we neglect the terms containing \( \tau_T \) and \( \tau_q \) into relation (4.22), we get
\[
4 \left( \frac{1 - \frac{v^2}{c_1^2}}{1 - \frac{v^2}{c_2^2}} \right)^2 \left( 2 - \frac{v^2}{c_1^2} \right)^2 = \left( 2 - \frac{v^2}{c_2^2} \right)^2,
\]
with
\[
\sigma_1 = -i \sqrt{1 - \frac{v^2}{c_1^2}}, \quad \sigma_3 = -i \sqrt{1 - \frac{v^2}{c_2^2}}.
\]

(4.23)

(4.24)

Remark 4.2. When we neglect the terms containing \( \tau_T \) and \( \tau_q \) into relation (4.22) we get the secular equation for the classical linear thermoelasticity as studied in [24,25] and which in our terms is
\[
\frac{\epsilon}{\kappa} \left\{ 4 \sigma_1 \sigma_3 \left( 2 - \frac{v^2}{c_1^2} \right) \left[ \frac{c_1^2}{c_2^2} (\sigma_1^2 + 1) - 2 \right] \right\} 
\times \left[ 1 - i \kappa \tau_T v - \frac{ic v}{\kappa k (\sigma_1^2 + 1)} \left( 1 - i \kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) \right] 
+ \frac{\epsilon c^2}{\kappa k c_2^2} v \left( 1 - i \kappa \tau_q v - \frac{1}{2} \kappa^2 \tau_q^2 v^2 \right) 
\times \left\{ \frac{4 c^2}{c_1^2} \sigma_2 \sigma_3 + \left( 2 - \frac{v^2}{c_1^2} \right) \left[ \frac{2 c_2^2}{c_1^2} - \frac{v^2}{c_1^2} - \sigma_2 \left( \sigma_2^2 + 1 - \frac{v^2}{c_1^2} \right) \right] \right\} = 0,
\]

(4.25)

where now \( \sigma_1 \) and \( \sigma_2 \) are solutions of the equation
\[
\left( \sigma^2 + 1 - \frac{v^2}{c_1^2} \right) \left( \sigma^2 + 1 - \frac{ic v}{\kappa k} \right) - \frac{ic \epsilon v}{\kappa k (\sigma^2 + 1)} = 0,
\]

(4.26)

and \( \sigma_3 \) is given by (4.24)_2.

5. Numerical simulations

In this section, we would like to investigate, from a numerical point of view, the behaviour of the complex parameter \( v \), or equivalently \( w \), as defined by (3.16). To this aim, we will take the following values for the relevant parameters of copper material (see e.g. [25,26]):
\[
c_1 = 4631.0 \text{ m s}^{-1},
\]
\[
c_2 = 2280.1 \text{ m s}^{-1},
\]
\[
\frac{c}{\kappa} = 8066.8 \text{ s m}^{-2}
\]
and
\[
\epsilon = 1.68 \times 10^{-2}.
\]
In particular, we look for a numerical solution of equation (3.17) regarding longitudinal plane harmonic waves, and equation (4.22) regarding surface waves.

For what concerns (3.17), using the parameters of copper given above and looking for numerical solution for different values of \( \tau_q \) in the range \([10^{-10} \text{ s}, 10^{-1} \text{ s}]\), and the same range for \( \tau_T \), we find that the first root is real and negative, as expected. Its value \( w_1 = -5.34639 \times 10^{-8} \text{ s} \) is almost constant, within 0.03\%, regardless of the values of \( \tau_q \) and \( \tau_T \).

Second and third roots are complex conjugates, but by restriction (3.3) we only consider the one with negative imaginary part, say \( w_2 \). It has same real part and imaginary part, within 0.03\%, furthermore this common value varies very little with respect to \( \tau_T \): the relative variation is always within 0.03\%, while the variation with respect to \( \tau_q \) is shown in figure 1.

Finally, fourth and fifth roots are the more interesting and, as before, we only need to consider the one with negative imaginary part, say \( w_4 \). The imaginary part of \( w_4 \) is almost constant (its relative variation is 0.00027\%) with value \( \text{Im}(w) = -2.048 \text{ s} \), while the real part ranges from \(-1.67 \times 10^{-7} \text{ s} \) to \(9.3 \times 10^{-11} \text{ s} \), and assumes also positive values that should be discarded by (3.4). In figure 2 (upper left), we can see the plateau where the real value is positive, while we can also see a minimum around \( \tau_q = 2.3 \times 10^{-4} \text{ s} \) and \( \tau_T = 10^{-1} \text{ s} \).
We turn now to relation (4.22). We note that this relation contains the unknown \( v \) both explicitly and implicitly through \( \sigma_i, i = 1, 2, 3 \), that should be taken as the solutions of relation (4.6). The solution of this system of two nonlinear equations is not easy, and we take another approach.

We define

\[
\mathcal{F}(\text{Re}(v), \text{Im}(v)) = \ln |D(v)|,
\]

where \( D(v) \) is the left-hand side of relation (4.22). The presence of the logarithm is convenient because the function has a wide range of variability. We can now make a graphics of the function \( \mathcal{F} \) looking for a minimum. In figure 3, we have such a graphics for \( \tau_q = \tau_T = 0.1 \text{ s} \) and \( \chi = 1 \text{ m}^{-1} \), where it is possible to see the presence of a minimum around \( v = 10(1 - i) \text{ m s}^{-1} \).

Figure 4 shows a similar graphics, but for \( \tau_q = \tau_T = 0.01 \text{ s} \) and \( \chi = 2 \text{ m}^{-1} \). Here the minimum is around \( v = 50(1 - i) \text{ m s}^{-1} \). These graphics show the same structure, but the scale is different.

An interesting feature that arises from plotting several graphics for different values of \( \tau_q, \tau_T \) and \( \chi \) is that the root is always around \( v = (1 - i)/(\chi \tau_q) \).

An important conclusion that we can draw from these examples is that the phenomena under consideration are by a large extent much more dependent on the value of \( \tau_q \) with respect to the
value of $\tau_T$, and this difference is found both in the case of plane waves as in the case of surface waves.

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