Scientific activity report for the research grant

Variational approaches to set-valued optimization and applications

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Concerning the scientific activity in the period July 2017 – December 2017, we obtained several preliminary results within the proposed objectives. These results are contained in two papers (see [1] and [5]). For more details, see

$$http://www.math.uaic.ro/\tilde{}zalinesc/ID-0188-ro.html$$

We briefly present next the main assertions from the two mentioned articles.

In the paper [5] we explore new, improved versions, of the celebrated Ekeland variational principle (EVP) which is known to be a very important tool in establishing many results in several domains of mathematics. One of its variants is the following:

Theorem 1 *Let (X, d) be a complete metric space and $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function which is bounded from below. Then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in X$ such that $f(\overline{x}) + d(\overline{x}, x_0) \leq f(x_0)$ and $f(\overline{x}) < f(u) + d(u, \overline{x})$ for $u \in X \setminus \{\overline{x}\}$.*

Taking x_0 such that $f(x_0) \leq \inf f + \varepsilon$, one gets rapidly $d(\overline{x}, x_0) \leq \varepsilon$, that is $\overline{x}$, solution of the perturbed problem, is close to the ε-solution of the minimization problem $\min f(x)$, $x \in X$.

Variants of EVP were established for $\overline{\mathbb{R}}$ replaced by a real separated topological vector space (tvs for short) Y quasi-ordered by a convex cone $K \subset Y$ ($y_1 \leq_K y_2$ if $y_2 - y_1 \in K$). The next step was to have EVP type results for f replaced by a set-valued function $\Gamma : X \rightrightarrows Y$ for Y a tvs quasi-ordered by the convex cone K and 2^Y quasi-ordered by $A_1 \leq_K^l A_2$ if $A_2 \subset A_1 + K$ ($A_1, A_2 \subset Y$). In this context several results were established by Ha [2], Hamel & Löhne [3], Liu & Ng [9], Qiu [10, 12], Khan, Tammer & Zalinescu [6]. Our aim is to provide a unified approach for establishing such results, getting as a by-product new sufficient conditions for having the usual conclusion.

As above, (X, d) is a metric space, Y is a tvs, Y^* is its topological dual, and $K \subseteq Y$ is a proper convex cone; as usual, K^+ is the positive dual cone of K:

$$K^+ = \{y^* \in Y^* \mid y^*(y) \geq 0 \ \forall y \in K\}.$$

As in [13] and [6], let $F : X \times X \rightrightarrows K$ satisfy the conditions:

(F1) $0 \in F(x, x)$ for all $x \in X$,

(F2) $F(x_1, x_2) + F(x_2, x_3) \subseteq F(x_1, x_3) + K$ for all $x_1, x_2, x_3 \in X$.

For F satisfying conditions (F1) and (F2), and $z^* \in K^+$, consider

$$\eta_{F,z^*} : X \times X \to \overline{\mathbb{R}}_+, \quad \eta_{F,z^*}(x, x') := \inf\{z^*(z) \mid z \in F(x, x')\}.$$

It follows immediately that

$$\eta_{F,z^*}(x, x) = 0 \ \text{ and } \ \eta_{F,z^*}(x, x'') \leq \eta_{F,z^*}(x, x') + \eta_{F,z^*}(x', x'') \quad \forall x, x', x'' \in X.$$

Using F we introduce the quasi-order $\preceq_F$ on $X \times 2^Y$ defined by

$$(x_1, A_1) \preceq_F (x_2, A_2) :\iff A_2 \subset A_1 + F(x_1, x_2) + K. \tag{1}$$

Of course,

$$(x_1, A_1) \preceq_F (x_2, A_2) \Rightarrow A_2 \subset A_1 + K \iff: A_1 \leq_K^l A_2;$$

moreover, by (F1), we have that

$$(x, A_1) \preceq_F (x, A_2) \iff A_2 \subset A_1 + K \iff A_1 \leq_K^l A_2.$$

Besides (F1) and (F2) we consider also the condition

(F3) there exists $z_F^* \in K^+$ such that

$$\eta(\delta) := \inf z_F^*(F_\delta) := \inf \{z_F^*(v) \mid v \in F_\delta\} > 0 \quad \forall \delta > 0,$$

where, for $\delta \geq 0$,

$$F_\delta := \cup\{F(x, x') \mid x, x' \in X,\ d(x, x') \geq \delta\};$$

clearly, condition (F3) can be rewritten as

$$\exists z_F^* \in K^+,\ \forall \delta > 0\ :\ \inf z_F^*(F_\delta) > 0.$$

A weaker condition is

$$\forall \delta > 0,\ \exists z^* \in K^+\ :\ \inf z^*(F_\delta) > 0. \tag{2}$$

An even weaker condition is the following

$$\forall \delta > 0,\ \forall (z_n) \subseteq F_\delta,\ \exists z^* \in K^+\ :\ \limsup z^*(z_n) > 0; \tag{3}$$

when (3) holds Qiu [11, Def. 3.5] says that F *is compatible with* d.

An important example of multifunction F satisfying conditions (F1) and (F2) is provided in the next result (proved in [6, Lem. 10.1.1]).

Lemma 2 *Let $\emptyset \neq H \subseteq K$ be a K–convex set. Consider*

$$F_H : X \times X \rightrightarrows K, \quad F_H(x, x') := d(x, x')H.$$

Then

(i) *F_H verifies (F1) and (F2).*

(ii) *F_H verifies condition (F3) iff F_H verifies condition (2) iff there exists $z_H^* \in K^+$ such that $\inf z_H^*(H) > 0$; if Y is a separated locally convex space, then F_H verifies condition (F3) iff $0 \notin \operatorname{cl}(H + K)$. Moreover, F_H verifies condition (3) iff*
$$\forall (h_n) \subseteq H,\ \exists z^* \in K^+\ :\ \limsup z^*(h_n) > 0.$$

For F satisfying conditions (F1) and (F2), and $z^* \in K^+$, we introduce the partial order $\preceq_{F,z^*}$ on $X \times 2^Y$ by

$$(x_1, A_1) \preceq_{F,z^*} (x_2, A_2) :\Longleftrightarrow \begin{cases} (x_1, A_1) = (x_2, A_2) \text{ or} \\ (x_1, A_1) \preceq_F (x_2, A_2) \text{ and } \inf z^*(A_1) < \inf z^*(A_2). \end{cases} \tag{4}$$

It is easy to verify that $\preceq_{F,z^*}$ is, indeed, reflexive, transitive, and antisymmetric. We denote by $\preceq_H$ (resp. $\preceq_{H,z^*}$) the partial order $\preceq_{F_H}$ (resp. $\preceq_{F_H,z^*}$) when $H \subset K$ is a K-convex set.

Below (X, d) is a metric space, Y is a separated topological vector space, $K \subset Y$ is a proper convex cone, and $F : X \times X \rightrightarrows K$ verifies conditions (F1) and (F2). On $X \times 2^Y$ we consider the quasi-order $\preceq_F$, as well as $\preceq_{F,z^*}$ for $z^* \in K^+$, defined in (1) and (4), respectively.

Moreover, we consider a nonempty set $\mathcal{A} \subseteq X \times 2^Y$. Because $(x, A) \preceq_F (x', \emptyset)$ for all $x, x' \in X$ and $A \in 2^Y$, in the sequel we assume that $A \neq \emptyset$ for every $A \in \operatorname{Pr}_{2^Y}(\mathcal{A})$; hence

$$Y_{\mathcal{A}} := \bigcup \{A \mid A \in \operatorname{Pr}_{2^Y}(\mathcal{A})\} \neq \emptyset.$$

An important example of set $\mathcal{A} \subseteq X \times 2^Y$ is

$$\mathcal{A}_\Gamma := \{((x, \Gamma(x))) \mid x \in \operatorname{dom} \Gamma\},$$

where $\Gamma : X \rightrightarrows Y$ with $\operatorname{dom} \Gamma \neq \emptyset$; of course, $Y_{\mathcal{A}_\Gamma} = \Gamma(X)$.

Ha [2] established an EVP type result for a set-valued function $\Gamma : X \rightrightarrows Y$ which corresponds to Kuroiwa optimality. Hamel [4] and Hamel–Löhne [3] established such results for subsets $\mathcal{A} \subseteq X \times 2^Y$ even for X a uniform space.

Theorem 3 *Assume that $(\mathcal{A}, \preceq_F)$ verifies condition*

(C0) $\forall\left((x_n, A_n)\right)_{n\geq 1} \subset \mathcal{A}$ $\preceq_F$*-decreasing :* $(x_n)_{n\geq 1}$ *is Cauchy and* $\exists (x, A) \in \mathcal{A}$ *such that* $(x, A) \preceq (x_n, A_n)$ $\forall n \geq 1$.

Then:

(i) *for every* $(x, A) \in \mathcal{A}$ *there exists* $(\overline{x}, \overline{A}) \in \mathcal{A}$ *such that* $(\overline{x}, \overline{A}) \preceq_F (x, A)$ *and* $\mathcal{A} \ni (x', A') \preceq_F (\overline{x}, \overline{A})$ *implies* $x' = \overline{x}$;

(ii) *assume that* $z^* \in K^+$ *is such that* $\inf z^*(A) > -\infty$ *for* $A \in \mathrm{Pr}_{2^Y}(\mathcal{A})$ *and* $\inf z^*(F(x, x')) > 0$ *for* $x, x' \in X$ *with* $x \neq x'$*; then for every* $(x, A) \in \mathcal{A}$ *there exists* $(\overline{x}, \overline{A}) \in \mathcal{A}$ *minimal with respect to* $\preceq_{F,z^*}$ *such that* $(\overline{x}, \overline{A}) \preceq_{F,z^*} (x, A)$ *and* $\mathcal{A} \ni (x', A') \preceq_F (\overline{x}, \overline{A})$ *implies* $x' = \overline{x}$.

Remark 4 *Note that, for having the conclusions of Theorem 3 (i) or (ii) only for a given* $(x_0, A_0) \in \mathcal{A}$*, it is sufficient to assume that (C0) is verified by the sets*

$$\mathcal{A}_F(x_0, A_0) := \{(x, A) \in \mathcal{A} \mid (x, A) \preceq_F (x_0, A_0)\},$$
$$\mathcal{A}_{F,z^*}(x_0, A_0) := \{(x, A) \in \mathcal{A} \mid (x, A) \preceq_{F,z^*} (x_0, A_0)\},$$

respectively.

Remark 5 *Taking* $F(x, x') := \{d(x, x')k^0\}$ *with* $k^0 \in K \setminus (-\operatorname{cl} K)$ *in Theorem 3 (using also Remark 4) one obtains [3, Th. 5.1] in the case* X *is a metric space. Indeed, on* $\mathcal{A}_0$*, if (M2) or (M2') is verified, then any* $\preceq_F$*–decreasing sequence in* $\mathcal{A}$ *is Cauchy. This together with (M3) shows that (C0) holds. In a similar way [3, Th. 6.1] can be obtained.*

Remark 6 *Taking* X*,* Y*,* K*,* H *and* $\Gamma : X \rightrightarrows Y$ *defined in Example 7 below,* $\mathcal{A}_\Gamma$ *verifies the conditions*

(C1) $\forall\left((x_n, A_n)\right)_{n\geq 1} \subset \mathcal{A}$ $\preceq$-decreasing with $(x_n)_{n\geq 1}$ Cauchy : $\exists (x, A) \in \mathcal{A}$ such that $(x, A) \preceq (x_n, A_n)$ $\forall n \geq 1$,

(C'1) $\forall\left((x_n, z_n)\right)_{n\geq 1} \subset \mathcal{A}$ $\preceq$-decreasing with $x_n \to x \in X$: $\exists z \in Z$ such that $(x, z) \in \mathcal{A}$ and $(x, z) \preceq (x_n, z_n)$ $\forall n \geq 1$,

for $F := F_H$, but not (C0). Moreover, the conclusion of Theorem 3 (i) does not hold. This shows that, in order to have the conclusion of Theorem 3 we need supplementary conditions besides (C'1) or (C1).

Example 7 *Let* $X := \mathbb{R}$ *and* $Y := \mathbb{R}^2$ *be endowed with their usual norms,* $K := \mathbb{R} \times \mathbb{R}_+$*,* $H := \{(y_1, y_2) \in \mathbb{R}^2_+ \mid y_1 y_2 \geq 1\}$*, and* $\Gamma : X \rightrightarrows Y$*,* $\Gamma(x) := \{(x, e^x)\}$*. It is clear that* H *is a closed convex subset of* $K \setminus \{(0, 0)\}$ *and* $K + \varepsilon H = \operatorname{int} K = \mathbb{R} \times \mathbb{R}^*_+$ *for* $\varepsilon > 0$*, where* $\mathbb{R}^*_+ := \mathbb{R}_+ \setminus \{0\}$*. One has* $\Gamma(x) + K = \mathbb{R} \times [e^x, \infty)$*, and so, for* $x, x' \in X$ *and* $\alpha > 0$*,*

$$\Gamma(x) \preceq_H \Gamma(x') \Leftrightarrow x \leq x' \Leftrightarrow \Gamma(x) \leq^l_K \Gamma(x'), \quad \Gamma(x') \subset \Gamma(x) + \alpha H + K \Leftrightarrow x < x';$$

moreover, $\Gamma(X) = \{(x, e^x) \mid x \in \mathbb{R}\} \subset K$*, which shows that* $\Gamma(X)$ *is* K*–bounded. So, for the sequence* $(x_n)_{n\geq 1} \subset X$ *with* $x_n \to x$ *and* $\Gamma(x_n) \subset \Gamma(x_{n+1}) + K$ *(that is* $\Gamma(x_{n+1}) \leq^l_K \Gamma(x_n)$*) for* $n \geq 1$*, we have that* $x_{n+1} \leq x_n$*, and so* $x \leq x_n$ *for* $n \geq 1$*, whence* $\Gamma(x_n) \subset \Gamma(x) + K$ *for* $n \geq 1$*. This shows that* (C'1) *and* (C1) *are verified; however, taking* $x_n := -n$ *(for* $n \geq 1$*) it is clear that* (C0) *is not verified.*

Remark 8 *Let* $\Gamma : X \rightrightarrows Y$ *have nonempty domain. Set* $x \preceq u$ *if* $(x, \Gamma(x)) \preceq_F (u, \Gamma(u))$*. Note that* $\mathcal{A}_\Gamma$ *verifies condition (C'1) if and only if* $S(u) := \{x \in X \mid x \preceq u\}$ *is* $\preceq$*–lower closed for every* $u \in X$*. This shows that condition (C'1) extends the dynamic closedness of a set-valued mapping as defined in [12] and elsewhere.*

Theorem 9 *Assume that the following two conditions hold:*

(i) F *verifies conditions* (F1)*,* (F2) *and* (F3)*;*

(ii) $\mathcal{A}$ *verifies* (C1) *and* $z^*_F(Y_\mathcal{A})$ *is bounded from below, where* $z^*_F \in K^+$ *is provided by* (F3)*.*

Then for every $(x, A) \in \mathcal{A}$ *there exists a minimal element* $(\overline{x}, \overline{A}) \in \mathcal{A}$ *with respect to* $\preceq_{F,z^*_F}$ *such that* $(\overline{x}, \overline{A}) \preceq_{F,z^*_F} (x, A)$*; moreover* $\mathcal{A} \ni (x', A') \preceq_F (\overline{x}, \overline{A})$ *implies* $x' = \overline{x}$.

In the case in which $F = F_H$ with $H \subset K$ a K-convex set, we provide several sufficient conditions for having the conclusion of the preceding theorem; these cover the majority of the EVP type results for set-valued mappings found in the literature.

In the paper [1] we explore some questions raised by Lasserre in [7] and [8], concerning the preservation of the necessary and sufficient optimality conditions from smooth convex optimization with inequalities constraints to the case where the feasible set is convex, but has no convex representation. The main results we obtain concerns, some relations between the hypotheses imposed by Lasserre and the Mangasarian-Fromowitz condition, and a barrier method based only on the geometric representation of the feasible set. Let X be a normed vector space and $f : X \to \mathbb{R}$ be a real-valued function. Take $\emptyset \neq M \subset X$ be a closed convex set and consider the standard geometrically constrained optimization problem

$$(P)\ \ \min f(x),\ \text{ s.t. } x \in M.$$

If f is convex and continuous, the Pshenichnyi-Rockafellar Theorem says that $\overline{x} \in M$ is a solution of (P) if and only if

$$\partial f(\overline{x}) \cap -N(M, \overline{x}) \neq \emptyset, \tag{5}$$

where ∂ denotes the convex subdifferential and N stands for the convex normal cone, that is

$$N(M, \overline{x}) := \{x^* \in X^* \mid \langle x^*, x - \overline{x}\rangle \leq 0, \forall x \in M\}.$$

Some interesting facts were derived in [7], [8] for the case when M is defined by functional inequalities, that is,

$$M := \{x \in X \mid g_i(x) \leq 0,\ \forall i \in \overline{1, n}\}, \tag{6}$$

where n is a nonzero natural number, and $g_i : X \to \mathbb{R}$ are scalar, not necessarily convex functions (that means one has a nonconvex representation for the convex set M). As usual, for $x \in M$ one denotes by $I(x)$ the set of active indices at x, that is, $I(x) := \{i \in \overline{1, n} \mid g_i(x) = 0\}$.

In order to present Karush-Kuhn-Tucker necessary and sufficient condition for this situation and the convergence of a barrier method in this setting, the works [7] and [8] use the following two assumptions in order get the results, that are:

- the Slater condition for M (there exists $x_0 \in M$ such that for all $i \in \overline{1, n}$, $g_i(x_0) < 0$) and
- the non-degeneracy condition: for every $i \in \overline{1, n}$,

$$x \in M, g_i(x) = 0 \Longrightarrow \nabla g_i(x) \neq 0. \tag{7}$$

We obtained the following results concerning the link between these conditions and the celebrated Mangasarian-Fromowitz constraint qualification condition.

Proposition 10 *Suppose that X is a Banach space, and $M \subset X$ is closed and convex. Take $x \in M$. Then the non-degeneracy condition at x and the Slater condition hold if and only if the Mangasarian-Fromowitz constraint qualification condition at x holds (that is, there exists $u \in X$ such that for every $i \in I(\overline{x})$ one has $\langle \nabla g_i(\overline{x}), u\rangle < 0$).*

Proposition 11 *If X is a Banach space and $M \subset X$ is closed and convex, under non-degeneracy condition, the Slater condition is equivalent to* $\operatorname{int} M \neq \emptyset$.

Next, we propose a barrier method for geometrically constrained problem (P) in the convex case (that is, f and M are convex) and we show its convergence to a minimum point. To this aim we use the oriented distance function associated to set $A \subset X$, $\Delta_A : X \to \mathbb{R}$, given as

$$\Delta_A(y) := d_A(y) - d_{Y \setminus A}(y).$$

Let $f : X \to \mathbb{R}$ be a continuous function and $\emptyset \neq M \subset X$ be a compact set with nonempty interior. Consider again the constrained optimization problem

$$(P)\ \ \min f(x),\ \text{ s.t. } x \in M.$$

Take $\mu > 0$ and define $\varphi_\mu : X \to \mathbb{R} \cup \{+\infty\}$ as

$$\varphi_\mu(x) = \begin{cases} f(x) - \mu \ln(-\Delta_M(x)), & \text{if } x \in \operatorname{int} M, \\ +\infty, & \text{otherwise.} \end{cases} \tag{8}$$

Observe that, in view of the properties of Δ_M, the function φ_μ is well defined.

Further, consider that $X = \mathbb{R}^p$ $(p \geq 1)$ and let $(x_n) \subset \operatorname{int} M$ be a sequence convergent to $x \in \operatorname{bd} M$. Then, from the continuity of Δ_M and f, one has that $\Delta_M(x_n) \to 0$ and $f(x_n) - \mu \ln(-\Delta_M(x_n)) \to +\infty$. This means that there is $x_\mu \in \operatorname{int} M$ which is a global minimum of φ_μ. Using the results in the previous results, under suitable qualification conditions, the point x is a Karush-Kuhn-Tucker point for (P). We prove the following results.

Theorem 12 *Suppose that $X = \mathbb{R}^p$ $(p \geq 1)$, f is a convex function, and M is a convex and compact set. Then for each $\mu > 0$, there exists a global minimizer x_μ of φ_μ in $\operatorname{int} M$, and if $\mu_n \downarrow 0$, every accumulation point of $\{x_{\mu_n}\}$ is a global minimizer of (P).*

Theorem 13 *Suppose that $X = \mathbb{R}^p$ $(p \geq 1)$, f is a convex function, and M given by (6) is a convex and compact set. If Slater and non-degeneracy conditions hold, then for each $\mu > 0$ there exists a global minimizer x_μ of φ_μ in $\operatorname{int} M$, and if $\mu_n \downarrow 0$, every accumulation point of $\{x_{\mu_n}\}$ is a global minimizer of (P) that satisfies Karush-Kuhn-Tucker conditions.*

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